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## LUPAŞ-DURRMEYER OPERATORS

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Abstract

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## Abstract

In the present paper, we obtain Stechkin-Marchaud-type inequalities for some approximation operators, more precisely for Lupaş-Durrmeyer operators defined as in (1.1).

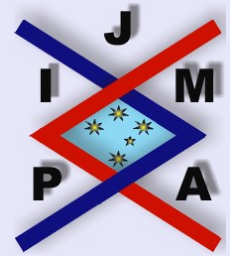
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*Key words:* Stechkin-Marchaud-type inequalities, Lupaş Operators, Durrmeyer Operators.

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# 1. Introduction

Lupaş proposed a family of linear positive operators mapping  $C[0, \infty)$  into  $C[0, \infty)$ , the class of all bounded and continuous functions on  $[0, \infty)$ , namely,

$$V_n(f, x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, \infty),$$

where  $p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$ .

Motivated by Derriennic [1], Sahai and Prasad [5] proposed modified Lupaş operators defined, for functions integrable on  $[0, \infty)$ , by

$$(1.1) \quad B_n(f, x) = (n-1) \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k}(t) f(t) dt.$$

Wicken discussed Stechkin-Marchaud-type inequalities in [2] for Bernstein polynomials and obtained the following results:

$$w_{\phi}^2\left(f, \frac{1}{\sqrt{n}}\right) \leq C n^{-1} \sum_{k=1}^n \|\phi^{-\alpha}(B_k f - f)\|_{\infty}.$$

The main object of this paper is to give Stechkin-Marchaud-type inequalities for Lupaş-Durrmeyer operators. In the end of this section we introduce some definitions and notations.

**Definition 1.1.** For  $0 \leq \lambda \leq 1$ ,  $0 < \alpha < 2r$ ,  $0 \leq \beta \leq 2r$ ,  $0 \leq \alpha(1-\lambda) + \beta \leq 2r$

$$(1.2) \quad \|f\|_0 = \|f\|_{0,\alpha,\beta,\lambda} = \sup_{x \in I} \left\{ \left| \phi^{\alpha(\lambda-1)-\beta}(x) f(x) \right| \right\},$$



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$$(1.3) \quad C_{\alpha,\beta,\lambda}^0 = \{f \in C_B(I), \|f\|_0 < \infty\},$$

$$(1.4) \quad \|f\|_r = \|f\|_{r,\alpha,\beta,\lambda} = \sup_{x \in I} \{|\phi^{2r+\alpha(\lambda-1)-\beta}(x)f^{(2r)}(x)|\}$$

and

$$(1.5) \quad C_{\alpha,\beta,\lambda}^r = \{f \in C_B(I), f^{(2r-1)} \in AC_{loc}, \|f\|_r < \infty\},$$

where  $\phi(x) = \sqrt{x(1+x)}$  and  $r = 0, 1, 2, \dots$

**Definition 1.2.** Peetre's  $K$ -functional is defined as

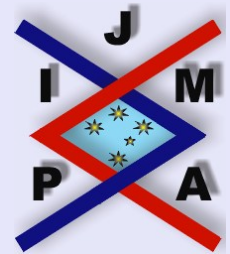
$$(1.6) \quad w_{\phi^\lambda}^{2r}(f, t)_{\alpha,\beta} = \sup_{0 < h \leq t} \sup_{x \pm rh\phi^\lambda(x) \in I} \{|\phi^{\alpha(\lambda-1)-\beta}(x)\Delta_{h\phi^\lambda}^{2r}f(x)|\}$$

and

$$(1.7) \quad K_{\phi^\lambda}(f, t^{2r})_{\alpha,\beta} = \inf_{g^{(2r-1)} \in AC_{loc}} \{\|f - g\|_0 + t^{2r} \|g\|_r\},$$

where  $AC_{loc}$  is the space of real valued absolute continuous and integrable functions on  $[0, 1]$ .

In second section of the paper, we will give some basic results, which will be useful in proving the main theorems; while in Section 3 the main results are given.



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## 2. Auxiliary Results

Some basic results are given here.

**Lemma 2.1.** *Suppose that for nonnegative sequences  $\{\sigma_n\}, \{\tau_n\}$  with  $\sigma_1 = 0$  the inequality  $\sigma \leq \left(\frac{k}{n}\right)^p \sigma_k + \tau_k$ , ( $1 \leq k \leq n$ ), is satisfied for  $n \in \mathbb{N}$ ,  $p > 0$ . Then one has*

$$(2.1) \quad \sigma_n \leq B_p n^{-p} \sum_{k=1}^n k^{p-1} \tau_k.$$

**Lemma 2.2.** *For  $f^{(2s)} \in C_{\alpha, \beta, \lambda}^0$ ,  $s \in N_0$ , the following inequalities hold*

$$(2.2) \quad \|B_n^{(2s)} f\|_r \leq C_1 n^r \|f^{(2s)}\|_0,$$

and

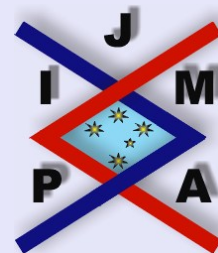
$$(2.3) \quad \|B_n^{(2s)} f\|_r \leq C_2 n^{r + \frac{\alpha(1-\lambda)}{2} + \frac{\beta}{2}} \|f^{(2s)}\|_\infty.$$

