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BEURLING VECTORS OF QUASIELLIPTIC SYSTEMS OF DIFFERENTIAL OPERATORS

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ABSTRACT. We show the iterate property in Beurling classes for quasielliptic systems of differential operators.

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1. INTRODUCTION

The aim of this work is to show the iterate property in Beurling classes for quasielliptic systems of differential operators. This property is proved for elliptic systems in [2]. A synthesis of results on the iterate problem is given in [1].

Let $(m_1,\dots,m_n)\in\mathbb{Z}_+^n, m_j\geq 1, 1\leq j\leq n$, we set $\mu=\prod_{j=1}^n m_j,\ m=\max\{m_j\}$, $q_j=\frac{m}{m_j}$ and $q=(q_1,\dots,q_n)$. If $\alpha\in\mathbb{Z}_+^n$ and $\beta\in\mathbb{Z}_+^n$, we denote $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$, $D^\alpha=D_1^{\alpha_1}\circ\cdots\circ D_n^{\alpha_n}$, where $D_j=\frac{1}{i}\cdot\frac{\partial}{\partial x_j}$, $\langle\alpha,q\rangle=\sum_{j=1}^n\alpha_j q_j$ and $\binom{\alpha}{\beta}=\prod_{j=1}^n\binom{\alpha_j}{\beta_j}$.

Let $(M_p)_{p=0}^{+\infty}$ be a sequence of real positive numbers such that

$$M_0=1,\ \exists a>0,\ 1\leq\frac{M_p}{M_{p-1}}\leq\frac{M_{p+1}}{M_p}\leq a^p,\ p\in\mathbb{Z}_+^*, \tag{1.1}$$

$$\exists b>0,\exists c>0,\ c\begin{pmatrix}p\\ j\end{pmatrix}M_{p-j}M_j\leq M_p\leq b^pM_{p-j}M_j,\ p,j\in\mathbb{Z}_+,j\leq p, \tag{1.2}$$

$$\forall m\geq 2,\exists d>0,\forall p,h\in\mathbb{Z}_+,h\leq m;\ (M_{pm})^{m-h}(M_{pm+m})^h\leq d\,(M_{pm+h})^m, \tag{1.3}$$

$$\forall m\geq 2,\exists H>0,\forall p,h\in\mathbb{Z}_+,h\leq p;\ \frac{M_{pm}}{M_{hm}}\leq H^{p-h}\left(\frac{M_p}{M_h}\right)^m. \tag{1.4}$$

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Let $(P_j(x,D))_{j=1}^N$ be $q-$quasihomogeneous differential operators of order m with C^∞ coefficients in an open subset Ω of $\mathbb{R}^n$, i.e.

$$P_j(x,D) = \sum_{\langle \alpha,q\rangle \leq m} a_{j\alpha}(x) D^\alpha .$$

We define the quasiprincipal symbol of the operator $P_j(x,D)$ by

$$P_{jm}(x,\xi) = \sum_{\langle \alpha,q\rangle = m} a_{j\alpha}(x) \xi^\alpha .$$

Definition 1.1. The system $(P_j)_{j=1}^N$ is said $q-$quasielliptic in Ω if for each $x_0 \in \Omega$ we have

$$\sum_{j=1}^N |P_{jm}(x_0,\xi)| \neq 0, \quad \forall \xi \in \mathbb{R}^n \backslash \{0\}. \tag{1.5}$$

Definition 1.2. Let $M = (M_p)$ be a sequence satisfying (1.1) – (1.4), the space of Beurling vectors of the system $(P_j(x,D))_{j=1}^N$ in Ω, denoted $B_M\left(\Omega, (P_j)_{j=1}^N\right)$, is the space of $u \in C^\infty(\Omega)$ such that $\forall K$ *compact of* $\Omega, \forall L > 0, \exists C > 0, \forall k \in \mathbb{Z}_+$,

$$\|P_{i_1} \ldots, P_{i_k} u\|_{L^2(K)} \leq C L^{km} M_{km}, \tag{1.6}$$

where $1 \leq i_l \leq N,\ l \leq k$.

