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A HÖLDER INEQUALITY FOR HOLOMORPHIC FUNCTIONS

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ABSTRACT. We prove a Hölder inequality for the L^p-spaces of analytic functions with respect to a complex Gaussian measure.

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1. INTRODUCTION

In this paper we will prove the following inequality: for any two entire analytic functions f, $g : \mathbb{C}^n \to \mathbb{C}$ and any positive numbers p, q, r, and s, such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we have:

$$\frac{1}{\pi^n} \int_{\mathbb{C}^n} |f(\sqrt{r}z)g(\sqrt{r}z)|^s e^{-|z|^2} dz \leq \left[\frac{1}{\pi^n} \int_{\mathbb{C}^n} |f(\sqrt{p}z)|^s e^{-|z|^2} dz \right] \left[\frac{1}{\pi^n} \int_{\mathbb{C}^n} |g(\sqrt{q}z)|^s e^{-|z|^2} dz \right], \tag{1.1}$$

provided that the integrals from the right side are both finite. This inequality is motivated by the following facts from White Noise Analysis. The S-transform is known to be a unitary isomorphism from the space of square integrable functions defined on a white noise space onto the space $\mathcal{H}L^2(E)$, where E is a separable complex Hilbert space (see [4, p. 39] for the definition of the S-transform, and page 337 for the stated isomorphism). The space of generalized functions in White Noise Analysis is the union of an increasing family of weighted L^2-functions. The S-transform maps such a weighted L^2-space onto $\Gamma(A)\mathcal{H}L^2(E)$, where A is an operator on E, and $\Gamma(A)\varphi(u) := \varphi(Au)$. In White Noise Analysis there is a product between two generalized functions, called the Wick product. It is defined in such a way that the S-transform of a Wick product of two generalized functions is the product of the S-transforms of the two generalized

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functions. A natural question is the following: knowing the smallest weighted space in which a generalized function φ lives and the smallest weighted space in which another generalized function ψ lives, what is the smallest weighted space in which the Wick product of φ and ψ lives? Applying the S-transform isomorphism, the question is reduced to the following question: If $f \in \Gamma(A)\mathcal{H}L^2(\mathbb{C}^n)$ and $g \in \Gamma(B)\mathcal{H}L^2(\mathbb{C}^n)$, then what are the operators C having the minimal operatorial norm such that $fg \in \Gamma(C)\mathcal{H}L^2(\mathbb{C}^n)$? This inequality, for $s = 2$ only, was proven in [5] and called "a Young inequality for White Noise Analysis". Although the inequality (1.1), for the space $\mathcal{H}L^2(\mathbb{C}^n)$ only, gives a satisfactory answer to this question, from a mathematical point of view it is important and interesting to extend this sharp inequality to all the other $\mathcal{H}L^p(\mathbb{C}^n)$ spaces. This is the purpose of this short paper and we do not know what applications it may have.

2. A COMPLEX HÖLDER INEQUALITY

For any $p \geq 1$, let $\mathcal{H}L^p(\mathbb{C}^n, \mu)$ denote the space of all holomorphic functions $f : \mathbb{C}^n \to \mathbb{C}$ such that:

$$\|f\|_p^p := \int_{\mathbb{C}^n} |f(z)|^p d\mu(z) < \infty,$$

where $d\mu(z) = (1/\pi^n)e^{-|z|^2}dz$. Here, if $z = x + iy$, then $dz = dxdy$ is the Lebesgue measure on the space $\mathbb{C}^n$ identified with $\mathbb{R}^{2n}$.
For any function $f : \mathbb{C}^n \to \mathbb{C}$ and complex number $a \in \mathbb{C}$, we define a new function $\Gamma(a)f : \mathbb{C}^n \to \mathbb{C}$, by $\Gamma(a)f(z) := f(az)$. Observe that if f is holomorphic, then $\Gamma(a)f$ is also holomorphic. The following hypercontractivity result gives us a relation between the spaces $\mathcal{H}L^p(\mathbb{C}^n, \mu)$, when $1 \leq p < \infty$.

Theorem 2.1. *For any $1 \leq p < q < \infty$ and any holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$, the following inequality holds provided that the right hand side is finite:*

$$\left\|\Gamma\left(\frac{1}{\sqrt{q}}\right) f\right\|_q \leq \left\|\Gamma\left(\frac{1}{\sqrt{p}}\right) f\right\|_p . \tag{2.1}$$

This theorem was first proven by Janson in [2]. Later Carlen in [1] and Zhou in [6] simultaneously proved the cases of equality. Using this theorem we will prove the following:

Theorem 2.2. *Let p, q, and r be strictly positive numbers (not necessarily larger than or equal to 1) such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Let $s \geq 1$. If f and g are holomorphic functions such that $\Gamma(\sqrt{p})f \in \mathcal{H}L^s(\mathbb{C}^n, \mu)$ and $\Gamma(\sqrt{q})g \in \mathcal{H}L^s(\mathbb{C}^n, \mu)$, then $\Gamma(\sqrt{r})(fg) \in \mathcal{H}L^s(\mathbb{C}^n, \mu)$ and

$$\left\|\Gamma(\sqrt{r})(fg)\right\|_s \leq \|\Gamma(\sqrt{p})f\|_s \cdot \|\Gamma(\sqrt{q})g\|_s . \tag{2.2}$$

The equality holds if and only if one of the functions f and g is identically equal to zero, or

$$f(z) = c_1 e^{\frac{1}{p}\sum_{j=1}^n a_j z_j}$$

$$g(z) = c_2 e^{\frac{1}{q}\sum_{j=1}^n a_j z_j},$$

where $c_1, c_2, a_1, a_2, \ldots, a_n$ are arbitrary complex numbers.