**Lemma 2.3.** *For  $f^{(2s)} \in C_{\alpha, \beta, \lambda}^r$ ,  $s \in N_0$ , the following inequality holds*

$$(2.4) \quad \|B_n^{(2s)} f\|_r \leq \|f^{(2s)}\|_r.$$

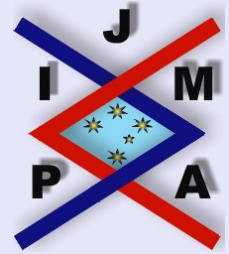
**Lemma 2.4.** *Let us suppose that  $f^{(2s)} \in C_{\alpha, \beta, \lambda}^0$ ,  $s \in N_0$ ,  $0 \leq \alpha(1-\lambda) + \beta \leq 2$ , then*

$$(2.5) \quad \|B_n^{(2s)} f\|_r \leq C \left( \sum_{k=1}^n k^{r-1} \|(B_k f - f)^{(2s)}\|_0 + \|f^{(2s)}\|_\infty \right).$$



**Lemma 2.5.** Suppose that  $r \in \mathbb{N}$ ,  $x \pm rt \in I$ ,  $0 \leq \beta \leq 2r$ ,  $0 \leq t \leq \frac{1}{16r}$ , then

$$(2.6) \quad \int_{-\frac{t}{2r}}^{\frac{t}{2r}} \cdots \int_{-\frac{t}{2r}}^{\frac{t}{2r}} \phi^{-\beta} \left( x + \sum_{j=1}^{2r} u_j \right) du_1 \cdots du_{2r} \leq C(\beta) t^{2r} \phi^{-\beta}(x).$$




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### 3. Main Results

We are now ready to prove the main results of this paper.

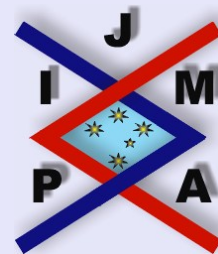
**Theorem 3.1.** *For the modulus of smoothness and  $K$ -functional*

$$(3.1) \quad K_{\phi^\lambda} \left( f^{(2s)}, \frac{1}{n^r} \right)_{\alpha, \beta} \leq C n^{-r} \left( \sum_{k=1}^n k^{r-1} \|(B_k f - f)^{(2s)}\|_0 + \|f^{(2s)}\|_\infty \right),$$

$$(3.2) \quad w_{\phi^\lambda}^{2r} \left( f^{(2s)}, \frac{1}{\sqrt{n}} \right)_{\alpha, \beta} \leq C n^{-\frac{r}{2-\lambda}} \left( \sum_{k=1}^{\left[ n^{\frac{1}{2-\lambda}} \right]} k^{-\frac{r-1}{2-\lambda}} \|(B_k f - f)^{(2s)}\|_0 + \|f^{(2s)}\|_\infty \right),$$

where  $\|\cdot\|_\infty$  denotes the supremum norm.

*Proof of (3.1).* Taking  $\frac{n}{2} \leq m \leq n$  such that  $\|(B_m f - f)^{(2s)}\|_0 \leq \|(B_k f - f)^{(2s)}\|_0$ ,



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( $\frac{n}{2} < k \leq n$ ), we have

$$\begin{aligned}
 K_{\phi^\lambda} \left( f^{(2s)}, \frac{1}{n^r} \right)_{\alpha, \beta} &\leq \| (B_m f - f)^{(2s)} \|_0 + n^{-r} \left\| f_m^{(2s)} \right\|_r \\
 &\leq \frac{2^r}{n^r} \sum_{k=\frac{n}{2}}^n k^{r-1} \| (B_k f - f)^{(2s)} \|_0 \\
 &\quad + C n^{-r} \left( \sum_{k=1}^m k^{r-1} \| (B_k f - f)^{(2s)} \|_0 + \| f^{(2s)} \|_\infty \right) \\
 &\leq C n^{-r} \left( \sum_{k=1}^n k^{r-1} \| (B_k f - f)^{(2s)} \|_0 + \| f^{(2s)} \|_\infty \right).
 \end{aligned}$$

□

*Proof of (3.2).* By definition of  $K$ -functional there exists  $g \in C_{\alpha, \beta, \lambda}^r$  such that

