



# POINCARÉ TYPE INEQUALITIES FOR VARIABLE EXPONENTS

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*Key words:* Poincaré inequality, variable exponent.

*Abstract:* We consider Poincaré type inequalities of integral form for variable exponents. We give conditions under which these inequalities do not hold as well as conditions under which they hold.

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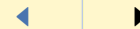
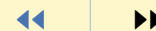
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# 1. Introduction and preliminaries

One of the classical Poincaré inequalities states

$$\int_G |\varphi(x)|^p dx \leq C(N, p, |G|) \int_G |\nabla\varphi(x)|^p dx, \quad \forall \varphi \in C_0^1(G),$$

where  $G$  is a bounded open set in  $\mathbb{R}^N$  ( $N \geq 1$ ) and  $p \geq 1$ .

In Fu [2], this inequality with  $p$  replaced by a bounded variable exponent  $p(x)$  is given as a lemma. Namely, let  $p(x)$  be a bounded measurable function on  $G$  such that  $p(x) \geq 1$  for all  $x \in G$ . We shall say that the Poincaré inequality (PI, for short) holds on  $G$  for  $p(\cdot)$  if there exists a constant  $C > 0$  such that

$$(PI) \quad \int_G |\varphi(x)|^{p(x)} dx \leq C \int_G |\nabla\varphi(x)|^{p(x)} dx$$

for all  $\varphi \in C_0^1(G)$ . Fu's lemma asserts that (PI) always holds. However, as was already remarked in [1, pp. 444-445, Example] in the one dimensional case, this is false. We shall give some types of  $p(\cdot)$  for which (PI) does not hold.

We remark here that the following norm-form of the Poincaré inequality holds for variable exponents (cf. [3, Theorem 3.10]):

$$\|\varphi\|_{L^{p(\cdot)}(G)} \leq C \|\nabla\varphi\|_{L^{p(\cdot)}(G)}$$

for all  $\varphi \in C_0^1(G)$  provided that  $p(x)$  is continuous on  $\overline{G}$ , where  $\|\cdot\|_{L^{p(\cdot)}(G)}$  denotes the (Luxemburg) norm in the variable exponent Lebesgue space  $L^{p(\cdot)}(G)$  (see [3] for definition). Thus, our results show that we must distinguish between norm-form and integral-form when we consider the Poincaré inequalities for variable exponents.

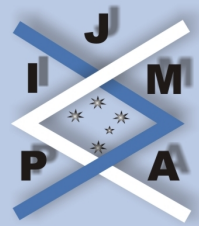
We also consider a slightly weaker form: we shall say that the weak Poincaré inequality (wPI, for short) holds on  $G$  for  $p(\cdot)$  if there exists a constant  $C > 0$  such

that

$$(wPI) \quad \int_G |\varphi(x)|^{p(x)} dx \leq C \left( 1 + \int_G |\nabla \varphi(x)|^{p(x)} dx \right)$$

for all  $\varphi \in C_0^1(G)$ . We shall see that this weak Poincaré inequality does not always hold either.

The main purpose of this paper is to give some sufficient conditions on  $p(\cdot)$  under which (PI) or (wPI) holds, and our results show that (PI) holds for a fairly large class of non-constant  $p(x)$  and (wPI) holds for  $p(x)$  in a larger class.



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## 2. Invalidity of Poincaré type inequalities

For a measurable function  $p(x)$  on  $G$  and  $E \subset G$ , let

$$p_E^+ = \operatorname{ess\,sup}_{x \in E} p(x) \quad \text{and} \quad p_E^- = \operatorname{ess\,inf}_{x \in E} p(x).$$

**Lemma 2.1.** *Let  $p(x)$  and  $q(x)$  be measurable functions on  $G$  such that  $0 < p_G^- \leq p_G^+ < \infty$  and  $0 < q_G^- \leq q_G^+ < \infty$ .*

- If there exist a compact set  $K$  and open sets  $G_1, G_2$  such that  $K \subset G_1 \Subset G_2 \subset G$ ,  $|K| > 0$  and  $q_K^- > p_{G_2 \setminus \overline{G_1}}^+$ , then there exists a sequence  $\{\varphi_n\}$  in  $C_0^1(G)$  such that*

$$\int_G |\varphi_n(x)|^{q(x)} dx \rightarrow \infty \quad \text{and} \quad \frac{\int_G |\nabla \varphi_n(x)|^{p(x)} dx}{\int_G |\varphi_n(x)|^{q(x)} dx} \rightarrow 0$$

as  $n \rightarrow \infty$ .