Definition 1.3. Let $l = (l_1, \ldots, l_n) \in \mathbb{R}_+^n$ and M be a sequence satisfying (1.1) – (1.4), we call anisotropic Beurling space in Ω, denoted $B_M^l(\Omega)$, the space of $u \in C^\infty(\Omega)$ such that $\forall K$ *compact of* $\Omega, \forall L > 0, \exists C > 0, \forall \alpha \in Z_+^n$,

$$\|D^\alpha u\|_{L^2(K)} \leq C L^{<\alpha,l>} \prod_{j=1}^n \left(M_{\alpha_j}\right)^{l_j}. \tag{1.7}$$

Remark 1.1. If $l_j = 1, j = 1, \ldots, n$, we obtain, thanks to (1.2) the definition of isotropic Beurling space $B_M(\Omega)$, (see [4]).

The principal result of this work is the following theorem:

Theorem 1.2. *Let M and M' be two sequences satisfying (1.1) – (1.4) and*

$$\lim_{p\to+\infty} \sum_{h=0}^p \frac{M'_{hm}}{M_{hm}} \frac{M_{pm+m}}{M'_{pm+m}} = 0. \tag{1.8}$$

Let $(P_j)_{j=1}^N$ be $q-$quasielliptic system with $B_M^q(\Omega)$ coefficients, then

$$B_{M'}\left(\Omega, (P_j)_{j=1}^N\right) \subset B_{M'}^q(\Omega).$$

2. PRELIMINARY LEMMAS

Let ω be an open neighbourhood of the origin, we set $\mathcal{K} = \left\{k = \langle \alpha, q\rangle, \alpha \in \mathbb{Z}_+^n\right\}$ and we define

$$|u|_{k,\omega} = \sum_{\langle \alpha,q\rangle = k} \|D^\alpha u\|_{L^2(\omega)}, \quad u \in C^\infty(\omega),\ k \in \mathcal{K}.$$

If $\rho > 0$ we set

$$B_\rho = \left\{ x \in \mathbb{R}^n,\ \left(\sum_{j=1}^n (x_j)^{\frac{2}{q_j}} \right)^{\frac{1}{2}} < \rho \right\}.$$

The two following lemmas are in [6].

Lemma 2.1. *Let* $u \in C^{\infty}(\Omega)$, $r \in \mathcal{K}$ *and* $p \in \mathbb{Z}_+$, *then*

$$|u|_{pm+r,\omega} \leq \sum_{\langle \alpha,q\rangle = pm} |D^{\alpha}u|_{r,\omega}. \tag{2.1}$$

Lemma 2.2. *Let* $k = pm + r < pm + jm$, *where* $k, r \in \mathcal{K}$ *and* $p, j \in \mathbb{Z}_+^*$, *then* $\exists c(j) > 0$, $\forall B_\rho \subset \omega, \forall \varepsilon \in]0,1[, \forall u \in C^{\infty}(\omega)$,

$$|u|_{k,B_\rho} \leq \varepsilon |u|_{(p+j)m,B_\rho} + c(j)\varepsilon^{-\frac{r}{jm-r}} |u|_{pm,B_\rho}. \tag{2.2}$$

If $a \in C^{\infty}(\omega)$, we denote $[a, D^{\alpha}]u = D^{\alpha}(au) - aD^{\alpha}u$ and if P is a differential operator, we define $[P, D^{\alpha}]u = D^{\alpha}(Pu) - P(D^{\alpha}u)$.

Lemma 2.3. *Let* B *be a bounded subset of* $\mathbb{R}^n$ *and* $a \in B_M^q(\overline{B})$, *then* $\forall L > 0, \exists C > 0, \forall u \in C^{\infty}(\overline{B}), \forall p \in \mathbb{Z}_+^*$,

$$\sum_{\langle \alpha,q\rangle = pm} |[a, D^{\alpha}]u|_{0,B} \leq C \sum_{\substack{k \leq pm-1 \\ k \in \mathcal{K}}} L^{pm-k} \left(\frac{M_{pm\mu}}{M_{k\mu}}\right)^{\frac{1}{\mu}} |u|_{k,B}. \tag{2.3}$$

