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CONTACT CR-DOUBLY WARPED PRODUCT SUBMANIFOLDS IN KENMOTSU SPACE FORMS

ANDREEA OLTEANU

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF BUCHAREST
STR. ACADEMIEI 14
011014 BUCHAREST, ROMANIA
andreea_d_olteanu@yahoo.com

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ABSTRACT. Recently, the author established general inequalities for CR-doubly warped products isometrically immersed in Sasakian space forms.

In the present paper, we obtain sharp estimates for the squared norm of the second fundamental form (an extrinsic invariant) in terms of the warping functions (intrinsic invariants) for contact CR-doubly warped products isometrically immersed in Kenmotsu space forms. The equality case is considered. Some applications are derived.

Key words and phrases: Doubly warped product, contact CR-doubly warped product, invariant submanifold, anti-invariant submanifold, Laplacian, mean curvature, Kenmotsu space form.

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1. INTRODUCTION

In 1978, A. Bejancu introduced the notion of CR-submanifolds which is a generalization of holomorphic and totally real submanifolds in an almost Hermitian manifold ([2]). Following this, many papers and books on the topic were published. The first main result on CR-submanifolds was obtained by Chen [4]: any CR-submanifold of a Kaehler manifold is foliated by totally real submanifolds. As non-trivial examples of CR-submanifolds, we can mention the (real) hypersurfaces of Hermitian manifolds.

Recently, Chen [5] introduced the notion of a CR-warped product submanifold in a Kaehler manifold and proved a number of interesting results on such submanifolds. In particular, he established a sharp relationship between the warping function f of a warped product CR-submanifold $M_1 \times_f M_2$ of a Kaehler manifold $\widetilde{M}$ and the squared norm of the second fundamental form $||h||^2$.

On the other hand, there are only a handful of papers about doubly warped product Riemannian manifolds which are the generalization of a warped product Riemannian manifold.

Recently, the author obtained a general inequality for CR-doubly warped products isometrically immersed in Sasakian space forms ([12]).

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In the present paper, we study contact CR-doubly warped product submanifolds in Kenmotsu space forms.

We prove estimates of the squared norm of the second fundamental form in terms of the warping function. Equality cases are investigated. Obstructions to the existence of contact CR-doubly warped product submanifolds in Kenmotsu space forms are derived.

2. PRELIMINARIES

A $(2m+1)$-dimensional Riemannian manifold $(\widetilde{M}, g)$ is said to be a Kenmotsu manifold if it admits an endomorphism ϕ of its tangent bundle $T\widetilde{M}$, a vector field ξ and a 1-form η satisfying:

$$\phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0,$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi), \tag{2.1}$$

$$\left(\widetilde{\nabla}_X \phi\right) Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \quad \widetilde{\nabla}_X \xi = X - \eta(X)\xi,$$

for any vector fields X, Y on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.

We denote by ω the fundamental 2-form of $\widetilde{M}$, i.e.,

$$\omega(X,Y) = g(\phi X, Y), \qquad \forall X, Y \in \Gamma\left(T\widetilde{M}\right). \tag{2.2}$$

It was proved that the pairing (ω, η) defines a locally conformal cosymplectic structure, i.e.,

$$d\omega = 2\omega \wedge \eta, \qquad d\eta = 0.$$

A plane section π in $T_p\widetilde{M}$ is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a ϕ-sectional curvature. A Kenmotsu manifold with constant ϕ-holomorphic sectional curvature c is said to be a Kenmotsu space form and is denoted by $\widetilde{M}(c)$.

The curvature tensor $\widetilde{R}$ of a Kenmotsu space form is given by [8]

$$\begin{aligned}\widetilde{R}(X,Y)Z = \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\{[\eta(X)Y - \eta(Y)X]\eta(Z) \\ + [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi + \omega(Y,Z)\phi X \\ - \omega(X,Z)\phi Y - 2\omega(X,Y)\phi Z\}.\end{aligned} \tag{2.3}$$

Let $\widetilde{M}$ be a Kenmotsu manifold and M an n-dimensional submanifold tangent to ξ. For any vector field X tangent to M, we put

$$\phi X = PX + FX, \tag{2.4}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of ϕX. Then P is an endomorphism of the tangent bundle TM and F is a normal bundle valued 1-form on TM.

The equation of Gauss is given by

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W), h(Y,Z)) - g(h(X,Z), h(Y,W)) \tag{2.5}$$

for any vectors X, Y, Z, W tangent to M.

