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UNIVALENT HARMONIC FUNCTIONS

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ABSTRACT. A necessary and sufficient coefficient is given for functions in a class of complex-valued harmonic univalent functions using the Dziok-Srivastava operator. Distortion bounds, extreme points, an integral operator, and a neighborhood of such functions are considered.

Key words and phrases: Harmonic functions, Dziok-Srivastava operator, Convolution, Integral operator, Distortion bounds, Neighborhood.

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1. INTRODUCTION

Let U denote the open unit disc and S_H denote the class of functions which are complex-valued, harmonic, univalent, sense-preserving in U normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in S_H$ can be expressed as $f = h + \overline{g}$, where h and g are analytic in U. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that $|h'(z)| > |g'(z)|$ in U (see [3]). Thus for $f = h + \overline{g} \in S_H$, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad (0 \le b_1 < 1). \tag{1.1}$$

Note that S_H reduces to S, the class of normalized analytic univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero.

For $\alpha_j \in C$ $(j = 1, 2, \dots, q)$ and $\beta_j \in C - \{0, -1, -2, \dots\}$ $(j = 1, 2, \dots, s)$, the generalized hypergeometric function is defined by

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{z^k}{k!},$$
$$(q \le s + 1; q, s \in N_0 = \{0, 1, 2, \dots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$$

for $n \in \mathbb{N} = \{1,2,\dots\}$ and 1 when $n = 0$. Corresponding to the function

$$h(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) = z_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z).$$

The Dziok-Srivastava operator [4], $H_{q,s}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)$ is defined by

$$\begin{aligned} H_{q,s}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z) &= h(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}} \frac{a_k z^k}{(k-1)!}, \end{aligned}$$

where "$*$" stands for convolution.

To make the notation simple, we write

$$H_{q,s}[\alpha_1]f(z) = H_{q,s}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z).$$

We define the Dziok-Srivastava operator of the harmonic function $f = h + \overline{g}$ given by (1.1) as

$$H_{q,s}[\alpha_1]f = H_{q,s}[\alpha_1]h + \overline{H_{q,s}[\alpha_1]g}. \tag{1.2}$$

Let $S_H^*(\alpha_1,\beta)$ denote the family of harmonic functions of the form (1.1) such that

$$\frac{\partial}{\partial\theta}(\arg H_{q,s}[\alpha_1]f) \geq \beta, \quad 0 \leq \beta < 1,\ |z| = r < 1. \tag{1.3}$$

For $q = s+1,\ \alpha_2 = \beta_1,\dots,\alpha_q = \beta_s,\ S_H^*(1,\beta) = SH(\beta)$ [6] is the class of orientation-preserving harmonic univalent functions f which are starlike of order β in U, that is, $\frac{\partial}{\partial\theta}(\arg f(re^{i\theta}) > \beta$.

Also, $S_H^*(n+1,\beta) = R_H(n,\beta)$ [7], is the class of harmonic univalent functions with $\frac{\partial}{\partial\theta}(\arg D^n f(z)) \geq \beta$, where D is the Ruscheweyh derivative (see [9]).

We also let $V_{\overline{H}}(\alpha_1,\beta) = S_H^*(\alpha_1,\beta) \cap V_H$, where V_H [5], the class of harmonic functions f of the form (1.1) and there exists ϕ so that, $\bmod 2\pi$,

$$\arg(a_k) + (k-1)\phi = \pi, \qquad \arg(b_k) + (k-1)\phi = 0 \qquad k \geq 2. \tag{1.4}$$

Jahangiri and Silverman [5] gave the sufficient and necessary conditions for functions of the form (1.1) to be in $V_H(\beta)$, where $0 \leq \beta < 1$.

Note for $q = s+1,\ \alpha_1 = 1,\ \alpha_2 = \beta_1,\dots,\alpha_q = \beta_s$ and the co-analytic part of $f = h + \overline{g}$ being zero, the class $V_{\overline{H}}(\alpha_1,\beta)$ reduces to the class studied in [10].