- If there exist a compact set  $K$  and open sets  $G_1, G_2$  such that  $K \subset G_1 \Subset G_2 \subset G$ ,  $|K| > 0$  and  $q_K^+ < p_{G_2 \setminus \overline{G_1}}^-$ , then there exists a sequence  $\{\psi_n\}$  in  $C_0^1(G) \setminus \{0\}$  such that*

$$\int_G |\nabla \psi_n(x)|^{p(x)} dx \rightarrow 0 \quad \text{and} \quad \frac{\int_G |\nabla \psi_n(x)|^{p(x)} dx}{\int_G |\psi_n(x)|^{q(x)} dx} \rightarrow 0$$

as  $n \rightarrow \infty$ .

*Proof.* Choose  $\varphi_1 \in C_0^1(G)$  such that  $\varphi_1 = 1$  on  $\overline{G_1}$  and  $\operatorname{Spt} \varphi_1 \subset G_2$ .

(1) Suppose  $q_K^- > p_{G_2 \setminus \overline{G_1}}^+$ . For simplicity, write  $q_1 = q_K^-$  and  $p_2 = p_{G_2 \setminus \overline{G_1}}^+$ . Let  $\varphi_n = n\varphi_1$ ,  $n = 1, 2, \dots$ . Then

$$\int_G |\nabla \varphi_n|^{p(x)} dx = \int_{G_2 \setminus \overline{G_1}} n^{p(x)} |\nabla \varphi_1|^{p(x)} dx \leq n^{p_2} \int_G |\nabla \varphi_1|^{p(x)} dx$$



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and

$$\int_G |\varphi_n|^{q(x)} dx \geq \int_K n^{q(x)} dx \geq n^{q_1} |K|.$$

These inequalities show that the sequence  $\{\varphi_n\}$  has the required properties.

(2) Suppose  $q_K^+ < p_{G_2 \setminus \overline{G_1}}^-$ . Write  $q_2 = q_K^+$  and  $p_1 = p_{G_2 \setminus \overline{G_1}}^-$ . Let  $\psi_n = (1/n)\varphi_1$ ,  $n = 1, 2, \dots$ . Then

$$\int_G |\nabla \psi_n|^{p(x)} dx = \int_{G_2 \setminus \overline{G_1}} n^{-p(x)} |\nabla \varphi_1|^{p(x)} dx \leq n^{-p_1} \int_G |\nabla \varphi_1|^{p(x)} dx$$

and

$$\int_G |\psi_n|^{q(x)} dx \geq \int_K n^{-q(x)} dx \geq n^{-q_2} |K|.$$

Thus the sequence  $\{\psi_n\}$  has the required properties.  $\square$

By taking  $p(x) = q(x)$  in this lemma, we readily obtain

### Proposition 2.2.

1. If there exist a compact set  $K$  and open sets  $G_1, G_2$  such that  $K \subset G_1 \Subset G_2 \subset G$ ,  $|K| > 0$  and  $p_K^- > p_{G_2 \setminus \overline{G_1}}^+$ , then (wPI) does not hold for  $p(\cdot)$  on  $G$ .
2. If there exist a compact set  $K$  and open sets  $G_1, G_2$  such that  $K \subset G_1 \Subset G_2 \subset G$ ,  $|K| > 0$  and  $p_K^+ < p_{G_2 \setminus \overline{G_1}}^-$ , then (PI) does not hold for  $p(\cdot)$  on  $G$ .



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### 3. Validity of Poincaré Type Inequalities in One-Dimensional Case

We shall say that  $f(t)$  on  $(t_0, t_1)$  is of type (L) if there is  $\tau \in (t_0, t_1)$  such that  $f(t)$  is non-increasing on  $(t_0, \tau)$  and non-decreasing on  $(\tau, t_1)$ .

**Proposition 3.1.** Let  $N = 1$  and  $G = (a, b)$ .

1. If  $p(t)$  is monotone (i.e., non-decreasing or non-increasing) or of type (L) on  $G$ , then

$$\int_a^b |f(t)|^{p(t)} dx \leq \frac{|G|}{2} + \max(|G|, |G|^{p^+}) \int_a^b |f'(t)|^{p(t)} dt$$

for  $f \in C_0^1(G)$ , where  $|G| = b - a$  and  $p^+ = p_G^+$ .

2. If  $p(t)$  is monotone on  $G$ , then

$$\int_a^b |f(t)|^{p(t)} dx \leq C \int_a^b |f'(t)|^{p(t)} dt$$

for  $f \in C_0^1(G)$ , where the constant  $C$  depends only on  $p^+$  and  $|G|$ .

*Proof.* (I) First, we consider the case  $G = (0, 1)$ . Let  $f \in C_0^1(G)$ .