Proof. Using Hölder's inequality $\left(\frac{r}{p}+\frac{r}{q}=1\right)$ we obtain:

$$\begin{aligned}
&\left\|\Gamma(\sqrt{r})(fg)\right\|_s \\
&\quad = \left[\int_{\mathbb{C}^n} |f(\sqrt{r}z)|^s |g(\sqrt{r}z)|^s d\mu(z)\right]^{\frac{1}{s}} \\
&\quad \le \left\{\left[\int_{\mathbb{C}^n} |f(\sqrt{r}z)|^{s\cdot\frac{p}{r}} d\mu(z)\right]^{\frac{r}{p}} \left[\int_{\mathbb{C}^n} |g(\sqrt{r}z)|^{s\cdot\frac{q}{r}} d\mu(z)\right]^{\frac{r}{q}}\right\}^{\frac{1}{s}} \\
&\quad = \left[\int_{\mathbb{C}^n} \left|f\left(\frac{\sqrt{r}}{\sqrt{sp}}(\sqrt{sp}z)\right)\right|^{\frac{sp}{r}} d\mu(z)\right]^{\frac{r}{sp}} \left[\int_{\mathbb{C}^n} \left|g\left(\frac{\sqrt{r}}{\sqrt{sq}}(\sqrt{sq}z)\right)\right|^{\frac{sq}{r}} d\mu(z)\right]^{\frac{r}{sq}}.
\end{aligned}$$

Observe that $\frac{sp}{r} > s \ge 1$ and $\frac{sq}{r} > s \ge 1$ and thus applying the "complex hypercontractivity" inequality (2.1) (which says that for any holomorphic functions h and any $1 \le u < v < \infty$, we have $\left\|\Gamma\left(\frac{1}{\sqrt{v}}\right)f\right\|_v \le \left\|\Gamma\left(\frac{1}{\sqrt{u}}\right)f\right\|_u$) to the holomorphic functions: $f(\sqrt{sp}z)$ with $u = s$ and $v = \frac{sp}{r}$, and $g(\sqrt{sq}z)$ with $u = s$ and $v = \frac{sq}{r}$ respectively, we obtain:

$$\begin{aligned}
&\left\|\Gamma(\sqrt{r})(fg)\right\|_s \\
&\quad \le \left[\int_{\mathbb{C}^n} \left|f\left(\frac{\sqrt{r}}{\sqrt{sp}}(\sqrt{sp}z)\right)\right|^{\frac{sp}{r}} d\mu(z)\right]^{\frac{r}{sp}} \left[\int_{\mathbb{C}^n} \left|g\left(\frac{\sqrt{r}}{\sqrt{sq}}(\sqrt{sq}z)\right)\right|^{\frac{sq}{r}} d\mu(z)\right]^{\frac{r}{sq}} \\
&\quad \le \left[\int_{\mathbb{C}^n} \left|f\left(\frac{1}{\sqrt{s}}(\sqrt{sp}z)\right)\right|^{s} d\mu(z)\right]^{\frac{1}{s}} \left[\int_{\mathbb{C}^n} \left|g\left(\frac{1}{\sqrt{s}}(\sqrt{sq}z)\right)\right|^{s} d\mu(z)\right]^{\frac{1}{s}} \\
&\quad = \left[\int_{\mathbb{C}^n} |f(\sqrt{p}z)|^s d\mu(z)\right]^{\frac{1}{s}} \left[\int_{\mathbb{C}^n} |g(\sqrt{q}z)|^s d\mu(z)\right]^{\frac{1}{s}} \\
&\quad = \left\|\Gamma(\sqrt{p})(f)\right\|_s \cdot \left\|\Gamma(\sqrt{q})(g)\right\|_s.
\end{aligned}$$

It is clear that if one of the functions f or g is identically equal to zero, then our inequality becomes an equality. Let us assume that both functions f and g are different from the zero functions.

From [1] and [6], we know that, in order to have equality in the "complex hypercontractivity" inequality, f and g must be functions of the form:

$$f(z) = c_1 e^{\sum_{j=1}^n \alpha_j z_j} \quad \text{and} \quad g(z) = c_2 e^{\sum_{j=1}^n \beta_j z_j},$$

where $c_1, c_2, \alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n$ are arbitrary complex numbers. To have equality in Hölder's inequality, there must be a constant k such that, for all $z \in \mathbb{C}^n$, $|f(z)|^{p/r} = k|g(z)|^{q/r}$. Since f and g are holomorphic we obtain the condition that, for all $1 \le j \le n$, $\frac{p\alpha_j}{r} = \frac{q\beta_j}{r}$. Denoting by a_j the common value of $p\alpha_j$ and $q\beta_j$, we obtain that the equality holds in our inequality only for a pair of functions of the form:

$$\begin{aligned}
f(z) &= c_1 e^{\frac{1}{p}\sum_{j=1}^n a_j z_j} \\
g(z) &= c_2 e^{\frac{1}{q}\sum_{j=1}^n a_j z_j},
\end{aligned}$$

where $c_1, c_2, a_1, a_2, \cdots, a_n$ are arbitrary complex numbers. □

We are thankful to Professor Svante Janson for adding the following:

Remark 2.3. The inequality (2.2) holds even for $0 < s < 1$.

This is true since in [2] the "complex hypercontractivity" inequality (2.1) is proved not only for $1 \leq s < \infty$, but also for any $0 < s < 1$.
The equality, for the case $0 < s < 1$, holds only for functions of the same form as above. This is true since the equality case in inequality (2.1) occurs only for exponential functions, even in the case $0 < s < 1$. This was proven in [1].

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