Let $p \in M$ and $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m+1}\}$ be an orthonormal basis of the tangent space $T_p\widetilde{M}$, such that $e_1, ..., e_n$ are tangent to M at p. We denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{2.6}$$

As is known, M is said to be minimal if H vanishes identically.

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, ..., n\}, r \in \{n+1, ..., 2m+1\} \tag{2.7}$$

as the coefficients of the second fundamental form h with respect to $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m+1}\}$, and

$$||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.8}$$

By analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in a Kenmotsu manifold were considered (see, for example, [13]).

A submanifold M tangent to ξ is called an invariant (resp. anti-invariant) submanifold if $\phi(T_pM) \subset T_pM, \ \forall\, p \in M$ (resp. $\phi(T_pM) \subset T_p^{\perp}M, \forall\, p \in M$).

A submanifold M tangent to ξ is called a contact CR-submanifold ([13]) if there exists a pair of orthogonal differentiable distributions D and $D^{\perp}$ on M, such that:

(1) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \{\xi\}$, where $\{\xi\}$ is the 1-dimensional distribution spanned by ξ;
(2) $\mathcal{D}$ is invariant by ϕ, i. e., $\phi(\mathcal{D}_p) \subset \mathcal{D}_p, \forall p \in M$;
(3) $\mathcal{D}^{\perp}$ is anti-invariant by ϕ, i. e., $\phi\left(\mathcal{D}_p^{\perp}\right) \subset \mathcal{D}_p^{\perp}, \forall p \in M$.

In particular, if $\mathcal{D}^{\perp} = \{0\}$ (resp. $\mathcal{D} = \{0\}$), M is an invariant (resp. anti-invariant) submanifold.

3. CONTACT CR-DOUBLY WARPED PRODUCT SUBMANIFOLDS

Singly warped products or simply warped products were first defined by Bishop and O'Neill in [3] in order to construct Riemannian manifolds with negative sectional curvature.

In general, doubly warped products can be considered as generalizations of singly warped products.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and let $f_1 : M_1 \to (0, \infty)$ and $f_2 : M_2 \to (0, \infty)$ be differentiable functions.

The doubly warped product $M =_{f_2} M_1 \times_{f_1} M_2$ is the product manifold $M_1 \times M_2$ endowed with the metric

$$g = f_2^2 g_1 + f_1^2 g_2. \tag{3.1}$$

More precisely, if $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ are natural projections, the metric g is defined by

$$g = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2. \tag{3.2}$$

The functions f_1 and f_2 are called warping functions. If either $f_1 \equiv 1$ or $f_2 \equiv 1$, but not both, then we obtain a warped product. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we have a Riemannian product manifold. If neither f_1 nor f_2 is constant, then we have a non-trivial doubly warped product.

We recall that on a doubly warped product one has

$$\nabla_X Z = Z(\ln f_2) X + X(\ln f_1) Z, \tag{3.3}$$

for any vector fields X tangent to M_1 and Z tangent to M_2.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = \frac{1}{f_1}\{(\nabla^1_X X) f_1 - X^2 f_1\} + \frac{1}{f_2}\{(\nabla^2_Z Z) f_2 - Z^2 f_2\}, \tag{3.4}$$

where ∇^1, ∇^2 are the Riemannian connections of the Riemannian metrics g_1 and g_2 respectively.

By reference to [12], a doubly warped product submanifold $M =_{f_2} M_1 \times_{f_1} M_2$ of a Kenmotsu manifold $\widetilde{M}$, with M_1 a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 a β-dimensional anti-invariant submanifold of $\widetilde{M}$ is said to be a contact CR-doubly warped product submanifold.

We state the following estimate of the squared norm of the second fundamental form for contact CR-doubly warped products in Kenmotsu manifolds.

Theorem 3.1. *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Kenmotsu manifold and $M =_{f_2} M_1 \times_{f_1} M_2$ an n-dimensional contact CR-doubly warped product submanifold, such that M_1 is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 is a β-dimensional anti-invariant submanifold of $\widetilde{M}(c)$. Then:*

(i) *The squared norm of the second fundamental form of M satisfies*

$$||h||^2 \geq 2\beta||\nabla(\ln f_1)||^2 - 1], \tag{3.5}$$

where $\nabla(\ln f_1)$ is the gradient of $\ln f_1$.