(I-1) Suppose  $p(t)$  is non-increasing on  $(0, \tau)$ ,  $0 < \tau \leq 1$ . Then, for  $0 < t < \tau$ ,

$$\begin{aligned} |f(t)|^{p(t)} &\leq \left( \int_0^t |f'(s)| ds \right)^{p(t)} \leq \int_0^t |f'(s)|^{p(s)} ds \\ &\leq \int_0^t (1 + |f'(s)|^{p(s)}) ds \leq t + \int_0^1 |f'(s)|^{p(s)} ds. \end{aligned}$$



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Hence

$$\int_0^\tau |f(t)|^{p(t)} dt \leq \frac{\tau^2}{2} + \tau \int_0^1 |f'(s)|^{p(s)} ds.$$

Similarly, if  $p(t)$  is non-decreasing on  $(\tau, 1)$ ,  $0 \leq \tau < 1$ , then

$$\int_\tau^1 |f(t)|^{p(t)} dt \leq \frac{(1-\tau)^2}{2} + (1-\tau) \int_0^1 |f'(s)|^{p(s)} ds.$$

Hence, if  $p(t)$  is monotone or of type (L) on  $G$ , then

$$(3.1) \quad \int_0^1 |f(t)|^{p(t)} dt \leq \frac{1}{2} + \int_0^1 |f'(t)|^{p(t)} dt.$$

(I-2) The case  $\|f'\|_1 := \int_0^1 |f'(t)| dt \geq 1$ .

In this case,

$$\begin{aligned} 1 &\leq \int_0^1 |f'(t)| dt = \frac{1}{2} \int_0^1 |2f'(t)| dt \\ &\leq \frac{1}{2} + \frac{1}{2} \int_0^1 |2f'(t)|^{p(t)} dt \leq \frac{1}{2} + 2^{p^+-1} \int_0^1 |f'(t)|^{p(t)} dt, \end{aligned}$$

so that

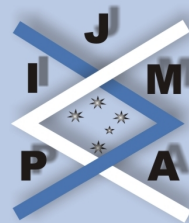
$$\frac{1}{2} \leq 2^{p^+-1} \int_0^1 |f'(t)|^{p(t)} dt.$$

Hence, by (3.1), we have

$$(3.2) \quad \int_0^1 |f(t)|^{p(t)} dt \leq (1 + 2^{p^+-1}) \int_0^1 |f'(t)|^{p(t)} dt$$

in case  $\|f'\|_1 \geq 1$ .





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(I-3) The case  $p(t)$  is monotone and  $\|f'\|_1 < 1$ .

We may assume that  $p(t)$  is non-decreasing. Set

$$E_1 = \{t \in (0, 1); |f'(t)| \leq 1\}, \quad E_2 = \{t \in (0, 1); |f'(t)| > 1\},$$

$$g_1(t) = \int_{(0,t) \cap E_1} |f'(s)| ds \quad \text{and} \quad g_2(t) = \int_{(0,t) \cap E_2} |f'(s)| ds.$$

Then for  $0 < t < 1$

$$\begin{aligned} |f(t)|^{p(t)} &\leq \left( \int_0^t |f'(s)| ds \right)^{p(t)} = (g_1(t) + g_2(t))^{p(t)} \\ &\leq 2^{p^+-1} (g_1(t)^{p(t)} + g_2(t)^{p(t)}). \end{aligned}$$

Since  $p(s) \leq p(t)$  for  $0 < s < t$  and  $|f(s)| \leq 1$  for  $s \in E_1$ ,

$$g_1(t)^{p(t)} \leq \int_{(0,t) \cap E_1} |f'(s)|^{p(t)} ds \leq \int_{(0,t) \cap E_1} |f'(s)|^{p(s)} ds \leq \int_{E_1} |f'(s)|^{p(s)} ds.$$

On the other hand, since  $g_2(t) \leq \|f'\|_1 < 1$  and  $|f'(s)| > 1$  for  $s \in E_2$ ,

$$g_2(t)^{p(t)} \leq g_2(t) = \int_{(0,t) \cap E_2} |f'(s)| ds \leq \int_{E_2} |f'(s)|^{p(s)} ds.$$

Hence

$$|f(t)|^{p(t)} \leq 2^{p^+-1} \int_0^1 |f'(s)|^{p(s)} ds$$

for all  $0 < t < 1$ , and hence

$$\int_0^1 |f(t)|^{p(t)} dt \leq 2^{p^+-1} \int_0^1 |f'(s)|^{p(s)} ds$$



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in case  $\|f'\|_1 < 1$ .

(I-4) Combining (I-2) and (I-3), we have (3.2) for all  $f \in C_0^1(G)$  if  $p(t)$  is monotone.