In this paper, we will give a sufficient condition for $f = h + \overline{g}$ given by (1.1) to be in $S_H^*(\alpha_1,\beta)$ and it is shown that this condition is also necessary for functions in $V_{\overline{H}}(\alpha_1,\beta)$. Distortion theorems, extreme points, integral operators and neighborhoods of such functions are considered.

2. MAIN RESULTS

In our first theorem, we introduce a sufficient coefficient bound for harmonic functions in $S_H^*(\alpha_1,\beta)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.1). If*

$$\sum_{k=2}^{\infty} \frac{1}{(k-1)!}\left(\frac{k-\beta}{1-\beta}|a_k| + \frac{k+\beta}{1-\beta}|b_k|\right)\Gamma(\alpha_1,k) \leq 1 - \frac{1+\beta}{1-\beta}|b_1|, \tag{2.1}$$

where $a_1 = 1$, $0 \leq \beta < 1$ *and* $\Gamma(\alpha_1, k) = \left|\frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}\right|$, *then* $f \in S_H^*(\alpha_1, \beta)$.

Proof. To prove that $f \in S_H^*(\alpha_1, \beta)$, we only need to show that if (2.1) holds, then the required condition (1.3) is satisfied. For (1.3), we can write

$$\frac{\partial}{\partial\theta}(\arg H_{q,s}[\alpha_1]f(z)) = \operatorname{Re}\left\{\frac{z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}}{H_{q,s}[\alpha_1]h + \overline{H_{q,s}[\alpha_1]g}}\right\}$$
$$= \operatorname{Re}\frac{A(z)}{B(z)}.$$

Using the fact that $\operatorname{Re}\omega \geq \beta$ if and only if $|1-\beta+\omega| \geq |1+\beta-\omega|$, it suffices to show that

$$|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)| \geq 0. \tag{2.2}$$

Substituting for $A(z)$ and $B(z)$ in (2.1) yields

$$\begin{aligned}
&|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)| \\
&\geq (2-\beta)|z| - \sum_{k=2}^{\infty}\frac{k+1-\beta}{(k-1)!}\Gamma(\alpha_1,k)|a_k||z|^k \\
&\quad - \sum_{k=1}^{\infty}\frac{k-1+\beta}{(k-1)!}\Gamma(\alpha_1,k)|b_k||z|^k - \beta|z| \\
&\quad - \sum_{k=2}^{\infty}\frac{k-1-\beta}{(k-1)!}\Gamma(\alpha_1,k)|a_k||z|^k - \sum_{k=1}^{\infty}\frac{k+1+\beta}{(k-1)!}\Gamma(\alpha_1,k)|b_k||z|^k \\
&\geq 2(1-\beta)|z|\left\{1 - \sum_{k=2}\frac{k-\beta}{(1-\beta)(k-1)!}\Gamma(\alpha_1,k)|a_k|\right. \\
&\qquad \left. - \sum_{k=1}^{\infty}\frac{k+\beta}{(1-\beta)(k-1)!}\Gamma(\alpha_1,k)|b_k|\right\} \\
&= 2(1-\beta)|z|\left\{1 - \frac{1+\beta}{1-\beta}|b_1|\right. \\
&\qquad \left. - \left[\sum_{k=2}^{\infty}\frac{1}{(k-1)!}\left(\frac{k-\beta}{1-\beta}|a_k| + \frac{k+\beta}{1-\beta}|b_k|\right)\Gamma(\alpha_1,k)\right]\right\}.
\end{aligned} \tag{2.3}$$

The last expression is non-negative by (2.1) and so $f \in S_H^*(\alpha_1, \beta)$. □

Now, we obtain the necessary and sufficient conditions for $f = h + \overline{g}$ given by (1.4).

Theorem 2.2. *Let* $f = h + \overline{g}$ *be given by (1.4). Then* $f \in V_{\overline{H}}(\alpha_1, \beta)$ *if and only if*

$$\sum_{k=2}^{\infty}\frac{1}{(k-1)!}\left(\frac{k-\beta}{1-\beta}|a_k| + \frac{k+\beta}{1-\beta}|b_k|\right)\Gamma(\alpha_1,k) \leq 1 - \frac{1+\beta}{1-\beta}|b_1|, \tag{2.4}$$

where $a_1 = 1,\ 0 \leq \beta < 1$ *and* $\Gamma(\alpha_1, k) = \left|\frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}\right|$.