(ii) *If the equality sign of (3.5) holds identically, then M_1 is a totally geodesic submanifold and M_2 is a totally umbilical submanifold of $\widetilde{M}$. Moreover, M is a minimal submanifold of $\widetilde{M}$.*

Proof. Let $M =_{f_2} M_1 \times_{f_1} M_2$ be a doubly warped product submanifold of a Kenmotsu manifold $\widetilde{M}$, such that M_1 is an invariant submanifold tangent to ξ and M_2 is an anti-invariant submanifold of $\widetilde{M}$.

For any unit vector fields X tangent to M_1 and Z, W tangent to M_2 respectively, we have:

$$\begin{aligned} g(h(\phi X, Z), \phi Z) &= g\left(\widetilde{\nabla}_Z \phi X, \phi Z\right) \\ &= g\left(\phi \widetilde{\nabla}_Z X, \phi Z\right) = g\left(\widetilde{\nabla}_Z X, Z\right) = g(\nabla_Z X, Z) = X \ln f_1, \\ g(h(\phi X, Z), \phi W) &= (X \ln f_1)\, g(Z, W). \end{aligned} \tag{3.6}$$

On the other hand, since the ambient manifold $\widetilde{M}$ is a Kenmotsu manifold, it is easily seen that

$$h(\xi, Z) = 0. \tag{3.7}$$

Obviously, (3.3) implies $\xi \ln f_1 = 1$. Therefore, by (3.6) and (3.7), the inequality (3.5) is immediately obtained.

Denote by h'' the second fundamental form of M_2 in M. Then, we get

$$g(h''(Z, W), X) = g(\nabla_Z W, X) = -(X \ln f_1)\, g(Z, W,)$$

or equivalently

$$h"(Z, W) = -g(Z, W)\, \nabla(\ln f_1). \tag{3.8}$$

If the equality sign of (3.5) identically holds, then we obtain

$$h(\mathcal{D}, \mathcal{D}) = 0, \quad h\left(\mathcal{D}^\perp, \mathcal{D}^\perp\right) = 0, \quad h\left(\mathcal{D}, \mathcal{D}^\perp\right) \subset \phi\mathcal{D}^\perp. \tag{3.9}$$

The first condition (3.9) implies that M_1 is totally geodesic in M. On the other hand, one has

$$g\left(h\left(X,\phi Y\right),\phi Z\right)=g\left(\widetilde{\nabla}_X\phi Y,\phi Z\right)=g\left(\nabla_X Y,Z\right)=0.$$

Thus M_1 is totally geodesic in $\widetilde{M}$.

The second condition in (3.9) and (3.8) imply that M_2 is a totally umbilical submanifold in $\widetilde{M}$.

Moreover, by (3.9), it follows that M is a minimal submanifold of $\widetilde{M}$. □

In particular, if the ambient space is a Kenmotsu space form, one has the following.

Corollary 3.2. *Let $\widetilde{M}\left(c\right)$ be a $(2m+1)$-dimensional Kenmotsu space form of constant ϕ-sectional curvature c and $M={}_{f_2}M_1\times_{f_1}M_2$ an n-dimensional non-trivial contact CR-doubly warped product submanifold, satisfying*

$$||h||^2=2\beta\left[||\nabla\left(\ln f_1\right)||^2-1\right].$$

Then, we have

(a) *M_1 is a totally geodesic invariant submanifold of $\widetilde{M}\left(c\right)$. Hence M_1 is a Kenmotsu space form of constant ϕ-sectional curvature c.*

(b) *M_2 is a totally umbilical anti-invariant submanifold of $\widetilde{M}\left(c\right)$. Hence M_2 is a real space form of sectional curvature $\varepsilon>\frac{c-3}{4}$.*

Proof. Statement (a) follows from Theorem 3.1.

Also, we know that M_2 is a totally umbilical submanifold of $\widetilde{M}\left(c\right)$. The Gauss equation implies that M_2 is a real space form of sectional curvature $\varepsilon\geq\frac{c-3}{4}$.

Moreover, by (3.3), we see that $\varepsilon=\frac{c-3}{4}$ if and only if the warping function f_1 is constant. □

4. ANOTHER INEQUALITY

In the present section, we will improve the inequality (3.5) for contact CR-doubly warped product submanifolds in Kenmotsu space forms. Equality case is characterized.