(II) The general case: Let  $G = (a, b)$  and  $f \in C_0^1(G)$ . Let

$$g(t) = f(a + t(b - a)) \quad \text{and} \quad q(t) = p(a + t(b - a))$$

for  $0 < t < 1$ . Then

$$\int_a^b |f(s)|^{p(s)} ds = (b - a) \int_0^1 |g(t)|^{q(t)} dt$$

and

$$\begin{aligned} \int_0^1 |g'(t)| dt &= \frac{1}{b - a} \int_a^b |(b - a)f'(s)|^{p(s)} ds \\ &\leq \max(1, (b - a)^{p^+ - 1}) \int_a^b |f'(s)|^{p(s)} ds. \end{aligned}$$

Hence, applying (3.1) and (3.2) to  $g(t)$  and  $q(t)$ , we obtain the required inequalities of the proposition. (In fact, we can take  $C = (1 + 2^{p^+ - 1}) \max(|G|, |G|^{p^+})$ .)  $\square$



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## 4. Validity of Poincaré Type Inequalities in Higher-Dimensional Case

**Theorem 4.1.** Let  $N \geq 2$  and  $G \subset G' \times (a, b)$  with a bounded open set  $G' \subset \mathbb{R}^{N-1}$  and set  $G_{x'} = \{t \in (a, b) : (x', t) \in G\}$  for  $x' \in G'$ .

1. If  $t \mapsto p(x', t)$  is monotone or of type (L) on each component of  $G_{x'}$  for a.e.  $x' \in G'$  (with respect to the  $(N - 1)$ -dimensional Lebesgue measure), then (wPI) holds for  $p(\cdot)$  on  $G$ .
2. If  $t \mapsto p(x', t)$  is monotone on each component of  $G_{x'}$  for a.e.  $x' \in G'$  (with respect to the  $(N - 1)$ -dimensional Lebesgue measure), then (PI) holds for  $p(\cdot)$  on  $G$ .

*Proof.* Fix  $x' \in G'$  for a moment and let  $I_j$  be the components of  $G_{x'}$ . If  $\varphi \in C_0^1(G)$ , then  $t \mapsto \varphi(x', t)$  belongs to  $C_0^1(I_j)$  for each  $j$ . Thus, by Proposition 3.1, if  $t \mapsto p(x', t)$  is monotone or of type (L) on each  $I_j$ , then

$$\int_{I_j} |\varphi(x', t)|^{p(x', t)} dt \leq |I_j| + \max(1, |I_j|^{p^+}) \int_{I_j} |\nabla \varphi(x', t)|^{p(x', t)} dt,$$

so that

$$\int_{G_{x'}} |\varphi(x', t)|^{p(x', t)} dt \leq |G_{x'}| + \max(1, (b - a)^{p^+}) \int_{G_{x'}} |\nabla \varphi(x', t)|^{p(x', t)} dt;$$

and if  $t \mapsto p(x', t)$  is monotone on each  $I_j$  then

$$\int_{I_j} |\varphi(x', t)|^{p(x', t)} dt \leq C(p^+, I_j) \int_{I_j} |\nabla \varphi(x', t)|^{p(x', t)} dt,$$

so that

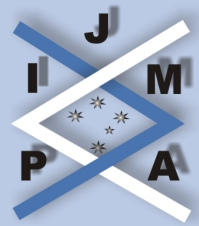
$$\int_{G_{x'}} |\varphi(x', t)|^{p(x', t)} dt \leq C(p^+, b - a) \int_{G_{x'}} |\nabla \varphi(x', t)|^{p(x', t)} dt.$$

Hence, integrating over  $G'$  with respect to  $x'$ , we obtain the assertion of the theorem.  $\square$

The following proposition is easily seen by a change of variables:

**Proposition 4.2.** *(PI) and (wPI) are diffeomorphically invariant. More precisely, let  $G_1$  and  $G_2$  be bounded open sets and  $\Phi(x) = (\phi_1(x), \dots, \phi_N(x))$  be a  $(C^1)$ -diffeomorphism of  $G_1$  onto  $G_2$ . Suppose  $|\nabla \phi_j|$ ,  $j = 1, \dots, N$  and  $|\nabla \psi_j|$ ,  $j = 1, \dots, N$  are all bounded, where  $\Phi^{-1}(y) = (\psi_1(y), \dots, \psi_N(y))$ , and suppose  $0 < \alpha \leq J_\Phi(x) \leq \beta$  for all  $x \in G_1$ . Let  $p_1(x) = p_2(\Phi(x))$  for  $x \in G_1$ . Then, (PI) (resp. (wPI)) holds for  $p_1(\cdot)$  on  $G_1$  if and only if it holds for  $p_2(\cdot)$  on  $G_2$ .*

Combining Theorem 4.1 with this Proposition, we can find a fairly large class of  $p(x)$  for which (PI) (as well as (wPI)) holds.



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