Proof. Since $V_{\overline{H}}(\alpha_1, \beta) \subset S_H^*(\alpha_1,\ \beta)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f \in V_{\overline{H}}(\alpha_1, \beta)$, we notice that the condition $\frac{\partial}{\partial\theta}(\arg H_{q,s}[\alpha_1]f(z)) \geq \beta$

is equivalent to

$$\frac{\partial}{\partial\theta}(\arg H_{q,s}[\alpha_1]f(z)) - \beta = \operatorname{Re}\left\{\frac{z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}}{H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}} - \beta\right\} \geq 0.$$

That is,

$$\operatorname{Re}\left[\frac{(1-\beta)z + \sum_{k=2}^{\infty}\frac{k-\beta}{(k-1)!}\Gamma(\alpha_1,k)|a_k|z^k - \sum_{k=1}^{\infty}\frac{k+\beta}{(k-1)!}\overline{\Gamma(\alpha_1,k)}|b_k|\overline{z}^k}{z + \sum_{k=2}^{\infty}\Gamma(\alpha_1,k)|a_k|z^k + \sum_{k=1}^{\infty}\overline{\Gamma(\alpha_1,k)}|b_k|\overline{z}^k}\right] \geq 0. \tag{2.5}$$

The above condition must hold for all values of z in U. Upon choosing ϕ according to (1.4), we must have

$$\frac{(1-\beta) - (1+\beta)|b_1| - \sum_{k=2}^{\infty}\left(\frac{k-\beta}{(k-1)!}|a_k| + \frac{k+\beta}{(k-1)!}|b_k|\right)\Gamma(\alpha_1,k)r^{k-1}}{1 + |b_1| + \sum_{k=2}^{\infty}(|a_k| + |b_k|)\Gamma(\alpha_1,k)r^{k-1}} \geq 0. \tag{2.6}$$

If condition (2.4) does not hold then the numerator in (2.6) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient of (2.6) is negative. This contradicts the fact that $f \in V_{\overline{H}}(\alpha_1,\beta)$ and so the proof is complete. □

The following theorem gives the distortion bounds for functions in $V_{\overline{H}}(\alpha_1,\beta)$ which yields a covering result for this class.

Theorem 2.3. *If $f \in V_{\overline{H}}(\alpha_1,\beta)$, then*

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{\Gamma(\alpha_1,2)}\left(\frac{1-\beta}{2-\beta} - \frac{1+\beta}{2-\beta}|b_1|\right)r^2 \qquad |z| = r < 1$$

and

$$|f(z)| \geq (1+|b_1|)r - \frac{1}{\Gamma(\alpha_1,2)}\left(\frac{1-\beta}{2-\beta} - \frac{1+\beta}{2+\beta}|b_1|\right)r^2 \qquad |z| = r < 1.$$

Proof. We will only prove the right hand inequality. The proof for the left hand inequality is similar.

Let $f \in V_{\overline{H}}(\alpha_1,\beta)$. Taking the absolute value of f, we obtain

$$|f(z)| \leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k \leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^2.$$

That is,

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \frac{1-\beta}{\Gamma(\alpha_1,2)(2-\beta)}\sum_{k=2}^{\infty}\left(\frac{2-\beta}{1-\beta}|a_k| + \frac{2-\beta}{1-\beta}|b_k|\right)\Gamma(\alpha_1,2)r^2 \\
&\leq (1+|b_1|)r + \frac{1-\beta}{\Gamma(\alpha_1,2)(2-\beta)}\left[1 - \frac{1+\beta}{1-\beta}|b_1|\right]r^2 \\
&\leq (1+|b_1|)r + \frac{1}{\Gamma(\alpha_1,2)}\left(\frac{1-\beta}{2-\beta} - \frac{1+\beta}{2-\beta}|b_1|\right)r^2.
\end{aligned}$$

□

Corollary 2.4. *Let f be of the form (1.1) so that $f \in V_{\overline{H}}(\alpha_1,\beta)$. Then*

$$\left\{\omega : |\omega| < \frac{2\Gamma(\alpha_1,2) - 1 - (\Gamma(\alpha_1,2)-1)\beta}{(2-\beta)\Gamma(\alpha_1,2)} - \frac{2\Gamma(\alpha_1,2) - 1 - (\Gamma(\alpha_1,2)-1)\beta}{(2+\beta)\Gamma(\alpha_1,2)}|b_1|\right\} \subset f(U). \tag{2.7}$$

Next, we examine the extreme points for $V_{\overline{H}}(\alpha_1,\beta)$ and determine extreme points of $V_{\overline{H}}(\alpha_1,\beta)$.

