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THE LOWER AND UPPER SOLUTIONS METHOD FOR FIRST ORDER DIFFERENTIAL INCLUSIONS WITH NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. In this paper a fixed point theorem for condensing maps combined with upper and lower solutions are used to investigate the existence of solutions for first order differential inclusions with general nonlinear boundary conditions.

Key words and phrases: Initial value problem, Convex multivalued map, Differential inclusions, Nonlinear boundary conditions, Condensing map, Fixed point, Truncation map, Upper and lower solutions.

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1. INTRODUCTION

This paper is concerned with the existence of solutions for the boundary multivalued problem:

$$y'(t) \in F(t, y(t)), \quad \text{for a.e. } t \in J = [0, T] \tag{1.1}$$

$$L(y(0), y(T)) = 0 \tag{1.2}$$

where $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact and convex valued multivalued map and $L : \mathbb{R}^2 \to \mathbb{R}$ is a continuous single-valued map.

The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems of first and second order.

This method has been used only in the context of single-valued differential equations. We refer to the books of Bernfeld-Lakshmikantham [4], Heikkila-Lakshmikantham [13], Ladde-

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Lakshmikantham-Vatsala [16], to the thesis of De Coster [7], to the papers of Carl-Heikkila-Kumpulainen [6], Cabada [5], Frigon [9], Frigon-O'Regan [10], Heikkila-Cabada [12], Lakshmikantham-Leela [17], Nkashama [20] and the references therein.

Using this method the authors obtained in [2] and [3] existence results for differential inclusions with periodic boundary conditions, for first and second order respectively.

In this paper we establish an existence result for the problem (1.1) – (1.2). Our approach is based on the existence of upper and lower solutions and on a fixed point theorem for condensing maps due to Martelli [19].

2. PRELIMINARIES

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel.

$AC(J,\mathbb{R})$ is the space of all absolutely continuous functions $y : J\to\mathbb{R}$.

Condition

$$y \le \bar{y} \quad \text{if and only if} \quad y(t) \le \bar{y}(t) \quad \text{for all} \quad t \in J$$

defines a partial ordering in $AC(J,\mathbb{R})$. If α, $\beta \in AC(J,\mathbb{R})$ and $\alpha \le \beta$, we denote

$$[\alpha,\beta] = \{y \in AC(J,\mathbb{R}) : \alpha \le y \le \beta\}.$$

$W^{1,1}(J,\mathbb{R})$ denotes the Banach space of functions $y : J\to\mathbb{R}$ which are absolutely continuous and whose derivative y' (which exists almost everywhere) is an element of $L^1(J,\mathbb{R})$ with the norm

$$\|y\|_{W^{1,1}} = \|y\|_{L^1} + \|y'\|_{L^1} \quad \text{for all } y \in W^{1,1}(J,\mathbb{R}).$$

Let $(X, |\cdot|)$ be a normed space. A multivalued map $G : X\to 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B}G(x)$ is bounded in X for all bounded subsets B of X (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set V of X containing $G(x_0)$, there exists an open neighbourhood U of x_0 such that $G(U) \subseteq V$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subset X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n\to x_*,\ y_n\to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $CC(X)$ denotes the set of all nonempty compact and convex subsets of X.

An upper semi-continuous map $G : X\to 2^X$ is said to be condensing [19] if for any bounded subset $B \subseteq X$, with $\mu(B) \ne 0$, we have $\mu(G(B)) < \mu(B)$, where μ denotes the Kuratowski measure of noncompacteness [1]. We remark that a compact map is the easiest example of a condensing map. For more details on multivalued maps see the books of Deimling [8] and Hu and Papageorgiou [15].

The multivalued map $F : J\to CC(\mathbb{R})$ is said to be measurable, if for every $y \in \mathbb{R}$, the function $t \longmapsto d(y,F(t)) = \inf\{|y-z| : z \in F(t)\}$ is measurable.

Definition 2.1. A multivalued map $F : J \times \mathbb{R}\to 2^{\mathbb{R}}$ is said to be an L^1-Carathéodory if

(i) $t \longmapsto F(t,y)$ is measurable for each $y \in \mathbb{R}$;

(ii) $y \longmapsto F(t,y)$ is upper semicontinuous for almost all $t \in J$;

(iii) For each $k > 0$, there exists $h_k \in L^1(J,\mathbb{R}_+)$ such that

$$\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \le h_k(t) \quad \text{for all } \ |y| \le k \ \text{ and for almost all } \ t \in J.$$

So let us start by defining what we mean by a solution of problem (1.1) – (1.2).