Theorem 4.1. *Let $\widetilde{M}\left(c\right)$ be a $(2m+1)$-dimensional Kenmotsu space form of constant ϕ-sectional curvature c and $M={}_{f_2}M_1\times_{f_1}M_2$ an n-dimensional contact CR-doubly warped product submanifold, such that M_1 is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 is a β-dimensional anti-invariant submanifold of $\widetilde{M}\left(c\right)$. Then:*

(i) The squared norm of the second fundamental form of M satisfies

$$||h||^2\geq2\beta\left[||\nabla\left(\ln f_1\right)||^2-\Delta_1(\ln f_1)-1\right]+\alpha\beta\left(c+1\right), \tag{4.1}$$

where Δ_1 denotes the Laplace operator on M_1.

(ii) The equality sign of (4.1) holds identically if and only if we have:

(a) M_1 is a totally geodesic invariant submanifold of $\widetilde{M}\left(c\right)$. Hence M_1 is a Kenmotsu space form of constant ϕ-sectional curvature c.

(b) M_2 is a totally umbilical anti-invariant submanifold of $\widetilde{M}\left(c\right)$. Hence M_2 is a real space form of sectional curvature $\varepsilon\geq\frac{c-3}{4}$.

Proof. Let $M={}_{f_2}M_1\times_{f_1}M_2$ be a contact CR-doubly warped product submanifold of a $(2m+1)$-dimensional Kenmotsu space form $\widetilde{M}\left(c\right)$, such that M_1 is an invariant submanifold tangent to ξ and M_2 is an anti-invariant submanifold of $\widetilde{M}\left(c\right)$.

We denote by ν be the normal subbundle orthogonal to $\phi\left(TM_2\right)$. Obviously, we have

$$T^{\perp}M=\phi\left(TM_2\right)\oplus\nu,\qquad\phi\nu=\nu.$$

For any vector fields X tangent to M_1 and orthogonal to ξ and Z tangent to M_2, equation (2.3) gives

$$\widetilde{R}(X,\phi X,Z,\phi Z)=\frac{c+1}{2}g(X,X)g(Z,Z).$$

On the other hand, by the Codazzi equation, we have

$$\text{(4.2)}\quad \widetilde{R}(X,\phi X,Z,\phi Z)=-g\left(\nabla^{\perp}_{X}h(\phi X,Z)-h(\nabla_X\phi X,Z)-h(\phi X,\nabla_X Z),\phi Z\right) \\ +g\left(\nabla^{\perp}_{\phi X}h(X,Z)-h(\nabla_{\phi X}X,Z)-h(X,\nabla_{\phi X}Z),\phi Z\right).$$

By using the equation (3.3) and structure equations of a Kenmotsu manifold, we get

$$\begin{aligned} &g\left(\nabla^{\perp}_{X}h(\phi X,Z),\phi Z\right)\\ &=Xg(h(\phi X,Z),\phi Z)-g\left(h(\phi X,Z),\nabla^{\perp}_{X}\phi Z\right)\\ &=Xg(\nabla_Z X,Z)-g\left(h(\phi X,Z),\phi\widetilde{\nabla}_X Z\right)\\ &=X((X\ln f_1)g(Z,Z))-(X\ln f_1)g(h(\phi X,Z),\phi Z)-g(h(\phi X,Z),\phi h_\nu(X,Z))\\ &=\left(X^2\ln f_1\right)g(Z,Z)+(X\ln f_1)^2g(Z,Z)-||h_\nu(X,Z)||^2, \end{aligned}$$

where we denote by $h_\nu(X,Z)$ the ν-component of $h(X,Z)$.

Also, by (3.6) and (3.3), we obtain respectively

$$g(h(\nabla_X\phi X,Z),\phi Z)=((\nabla_X X)\ln f_1)g(Z,Z),$$

$$g(h(\phi X,\nabla_X Z),\phi Z)=(X\ln f_1)g(h(\phi X,Z),\phi Z)=(X\ln f_1)^2g(Z,Z).$$

Substituting the above relations in (4.2), we find

$$\text{(4.3)}\quad \widetilde{R}(X,\phi X,Z,\phi Z)=2||h_\nu(X,Z)||^2-\left(X^2\ln f_1\right)g(Z,Z)+((\nabla_X X)\ln f_1)g(Z,Z) \\ -\left((\phi X)^2\ln f_1\right)g(Z,Z)+((\nabla_{\phi X}\phi X)\ln f_1)g(Z,Z).$$