Theorem 2.5. *Set*

$$\lambda_k = \frac{(1-\beta)(k-1)!}{(k-\beta)\Gamma(\alpha_1,k)} \quad \textit{and} \quad \mu_k = \frac{(1-\beta)(k-1)!}{(k+\beta)\Gamma(\alpha_1,k)}.$$

For b_1 fixed, the extreme points for $V_{\overline{H}}(\alpha_1,\beta)$ are

$$\{z+\lambda_k x z^k + \overline{b_1 z}\} \cup \left\{z + \overline{b_1 z + \mu_k x z^k}\right\}, \tag{2.8}$$

where $k \geq 2$ and $|x| = 1-|b_1|$.

Proof. Any function $f \in V_{\overline{H}}(\alpha_1,\beta)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} |a_k| e^{i\gamma_k} |z^k + \overline{b_1 z} + \overline{\sum_{k=2} |b_k| e^{i\delta_k} z^k},$$

where the coefficients satisfy the inequality (2.1). Set

$$h_1(z) = z, g_1(z) = b_1 z, h_k(z) = z + \lambda_k e^{i\gamma_k} z^k$$
$$g_k = b_1 z + \mu_k e^{i\delta_k} z^k, \quad \text{for} \quad k = 2,3,\dots$$

Writing $X_k = \frac{|a_k|}{\lambda_k}, Y_k = \frac{|b_k|}{\mu_k}, \quad k = 2,3,\dots$ and

$$X_1 = 1 - \sum_{k=2}^{\infty} X_k; \qquad Y_1 = 1 - \sum_{k=2}^{\infty} Y_k,$$

we have

$$f(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)).$$

In particular, setting

$$f_1(z) = z + \overline{b_1 z} \quad \text{and} \quad f_k(z) = z + \lambda_k x z^k + \overline{b_1 z} + \overline{\mu_k y z^k}$$
$$(k \geq 2, |x| + |y| = 1 - |b_1|),$$

we see that the extreme points of $V_{\overline{H}}(\alpha_1,\beta)$ are contained in $\{f_k(z)\}$.

To see that f_1 is not an extreme point, note that f_1 may be written as

$$f_1(z) = \frac{1}{2}\{f_1(z) + \lambda_2(1-|b_1|)z^2\} + \frac{1}{2}\{f_1(z) - \lambda_2(1-|b_1|)z^2\},$$

a convex linear combination of functions in $V_{\overline{H}}(\alpha_1,\beta)$. If both $|x| \neq 0$ and $|y| \neq 0$, we will show that it can also be expressed as a convex linear combination of functions in $V_{\overline{H}}(\alpha_1,\beta)$. Without loss of generality, assume $|x| \geq |y|$. Choose $\epsilon > 0$ small enough so that $\epsilon < \frac{|x|}{|y|}$. Set $A = 1+\epsilon$ and $B = 1 - \left|\frac{\epsilon x}{y}\right|$. We then see that both

$$t_1(z) = z + \lambda_k A x z^k + \overline{b_1 z + \mu_k y B z^k}$$

and

$$t_2(z) = z + \lambda_k(2-A)xz^k + \overline{b_1 z + \mu_k y(2-B)z^k}$$

are in $V_{\overline{H}}(\alpha_1,\beta)$ and note that

$$f_n(z) = \frac{1}{2}\{t_1(z) + t_2(z)\}.$$

The extremal coefficient bound shows that the functions of the form (2.8) are the extreme points for $V_{\overline{H}}(\alpha_1,\beta)$ and so the proof is complete. □

For $q = s+1, \alpha_2 = \beta_1, \dots, \alpha_q = \beta_s, \alpha_1 = n+1$, Theorems 2.1 to 2.5 give Theorems 1, 2, 3 and 4 in [7].