Definition 2.2. A function $y \in AC(J, \mathbb{R})$ is said to be a solution of (1.1) – (1.2) if there exists a function $v \in L^1(J, \mathbb{R})$ such that $v(t) \in F(t, y(t))$ a.e. on J, $y'(t) = v(t)$ a.e. on J and $L(y(0), y(T)) = 0$.

The following concept of lower and upper solutions for (1.1) – (1.2) has been introduced by Halidias and Papageorgiou in [14] for second order multivalued boundary value problems. It will be the basic tools in the approach that follows.

Definition 2.3. A function $\alpha \in AC(J, \mathbb{R})$ is said to be a lower solution of (1.1) – (1.2) if there exists $v_1 \in L^1(J, \mathbb{R})$ such that $v_1(t) \in F(t, \alpha(t))$ a.e. on J, $\alpha'(t) \leq v_1(t)$ a.e. on J and $L(\alpha(0), \alpha(T)) \leq 0$.
Similarly, a function $\beta \in AC(J, \mathbb{R})$ is said to be an upper solution of (1.1) – (1.2) if there exists $v_2 \in L^1(J, \mathbb{R})$ such that $v_2(t) \in F(t, \beta(t))$ a.e. on J, $\beta'(t) \geq v_2(t)$ a.e. on J and $L(\beta(0), \beta(T)) \geq 0$.

For the multivalued map F and for each $y \in C(J, \mathbb{R})$ we define $S^1_{F,y}$ by

$$S^1_{F,y} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

Our main result is based on the following:

Lemma 2.1. [18]. *Let I be a compact real interval and X be a Banach space. Let $F : I \times X \to CC(X); (t, y) \to F(t, y)$ measurable with respect to t for any $y \in X$ and u.s.c. with respect to y for almost each $t \in I$ and $S^1_{F,y} \neq \emptyset$ for any $y \in C(I, X)$ and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$, then the operator*

$$\Gamma \circ S^1_F : C(I, X) \to CC(C(I, X)),\ y \longmapsto (\Gamma \circ S^1_F)(y) := \Gamma(S^1_{F,y})$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

Lemma 2.2. [19]. *Let $G : X \to CC(X)$ be an u.s.c. condensing map. If the set*

$$M := \{v \in X : \lambda v \in G(v) \text{ for some } \lambda > 1\}$$

is bounded, then G has a fixed point.

3. MAIN RESULT

We are now in a position to state and prove our existence result for the problem (1.1) – (1.2).

Theorem 3.1. *Suppose $F : J \times \mathbb{R} \to CC(\mathbb{R})$ is an L^1-Carathéodory multivalued map. In addition assume the following conditions*

- (H1) *there exist α and β in $W^{1,1}(J, \mathbb{R})$ lower and upper solutions respectively for the problem (1.1) – (1.2) such that $\alpha \leq \beta$,*
- (H2) *L is a continuous single-valued map in $(x, y) \in [\alpha(0), \beta(0)] \times [\alpha(T), \beta(T)]$ and nonincreasing in $y \in [\alpha(T), \beta(T)]$,*

are satisfied. Then the problem (1.1) – (1.2) has at least one solution $y \in W^{1,1}(J, \mathbb{R})$ such that

$$\alpha(t) \leq y(t) \leq \beta(t) \text{ for all } t \in J.$$

Proof. Transform the problem into a fixed point problem. Consider the following modified problem (see [5])

$$y'(t) + y(t) \in F_1(t, y(t)), \text{ a.e. } t \in J, \tag{3.1}$$

$$y(0) = \tau(0, y(0) - L(\overline{y}(0), \overline{y}(T))) \tag{3.2}$$

where $F_1(t, y) = F(t, \tau(t, y)) + \tau(t, y)$, $\tau(t, y) = \max\{\alpha(t), \min\{y, \beta(t)\}\}$ and $\overline{y}(t) = \tau(t, y(t))$.