Proof. Let $L > 0$, as $a \in B_M^q(\overline{B})$, there exists $C_1 > 0$ such that

$$|D^{\alpha}a| \leq C_1 L^{\langle \alpha,q\rangle} \prod_{j=1}^{n} (M_{\alpha_j})^{q_j}, \ \forall \alpha \in \mathbb{Z}_+^n,$$

therefore, with the Leibniz formula, we get

$$|[a, D^{\alpha}]u|_{0,B} \leq \sum_{\beta<\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} |D^{\beta}u|_{0,B} C_1 L^{\langle \alpha-\beta,q\rangle} \prod_{j=1}^{n} (M_{\alpha_j-\beta_j})^{q_j}. \tag{2.4}$$

We need the following easy inequality

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leq \left(\prod_{j=1}^{n} \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}^{q_j\mu}\right)^{\frac{1}{\mu}} \leq \left(\begin{pmatrix} \langle \alpha,q\rangle \mu \\ \langle \beta,q\rangle \mu \end{pmatrix}\right)^{\frac{1}{\mu}}. \tag{2.5}$$

It is easy to check that from condition (1.2) we have

$$c^{l-1} \prod_{j=1}^{l} M_{h_j} \leq M_{\sum_{j=1}^{l} h_j} \leq b^{(l-1)\sum_{j=1}^{l} h_j} \prod_{j=1}^{l} M_{h_j}, \tag{2.6}$$

hence

$$\prod_{j=1}^{n} (M_{\alpha_j-\beta_j})^{q_j\mu} \leq \left(\frac{1}{c}\right)^{\sum_{j=1}^{n} q_j\mu - 1} M.$$

This inequality with (1.2) and (2.5) imply

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \prod_{j=1}^{n} (M_{\alpha_j-\beta_j})^{q_j} \leq \left(\frac{1}{c}\right)^{\sum_{j=1}^{l} q_j} \left(\frac{M_{\langle \alpha,q\rangle \mu}}{M_{\langle \beta,q\rangle \mu}}\right)^{\frac{1}{\mu}} |u|_{k,B}. \tag{2.7}$$

As the number of $\alpha \in \mathbb{Z}_+^*$ satisfying $\langle \alpha, q\rangle = pm$ and $\alpha > \beta$, is limited by $C_2^{pm-\langle \beta,q\rangle}$, where C_2 depends only of n, then (2.4) and (2.7) give

$$\sum_{\langle \alpha,q\rangle = pm} |[a, D^{\alpha}]u|_{0,B} \leq \sum_{\substack{k \leq pm-1 \\ k \in \mathcal{K}}} C_1 \left(\frac{1}{c}\right)^{\sum_{j=1}^{l} q_j} (C_2L)^{pm-k} \left(\frac{M_{pm\mu}}{M_{k\mu}}\right)^{\frac{1}{\mu}} |u|_{k,B},$$

from which the desired estimate is obtained. $\square$

3. LOCAL ESTIMATES

Let $(P_j)_{j=1}^N$ be a $q-$quasielliptic system with coefficients in $B_M^q\left(\overline{B}\right)$, where B is a neighbourhood of the origin. The following lemma is a light modification of an analogous lemma in [6, Lemma 2.3].

Lemma 3.1. *Let ω be a small neighbourhood of the origin, $\rho > 0$ and $\delta \in]0,1[$, such that $\overline{B}_{\rho+\delta} \subset \omega$. Then there exists $C > 0$, not depending on ρ and δ, such that for any $u \in C^\infty(\overline{\omega})$,*

$$|u|_{m,B_\rho} \leq C\left(\sum_{j=1}^N |P_j u|_{0,B_{\rho+\delta}} + \sum_{\substack{k\leq m-1\\ k\in\mathcal{K}}} \delta^{-m+k}|u|_{k,B_{\rho+\delta}}\right). \tag{3.1}$$