Now, we will examine the closure properties of the class $V_{\overline{H}}(\alpha_1, \beta)$ under the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ which is defined by

$$L_c(f(z)) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \qquad c > -1.$$

Theorem 2.6. *Let $f \in V_{\overline{H}}(\alpha_1, \beta)$. Then $L_c(f(z))$ belongs to the class $V_{\overline{H}}(\alpha_1, \beta)$.*

Proof. From the representation of $L_c(f(z))$, it follows that

$$\begin{aligned} L_c(f(z)) &= \frac{c+1}{z^c} \int_0^z t^{c-1}\{h(t) + \overline{g}(t)\}dt \\ &= \frac{c+1}{z^c} \left(\int_0^z t^{c-1} \left(t + \sum_{k=2}^{\infty} a_k t^k \right) dt + \overline{\int_0^z t^{c-1} \left(\sum_{k=1}^{\infty} b_k t^k \right) dt} \right) \\ &= z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B_k z^k}, \end{aligned}$$

where

$$A_k = \frac{c+1}{c+k} a_k, \qquad B_k = \frac{c+1}{c+k}\, b_k.$$

Therefore,

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} &\left(\frac{(k-\beta)(c+1)}{(1-\beta)(c+k)} |a_k| + \frac{(k+\beta)(c+1)}{(1-\beta)(c+k)} |b_k| \right) \Gamma(\alpha_1, k) \\ &\le \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \left(\frac{k-\beta}{1-\beta} |a_k| + \frac{k+\beta}{1-\beta} |b_k| \right) \Gamma(\alpha_1, k) \\ &\le 1 - \frac{1+\beta}{1-\beta} b_1. \end{aligned}$$

Since $f \in V_{\overline{H}}(\alpha_1, \beta)$, therefore by Theorem 2.2, $L_c(f(z)) \in V_{\overline{H}}(\alpha_1, \beta)$. □

The next theorem gives a sufficient coefficient bound for functions in $S^*(\alpha_1, \beta)$.

Theorem 2.7. *$f \in S_H^*(\alpha_1, \beta)$ if and only if*

$$\begin{aligned} H_{q,s}[\alpha_1] h(z) * &\left[\frac{2(1-\beta)z + (\xi - 1 + 2\beta)z^2}{(1-z)^2} \right] \\ &+ \overline{H_{q,s}[\alpha_1] g} * \left[\frac{2(\xi+\beta)\overline{z} - (\xi - 1 + 2\beta)\overline{z}^2}{(1-\overline{z})^2} \right] \ne 0, \qquad |\xi| = 1, \quad z \in U. \end{aligned}$$

Proof. From (1.3), $f \in S_H^*(\alpha_1, \beta)$ if and only if for $z = re^{i\theta}$ in U, we have

$$\frac{\partial}{\partial\theta}(\arg(H_{q,s}[\alpha_1] f(re^{i\theta})) = \frac{\partial}{\partial\theta}\left[\arg\left(H_{q,s}[\alpha_1] h(re^{i\theta}) + \overline{H_{q,s}[\alpha_1] g(re^{i\theta})} \right) \right] \ge \beta.$$

Therefore, we must have

$$\operatorname{Re}\left\{ \frac{1}{1-\beta} \left[\frac{z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}}{H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}} - \beta \right] \right\} \ge 0.$$

Since

$$\frac{1}{1-\beta}\left[\frac{z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}}{H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}} - \beta\right] = 1 \quad \text{at} \quad z = 0,$$

the above required condition is equivalent to

$$\frac{1}{1-\beta}\left[\frac{z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}}{H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}} - \beta\right] \neq \frac{\xi-1}{\xi+1}, \tag{2.9}$$
$$|\xi| = 1,\ \xi \neq -1,\ 0 < |z| < 1.$$