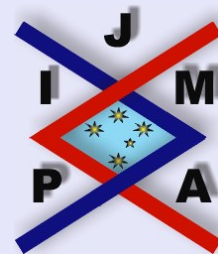
$$(3.3) \quad \| f^{(2s)} - g \|_0 + n^{-\frac{r}{2-\lambda}} \| g \|_r \leq K_{\phi^\lambda} \left( f, n^{-\frac{r}{2-\lambda}} \right)_{\alpha, \beta}$$

and

$$(3.4) \quad \left| \Delta_{h\phi^\lambda}^{2r} f^{(2s)}(x) \right| \leq C \phi^{\alpha(1-\lambda)+\beta}(x) \| f^{(2s)} \|_0$$

by Lemma 2.5 for above  $g$ ,  $0 < h\phi^\lambda(x) < \frac{1}{16r}$ ,  $x \pm rh\phi^\lambda(x) \in I$ ,

$$(3.5) \quad \left| \Delta_{h\phi^\lambda}^{2r} g(x) \right| \leq C h^{2r} \phi^{(-2r+\alpha)(1-\lambda)+\beta}(x) \| g \|_r.$$



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Using (3.4) and (3.5), again for  $0 < h\phi^\lambda(x) < \frac{1}{16r}$ ,  $x \pm rh\phi^\lambda(x) \in I$ , we get

$$(3.6) \quad \left| \Delta_{h\phi^\lambda(x)}^{2r} f^{(2s)}(x) \right| \leq C\phi^{\alpha(1-\lambda)+\beta}(x) \left\{ \|f^{(2s)} - g\|_0 + h^{2r}\phi^{2r(\lambda-1)}(x) \|g\|_r \right\}.$$

For  $x \pm rh\phi^\lambda(x) \in I$ , we obtain

$$(3.7) \quad h^2\phi^{2(\lambda-1)}(x) \leq \left[ \frac{1}{2}n^{\frac{1}{2-\lambda}} \right]^{-1}.$$

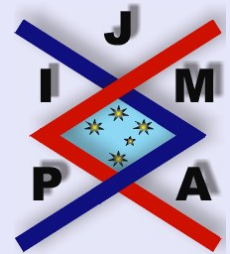
From (3.6) and (3.7) we have

$$(3.8) \quad \left| \Delta_{h\phi^\lambda(x)}^{2r} f^{(2s)}(x) \right| \leq C\phi^{\alpha(1-\lambda)+\beta}(x) K_{\phi^\lambda} \left( f^{(2s)}, \left[ \frac{1}{2}n^{\frac{1}{2-\lambda}} \right]^{-1} \right)_{\alpha,\beta} \leq C\phi^{\alpha(1-\lambda)+\beta}(x) n^{-\frac{r}{2-\lambda}} \times \left( \sum_{k=1}^{\left[ n^{\frac{r}{2-\lambda}} \right]} k^{-\frac{r-1}{2-\lambda}} \| (B_k f - f)^{(2s)} \|_0 + \| f^{(2s)} \|_\infty \right).$$

□

**Corollary 3.2.** *If  $0 < \alpha < 2$ ,  $f \in C_B(I)$ , then*

$$|(B_n f)(x) - f(x)| = O\left((n^{-1/2}\phi^{1-\lambda}(x))^\alpha\right) \Rightarrow w_{\phi^\lambda}^2(f, t) = O(t^\alpha),$$



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where

$$w_{\phi^\lambda}^2(f, t) = \sup_{0 < h \leq t} \sup_{x \pm h\phi^\lambda(x) \in I} \{ |\Delta_{h\phi^\lambda}^2 f(x)| \}.$$

This is the inverse part in [3].

In (1.4) and (1.5), for  $\delta_n(x) = \phi(x) + \frac{1}{\sqrt{n}}$ ,  $\phi(x)$  replaced by  $\delta_n(x)$ , (3.1) also holds.

**Corollary 3.3.** *If  $0 < \alpha < 2r$ ,  $f \in C_B(I)$ , then*

$$|(M_n f)(x) - f(x)| = O((n^{-1/2} \phi^{1-\lambda}(x))^\alpha) \Rightarrow w_{\phi^\lambda}^{2r}(f, t) = O(t^\alpha),$$

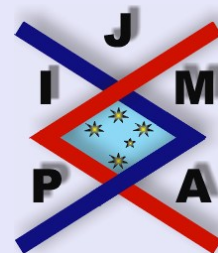
where  $(M_n f)(x)$  is linear combination of  $(B_n f)(x)$ .

This is the inverse parts in [4].

**Remark 3.1.** *We also propose some other modifications of Lupaş operators as*

$$M_n(f, x) = n \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} s_{n,k}(t) f(t) dt$$

where  $s_{n,k}(t) = e^{-nt} \frac{(nt)^k}{k!}$  and  $p_{n,k}(x)$  is defined in (1.1) for these operators  $M_n$ .



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