Lemma 3.2. *Let ω, ρ and δ be as in Lemma 3.1, then $\exists C > 0, \forall L > 0, \exists A > 0, \forall p \in \mathbb{Z}_+^*, \forall u \in C^\infty(\overline{\omega})$*

$$\begin{aligned}|u|_{(p+1)m,B_\rho} \leq C\Bigg(&\sum_{j=1}^N |P_j u|_{pm,B_{\rho+\delta}} + \delta^{-m}|u|_{pm,B_{\rho+\delta}} + \frac{1}{(4e)^m}|u|_{(p+1)m,B_{\rho+\delta}}\\ &+A\sum_{h=0}^p L^{(p+1-h)m}\frac{M_{pm+m}}{M_{hm}}|u|_{hm,B_{\rho+\delta}}\Bigg),\end{aligned} \tag{3.2}$$

and

$$|u|_{m,B_\rho} \leq C\left(\sum_{j=1}^N |P_j u|_{0,B_{\rho+\delta}} + \delta^{-m}|u|_{0,B_{\rho+\delta}} + \frac{1}{(4e)^m}|u|_{m,B_{\rho+\delta}}\right). \tag{3.3}$$

Proof. From (2.1) and (3.1) we obtain

$$\begin{aligned}|u|_{(p+1)m,B_\rho} \leq C\Bigg(&\sum_{j=1}^N |P_j u|_{pm,B_{\rho+\delta}} + \sum_{j=1}^N\sum_{\langle\alpha,q\rangle=pm}|[P_j,D^\alpha]u|_{0,B_{\rho+\delta}}\\ &+\sum_{\substack{k\leq m-1\\ k\in\mathcal{K}}}\delta^{-m+k}|u|_{pm+k,B_{\rho+\delta}}\Bigg),\end{aligned} \tag{3.4}$$

Following the same idea as in the proof of Lemma 2.2 of [2], we get

$$\sum_{\langle\alpha,q\rangle=pm}|[P_j,D^\alpha]u|_{0,B_{\rho+\delta}} \leq C'\sum_{\substack{s\leq pm+m-1\\ s\in\mathcal{K}}} L^{pm+m-s}\left(\frac{M_{(pm+m)\mu}}{M_{s\mu}}\right)^{\frac{1}{\mu}}|u|_{s,B_{\rho+\delta}}. \tag{3.5}$$

On the other hand, there exists $h \in \mathbb{Z}_+$ and $r \in \mathcal{K}$ such that $s = hm + r$, $r < nm - n$, (see [6, (1.3)]). As $s \leq pm + m - 1$, then $h \leq p$. From (2.2) we have

$$|u|_{s,B_{\rho+\delta}} \leq \varepsilon|u|_{(h+n)m,B_{\rho+\delta}} + C_2\varepsilon^{-\frac{r}{nm-r}}|u|_{hm,B_{\rho+\delta}} \tag{3.6}$$

if $s = hm + r$, where $0 \leq h \leq p - n + 1$ and $0 \leq r < nm - n$, and

$$|u|_{s,B_{\rho+\delta}} \leq \varepsilon|u|_{pm+m,B_{\rho+\delta}} + C_2\varepsilon^{-\frac{r}{jm-r}}|u|_{hm,B_{\rho+\delta}} \tag{3.7}$$

if $s = hm + r$ where $h = p + 1 - j,\ 1 \leq j \leq n - 1$ and $0 \leq r \leq jm - 1$.

Let $\varepsilon' \in]0,1[$ and put

$$\varepsilon = \varepsilon' \left(\frac{M_{s\mu}}{M_{(h+n)m\mu}} \right)^{\frac{1}{\mu}} L^{-nm+r} \text{ in } (3.6)$$

and

$$\varepsilon = \varepsilon' \left(\frac{M_{s\mu}}{M_{(p+1)m\mu}} \right)^{\frac{1}{\mu}} L^{-jm+r} \text{ in } (3.7).$$