By a simple algebraic manipulation, inequality (2.9) yields

$$\begin{aligned}
0 &\neq (\xi+1)[z(H_{q,s}[\alpha_1]h(z))' - \overline{z(H_{q,s}[\alpha_1]g(z))'}] \\
&\quad - (\xi - 1 + 2\beta)[H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}] \\
&= H_{q,s}[\alpha_1]h(z) * \left[\frac{(\xi+1)z}{(1-z)^2} - \frac{\xi-1+2\beta}{1-z}\right] \\
&\quad - \overline{H_{q,s}[\alpha_1]g(z)} * \left[\overline{\frac{(\bar{\xi}+1)z}{(1-z)^2} + \frac{(\bar{\xi}-1+2\beta)z}{1-z}}\right] \\
&= H_{q,s}[\alpha_1]h(z) * \left[\frac{2(1-\beta)z + (\xi-1+2\beta)z^2}{(1-z)^2}\right] \\
&\quad + \overline{H_{q,s}[\alpha_1]g(z)} * \left[\overline{\frac{2(\bar{\xi}+\beta)z - (\bar{\xi}-1+2\beta)z^2}{(1-z)^2}}\right],
\end{aligned}$$

which is the condition required by Theorem 2.7. □

Finally, for f given by (1.1), the δ-neighborhood of f is the set

$$N_\delta(f) = \left\{F = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B_k z^k} : \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k|) + |b_1 - B_1| \leq \delta\right\}$$

(see [1] [8]). In our case, let us define the generalized δ-neighborhood of f to be the set

$$N(f) = \left\{F : \sum_{k=2}^{\infty} \frac{\Gamma(\alpha_1,k)}{(k-1)!}\left[(k-\beta)|a_k - A_k| + (k+\beta)|b_k - B_k|\right] + (1+\beta)|b_1 - B_1| \leq (1-\beta)\delta\right\}.$$

Theorem 2.8. *Let f be given by (1.1). If f satisfies the conditions*

$$\sum_{k=2}^{\infty} \frac{k(k-\beta)}{(k-1)!}|a_k|\Gamma(\alpha_1,k) + \sum_{k=1}^{\infty} \frac{k(k+\beta)}{(k-1)!}|b_k|\Gamma(\alpha_1,k) \leq 1-\beta,\ 0 \leq \beta < 1 \tag{2.10}$$

and

$$\delta \leq \frac{1-\beta}{2-\beta}\left(1 - \frac{1+\beta}{1-\beta}|b_1|\right),$$

then $N(f) \subset S_H^*(\alpha_1, \beta)$.

Proof. Let f satisfy (2.10) and

$$F(z) = z + \overline{B_1 z} + \sum_{k=2}^{\infty} \left(A_k z^k + \overline{B_k z^k} \right)$$

belong to $N(f)$. We have

$$\begin{aligned}
&(1+\beta)|B_1| + \sum_{k=2}^{\infty} \frac{\Gamma(\alpha_1, k)}{(k-1)!} \left((k-\beta)|A_k| + (k+\beta)|B_k|\right) \\
&\le (1+\beta)|B_1 - b_1| + (1+\beta)|b_1| + \sum_{k=2}^{\infty} \frac{\Gamma(\alpha_1, k)}{(k-1)!} \left[(k-\beta)|A_k - a_k| + (k+\beta)|B_k - b_k|\right] \\
&\qquad + \sum_{k=2}^{\infty} \frac{\Gamma(\alpha_1, k)}{(k-1)!} \left[(k-\beta)|a_k| + (k+\beta)|b_k|\right] \\
&\le (1-\beta)\delta + (1+\beta)|b_1| + \frac{1}{2-\beta} \sum_{k=2} k \frac{\Gamma(\alpha_1, k)}{(k-1)!} \left[(k-\beta)|a_k| + (k+\beta)|b_k|\right] \\
&\le (1-\beta)\delta + (1+\beta)|b_1| + \frac{1}{2-\beta} \left[(1-\beta) - (1+\beta)|b_1|\right] \\
&\le 1 - \beta.
\end{aligned}$$

Hence, for

$$\delta \le \frac{1-\beta}{2-\beta} \left[1 - \frac{1+\beta}{1-\beta} |b_1| \right],$$

we have $F(z) \in S_H^*(\alpha_1, \beta)$. □

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