Then the equation (4.3) becomes

$$\text{(4.4)}\quad 2||h_\nu(X,Z)||^2=\Big[\frac{c+1}{2}g(X,X)+\left(X^2\ln f_1\right)-((\nabla_X X)\ln f_1) \\ +\left((\phi X)^2\ln f_1\right)-((\nabla_{\phi X}\phi X)\ln f_1)\Big]g(Z,Z).$$

Let

$$\{X_0=\xi,X_1,...,X_\alpha,X_{\alpha+1}=\phi X_1,...,X_{2\alpha}=\phi X_\alpha,Z_1,...,Z_\beta\}$$

be a local orthonormal frame on M such that $X_0,...,X_{2\alpha}$ are tangent to M_1 and $Z_1,...,Z_\beta$ are tangent to M_2.

Therefore

$$\text{(4.5)}\quad 2\sum_{j=1}^{2\alpha}\sum_{t=1}^{\beta}||h_\nu(X_j,Z_t)||^2=\alpha\beta(c+1)-2\beta\Delta_1(\ln f_1).$$

Combining (3.5) and (4.5), we obtain the inequality (4.1).

The equality case can be solved similarly to Corollary 3.2. □

Corollary 4.2. *Let $\widetilde{M}(c)$ be a Kenmotsu space form with $c<-1$. Then there do not exist contact CR-doubly warped product submanifolds ${}_{f_2}M_1\times_{f_1}M_2$ in $\widetilde{M}(c)$ such that $\ln f_1$ is a harmonic function on M_1.*

Proof. Assume that there exists a contact CR-doubly warped product submanifold ${}_{f_2}M_1 \times_{f_1} M_2$ in a Kenmotsu space form $\widetilde{M}(c)$ such that $\ln f_1$ is a harmonic function on M_1.Then (4.5) implies $c \geq -1$. □

Corollary 4.3. *Let $\widetilde{M}(c)$ be a Kenmotsu space form with $c \leq -1$. Then there do not exist contact CR-doubly warped product submanifolds ${}_{f_2}M_1 \times_{f_1} M_2$ in $\widetilde{M}(c)$ such that $\ln f_1$ is a non-negative eigenfunction of the Laplacian on M_1 corresponding to an eigenvalue $\lambda > 0$.*

REFERENCES

[1] K. ARSLAN, R. EZENTAS, I. MIHAI AND C. MURATHAN, Contact CR-warped product submanifolds in Kenmotsu space forms, *J. Korean Math. Soc.*, **42**(5) (2005), 1101–1110.

[2] A. BEJANCU, CR-submanifolds of a Kaehler manifold I, *Proc. Amer. Math. Soc.*, **69**(1) (1978), 135–142.

[3] R.L. BISHOP AND B. O'NEILL, Manifolds of negative curvature, *Trans. Amer. Math. Soc.*, **145** (1969), 1–49.

[4] B.Y. CHEN, CR-submanifolds of a Kaehler manifold, *J. Differential Geom.*, **16** (1981), 305–323.

[5] B.Y. CHEN, Geometry of warped product CR-submanifolds in Kaehler Manifolds, *Monatsh. Math.*, **133** (2001), 177–195.

[6] B.Y. CHEN, On isometric minimal immersions from warped products into real space forms, *Proc. Edinburgh Math. Soc.*, **45** (2002), 579–587.

[7] I. HASEGAWA AND I. MIHAI, Contact CR-warped product submanifolds in Sasakian manifolds, *Geom. Dedicata*, **102** (2003), 143–150.

[8] K. KENMOTSU, A class of almost contact Riemannian Manifolds, *Tohoku Math. J.*, **24** (1972), 93–103.

[9] K. MATSUMOTO AND V. BONANZINGA, Doubly warped product CR-submanifolds in a locally conformal Kaehler space form, *Acta Mathematica Academiae Paedagogiace Nyíregyháziensis*, **24** (2008), 93–102.

[10] I. MIHAI, Contact CR-warped product submanifolds in Sasakian space forms, *Geom. Dedicata*, **109** (2004), 165–173.

[11] M.I. MUNTEANU, Doubly warped product CR-submanifolds in locally conformal Kaehler manifolds, *Monatsh. Math.*, **150** (2007), 333–342.

[12] A. OLTEANU, CR-doubly warped product submanifolds in Sasakian space forms, *Bulletin of the Transilvania University of Brasov*, **1** (50), III-2008, 269–278.

[13] K. YANO AND M. KON, *Structures on Manifolds*, World Scientific, Singapore, 1984.