According to (1.3) we obtain for any s satisfying (3.6),

$$\frac{L^{-s}}{(M_{s\mu})^{\frac{1}{\mu}}} |u|_{s,B_{\rho+\delta}} \leq \varepsilon' \frac{L^{-(h+n)m}}{\left(M_{(h+n)m\mu}\right)^{\frac{1}{\mu}}} |u|_{(h+n)m,B_{\rho+\delta}} + C_2 d' \varepsilon'^{-m} \frac{L^{-hm}}{(M_{hm\mu})^{\frac{1}{\mu}}} |u|_{hm,B_{\rho+\delta}}$$

and for any s satisfying (3.7),

$$\frac{L^{-s}}{(M_{s\mu})^{\frac{1}{\mu}}} |u|_{s,B_{\rho+\delta}} \leq \varepsilon' \frac{L^{-(p+1)m}}{\left(M_{(p+1)m\mu}\right)^{\frac{1}{\mu}}} |u|_{(p+1)m,B_{\rho+\delta}} + C_2 d'' \varepsilon'^{-nm} \frac{L^{-hm}}{(M_{hm\mu})^{\frac{1}{\mu}}} |u|_{hm,B_{\rho+\delta}}.$$

These inequalities and (3.5) give

$$\sum_{\langle \alpha,q\rangle = pm} |[P_j, D^\alpha]\, u|_{0,B_{\rho+\delta}} \leq C' \Bigg(n\varepsilon' |u|_{(p+1)m,B_{\rho+\delta}} + c(\varepsilon') \sum_{h=0}^{p} L^{(p+1-h)m} \left(\frac{M_{(pm+m)\mu}}{M_{hm\mu}} \right)^{\frac{1}{\mu}} |u|_{hm,B_{\rho+\delta}} \Bigg).$$

Choosing $\varepsilon' = (2CC'Nn\,(4e)^m)^{-1}$, then we obtain, with (1.4),

$$\sum_{J=1}^{N} \sum_{\langle \alpha,q\rangle = pm} |[P_j, D^\alpha]\, u|_{0,B_{\rho+\delta}} \leq \frac{1}{2C} \frac{1}{(4e)^m} |u|_{(p+1)m,B_{\rho+\delta}} + A \sum_{h=0}^{p} (HL)^{(p+1-h)m} \frac{M_{pm+m}}{M_{hm}} |u|_{hm,B_{\rho+\delta}}.$$

It follows from this inequality: $\forall L > 0,\ \exists A > 0$,

$$\sum_{J=1}^{N} \sum_{\langle \alpha,q\rangle = pm} |[P_j, D^\alpha]\, u|_{0,B_{\rho+\delta}} \leq \frac{1}{2C} \frac{1}{(4e)^m} |u|_{(p+1)m,B_{\rho+\delta}} + A \sum_{h=0}^{p} L^{(p+1-h)m} \frac{M_{pm+m}}{M_{hm}} |u|_{hm,B_{\rho+\delta}}. \tag{3.8}$$

It remains the estimate of the third term of the right-hand side of (3.4). From (2.2), we have

$$|u|_{pm+k,B_{\rho+\delta}} \leq \varepsilon |u|_{pm+m,B_{\rho+\delta}} + C_2 \varepsilon^{-\frac{k}{m-k}} |u|_{pm,B_{\rho+\delta}}.$$

Setting $\varepsilon = \varepsilon' \delta^{m-k}$ and choosing $\varepsilon' = (2C_1 C\,(4e)^m)^{-1}$, then we obtain

$$\sum_{\substack{k \leq m-1 \\ k \in \mathcal{K}}} \delta^{-m+k} |u|_{pm+k,B_{\rho+\delta}} \leq \frac{1}{2C} \frac{1}{(4e)^m} |u|_{(p+1)m,B_{\rho+\delta}} + C_2' \delta^{-m} |u|_{pm,B_{\rho+\delta}}. \tag{3.9}$$

The estimates (3.4), (3.8) and (3.9) imply (3.2). The estimate (3.3) is obtained from (3.1) and (3.9) with $p = 0$. □

4. THE MAIN RESULT

Let $R>0,$ to every sequence M satisfying (1.1) – (1.4) we define

$$\sigma_{M}^{p}\left(u\right)=\frac{1}{M_{pm}}\sup_{R/2\leq\rho<R}\left(R-\rho\right)^{pm}\left|u\right|_{pm,B_{\rho}}.$$

The following lemma is in [2].