Remark 3.2. (i) Notice that F_1 is an L^1-Carathéodory multivalued map with compact convex values and there exists $\phi \in L^1(J, \mathbb{R}_+)$ such that

$$\|F_1(t, y(t))\| \leq \phi(t) + \max(\sup_{t \in J} |\alpha(t)|, \sup_{t \in J} |\beta(t)|) \text{ for a.e. } t \in J \text{ and all } y \in C(J, \mathbb{R}).$$

(ii) By the definition of τ it is clear that $\alpha(0) \leq y(0) \leq \beta(0)$.

Clearly a solution to (3.1) – (3.2) is a fixed point of the operator $N : C(J, \mathbb{R}) \to 2^{C(J,\mathbb{R})}$ defined by

$$N(y) := \left\{ h \in C(J, \mathbb{R}) : h(t) = y(0) + \int_0^t [v(s) + \overline{y}(s) - y(s)] ds, \ v \in \tilde{S}^1_{F,\overline{y}} \right\}$$

where

$$\tilde{S}^1_{F,\overline{y}} = \{v \in S^1_{F,\overline{y}} : v(t) \geq v_1(t) \text{ a.e. on } A_1 \text{ and } v(t) \leq v_2(t) \text{ a.e. on } A_2\},$$

$$S^1_{F,\overline{y}} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, \overline{y}(t)) \text{ for a.e. } t \in J\},$$

$$A_1 = \{t \in J : y(t) < \alpha(t) \leq \beta(t)\}, \ \ A_2 = \{t \in J : \alpha(t) \leq \beta(t) < y(t)\}.$$

Remark 3.3. (i) For each $y \in C(J, \mathbb{R})$ the set $S^1_{F,y}$ is nonempty (see Lasota and Opial [18]).

(ii) For each $y \in C(J, \mathbb{R})$ the set $\tilde{S}^1_{F,\overline{y}}$ is nonempty. Indeed, by (i) there exists $v \in S^1_{F,y}$. Set

$$w = v_1 \chi_{A_1} + v_2 \chi_{A_2} + v \chi_{A_3},$$

where

$$A_3 = \{t \in J : \alpha(t) \leq y(t) \leq \beta(t)\}.$$

Then by decomposability $w \in \tilde{S}^1_{F,\overline{y}}$.

We shall show that N is a completely continuous multivalued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in C(J, \mathbb{R})$.*

Indeed, if $h, \ \overline{h}$ belong to $N(y)$, then there exist $v \in \tilde{S}^1_{F,\overline{y}}$ and $\overline{v} \in \tilde{S}^1_{F,\overline{y}}$ such that

$$h(t) = y(0) + \int_0^t [v(s) + \overline{y}(s) - y(s)] ds, \ \ t \in J$$

and

$$\overline{h}(t) = y(0) + \int_0^t [\overline{v}(s) + \overline{y}(s) - y(s)] ds, \ \ t \in J.$$

Let $0 \leq k \leq 1$. Then for each $t \in J$ we have

$$[kh + (1-k)\overline{h}](t) = y(0) + \int_0^t [kv(s) + (1-k)\overline{v}(s) + \overline{y}(s) - y(s)] ds.$$

Since $\tilde{S}^1_{F,\overline{y}}$ is convex (because F has convex values) then

$$kh + (1-k)\overline{h} \in G(y).$$

Step 2: *N sends bounded sets into bounded sets in $C(J, \mathbb{R})$.*

Let $B_r := \{y \in C(J, \mathbb{R}) : \|y\|_\infty \leq r\}$, $(\|y\|_\infty := sup\{|y(t)| : t \in J\})$ be a bounded set in $C(J, \mathbb{R})$ and $y \in B_r$, then for each $h \in N(y)$ there exists $v \in \tilde{S}^1_{F,\overline{y}}$ such that

$$h(t) = y(0) + \int_0^t [v(s) + \overline{y}(s) - y(s)] ds, \ \ t \in J.$$

Thus for each $t \in J$ we get

$$\begin{aligned} |h(t)| &\leq |y(0)| + \int_0^t [|v(s)| + |\overline{y}(s)| + |y(s)|]ds \\ &\leq \max(\alpha(0), \beta(0)) + \|\phi_r\|_{L^1} + T\max(r, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|) + Tr. \end{aligned}$$