Lemma 4.1. *Let ω be as in Lemma 3.1, $R\in]0,1[$ such that $\overline{B}_{R}\subset\omega,$ M,M' two sequences satisfying (1.1) – (1.4) and $u\in B_{M'}\left(\overline{\omega},\left(P_{j}\right)_{j=1}^{N}\right)$, then for any $L>0,$ there exists an increasing positive sequence $\left(C_{p}\right)_{p=0}^{+\infty}$ such that $\forall p,l\in\mathbb{Z}_{+},$*

$$\sigma_{M'}^{p}\left(P_{i_{0}}\cdots P_{i_{l}}u\right)\leq C_{p}\frac{M'_{pm+lm}}{M'_{pm}}L^{pm+lm}. \tag{4.1}$$

where the sequence $\left(C_{p}\right)$ is constructed by recurrence,

$$C_{p+1}=C_{p}\left(NC+A\sum_{h=0}^{p}\frac{M'_{hm}}{M_{hm}}\frac{M_{pm+m}}{M'_{pm+m}}\right),$$

where C and A are the constants of Lemma 3.2 and C_{0} is the constant satisfying

$$\left\Vert P_{i_{0}}\cdots P_{i_{l}}u\right\Vert_{L^{2}\left(B_{R}\right)}\leq C_{0}L^{lm}M'_{lm}.$$

Theorem 4.2. *Let M and M' be two sequences satisfying (1.1) – (1.4) and*

$$\lim_{p\to+\infty}\sum_{h=0}^{p}\frac{M'_{hm}}{M_{hm}}\frac{M_{pm+m}}{M'_{pm+m}}=0. \tag{4.2}$$

Let $\left(P_{j}\right)_{j=1}^{N}$ be q-quasielliptic system with coefficients in $B_{M}^{q}\left(\Omega\right)$, then

$$B_{M'}\left(\Omega,\left(P_{j}\right)_{j=1}^{N}\right)\subset B_{M'}^{q}\left(\Omega\right).$$

Proof. We must verify (1.7) near every point x of Ω. By a translation of x at the origin, there exists a neighbourhood ω of the origin for which the precedent lemmas are true. Let $L>0$ and let $\left(C_{p}\right)_{p=0}^{+\infty}$ be as in Lemma 4.1, then from (4.2) there exists $p_{0}\in\mathbb{Z}_{+}$ such that $C_{p+1}\leq 2NCC_{p},\ p\geq p_{0}$, hence

$$C_{p}\leq C_{p_{0}}\left(2NC\right)^{p-p_{0}}\leq C_{p_{0}}\left(2NC\right)^{pm+lm},\ \forall l\in\mathbb{Z}_{+}.$$

For $p\leq p_{0}$, this inequality is true because the sequence $\left(C_{p}\right)_{p=0}^{+\infty}$ is increasing.
Let $R\in]0.1[$ such that $\overline{B}_{R}\subset\omega$, from (4.1) we obtain

$$\sigma_{M'}^{p}\left(P_{i_{0}}\cdots P_{i_{l}}u\right)\leq C_{p_{0}}\frac{M'_{pm+lm}}{M'_{pm}}\left(2NCL\right)^{pm+lm},\ \forall p,l\in\mathbb{Z}_{+}.$$

In particular for $l=0$,

$$\left(\frac{R}{2}\right)^{pm}\frac{1}{M'_{pm}}\left|u\right|_{pm,B_{R/2}}\leq\sigma_{M'}^{p}\left(u\right)\leq C_{p_{0}}\left(2NCL\right)^{pm},$$

hence

$$\left|u\right|_{pm,B_{R/2}}\leq C_{p_{0}}\left(\frac{4NC}{R}L\right)^{pm}M'_{pm},$$

which can be rewritten as

$$\forall L>0,\ \exists C>0,\ \left|u\right|_{pm,B_{R/2}}\leq CL^{pm}M'_{pm}. \tag{4.3}$$

The last inequality will allow us to conclude. In fact let $k \in \mathcal{K}$, then there exists $p \in \mathbb{Z}_+$ and $r \in \mathcal{K},\ r < nm - n$, such that $k = pm + r$. From (2.2) , (4.3) and (2.6) , we obtain

$$|u|_{k,B_{R/2}} \leq \varepsilon C' L^{(p+n)m} M'_{(p+n)m} + C'C''\varepsilon^{-\frac{r}{nm-r}} L^{pm} M'_{pm}$$
$$\leq \varepsilon C' L^{(p+n)m} \frac{1}{c} \left(M'_{(p+n)m\mu}\right)^{\frac{1}{\mu}} + C'C''\varepsilon^{-\frac{r}{nm-r}} L^{pm} \frac{1}{c} \left(M'_{pm\mu}\right)^{\frac{1}{\mu}} .$$

Setting

$$\varepsilon = \left(\frac{M'_{(pm+r)\mu}}{M'_{(p+n)m\mu}}\right)^{\frac{1}{\mu}} L^{-nm+r},$$

then from (1.3) we get

$$|u|_{k,B_{R/2}} \leq C_1 L^k \left(M'_{k\mu}\right)^{\frac{1}{\mu}} . \tag{4.4}$$

By an imbedding theorem of anisotropic Sobolev spaces (see [5]), from (4.4) and (1.2) we obtain

$$\sup_{B_{R/2}} |D^\alpha u(x)| \leq C_2 (bL)^{\langle \alpha,q\rangle} \left(M'_{\langle \alpha,q\rangle\mu}\right)^{\frac{1}{\mu}} .$$

The last estimate, with (2.6) gives

$$\sup_{B_{R/2}} |D^\alpha u(x)| \leq C_3 (bL)^{\langle \alpha,q\rangle} \left(b^{\langle \alpha,q\rangle n\mu} \prod_{j=1}^{n} M'_{\alpha_j q_j \mu}\right)^{\frac{1}{\mu}}$$
$$\leq C_3 (bL)^{\langle \alpha,q\rangle} \left(b^{\langle \alpha,q\rangle n\mu} \prod_{j=1}^{n} b^{q_j\mu(\alpha_j q_j \mu)} \left(M'_{\alpha_j}\right)^{q_j\mu}\right)^{\frac{1}{\mu}}$$
$$\leq C_3 \left(b^{(1+n+m\mu)} L\right)^{\langle \alpha,q\rangle} \prod_{j=1}^{n} \left(M'_{\alpha_j}\right)^{q_j} ,$$

from there $u \in B^q_{M'}\left(B_{R/2}\right)$. □

As a corollary we obtain from Theorem 1.2, the principal result of [2]. Theorem 1.2 also gives a result of regularity of solutions of differential equations in Beurling classes.

Corollary 4.3. *Under the assumptions of Theorem 1.2, the following assertions are equivalent:*

i) $u \in \mathfrak{D}'(\Omega)$ *and* $P_j u \in B^q_{M'}(\Omega)$,
ii) $u \in B^q_{M'}(\Omega)$.

For anisotropic projective Gevrey classes $G^{\{s\},q}(\Omega) = B^q_M(\Omega)$, $M_p = (p!)^s$, $s \geq 1$, we have the same result.

Corollary 4.4. *Let* s, s' *be such that* $s' > s \geq 1$ *and* $(P_j)_{j=1}^N$ $q-$*quasielliptic system with coefficients in* $G^{\{s\},q}(\Omega)$, *then*

$$G^{\{s\}}\left(\Omega, (P_j)_{j=1}^N\right) \subset G^{\{s\},q}(\Omega) .$$

Corollary 4.5. *Let* M *and* M' *be two sequences satisfying (1.1) – (1.4) and*

$$\lim_{p\to+\infty} \sum_{h=0}^{p} \frac{M'_{hm}}{M_{hm}} \frac{M_{pm+m}}{M'_{pm+m}} = 0, \tag{4.5}$$

and let $(P_j)_{j=1}^{N}$ *be an elliptic system with coefficients in* $B_M(\Omega)$ *, then*

$$B_{M'}\left(\Omega, (P_j)_{j=1}^{N}\right) \subset B_{M'}(\Omega).$$

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