Step 3: *N sends bounded sets in $C(J,\mathbb{R})$ into equicontinuous sets.*
Let $u_1, u_2 \in J,\ u_1 < u_2$, $B_r := \{y \in C(J,\mathbb{R}) : \|y\|_\infty \leq r\}$ be a bounded set in $C(J,\mathbb{R})$ and $y \in B_r$. For each $h \in N(y)$ there exists $v \in \tilde{S}^1_{F,\overline{y}}$ such that

$$h(t) = y(0) + \int_0^t [v(s) + \overline{y}(s) - y(s)]ds, \ \ t \in J.$$

We then have

$$\begin{aligned} |h(u_2) - h(u_1)| &\leq \int_{u_1}^{u_2} [|v(s) + \overline{y}(s)| + |y(s)|]ds \\ &\leq \int_{u_1}^{u_2} |\phi_r(s)|ds + (u_2 - u_1)\max(r, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|) + r(u_2 - u_1). \end{aligned}$$

As a consequence of Step 2, Step 3 together with the Ascoli-Arzela theorem we can conclude that $N : C(J,\mathbb{R}) \rightarrow 2^{C(J,\mathbb{R})}$ is a compact multivalued map, and therefore, a condensing map.

Step 4: *N has a closed graph.*
Let $y_n \rightarrow y_0,\ h_n \in N(y_n)$ and $h_n \rightarrow h_0$. We shall prove that $h_0 \in N(y_0)$.
$h_n \in N(y_n)$ means that there exists $v_n \in \tilde{S}_{F,\overline{y}_n}$ such that

$$h_n(t) = y(0) + \int_0^t [v_n(s) + \overline{y}_n(s) - y_n(s)]ds, \ \ t \in J.$$

We must prove that there exists $v_0 \in \tilde{S}^1_{F,\overline{y}_0}$ such that

$$h_0(t) = y(0) + \int_0^t [v_0(s) + \overline{y}_0(s) - y_0(s)]ds, \ \ t \in J.$$

Consider the linear continuous operator $\Gamma : L^1(J,\mathbb{R}) \rightarrow C(J,\mathbb{R})$ defined by

$$(\Gamma v)(t) = \int_0^t v(s)ds.$$

We have

$$\left\| \left(h_n - y(0) - \int_0^t [\overline{y}_n(s) - y_n(s)]\, ds \right) - \left(h_0 - y(0) + \int_0^t [\overline{y}_0(s) - y_0(s)]\, ds \right) \right\|_\infty \rightarrow 0.$$

From Lemma 2.1, it follows that $\Gamma \circ \tilde{S}^1_F$ is a closed graph operator.
Also from the definition of Γ we have that

$$\left(h_n(t) - y(0) - \int_0^t [\overline{y}_n(s) - y_0(s)]\, ds \right) \in \Gamma\left(\tilde{S}^1_{F,\overline{y}_n} \right).$$

Since $y_n \rightarrow y_0$ it follows from Lemma 2.1 that

$$h_0(t) = y(0) + \int_0^t [v_0(s) + \overline{y}_0(s) - y_0(s)]ds, \ \ t \in J$$

for some $v_0 \in \tilde{S}^1_{F,\overline{y}_0}$.

Next we shall show that N has a fixed point, by proving that

Step 5: *The set*

$$M := \{v \in C(J,\mathbb{R}) : \lambda v \in N(v) \textit{ for some } \lambda > 1\}$$

is bounded.

Let $y \in M$ then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $v \in \tilde{S}^1_{F,\overline{y}}$ such that

$$y(t) = \lambda^{-1}y(0) + \lambda^{-1}\int_0^t [v(s) + \overline{y}(s) - y(s)]ds, \ \ t \in J.$$

Thus

$$|y(t)| \le |y(0)| + \int_0^t |v(s) + \overline{y}(s) - y(s)|ds, \quad t \in J.$$

From the definition of τ there exists $\phi \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,\overline{y}(t))\| = \sup\{|v| : v \in F(t,\overline{y}(t))\} \le \phi(t) \text{ for each } y \in C(J,\mathbb{R}),$$

$$|y(t)| \le \max(\alpha(0),\beta(0)) + \|\phi\|_{L^1} + T\max(\sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|) + \int_0^t |y(s)|ds.$$

Set

$$z_0 = \max(\alpha(0),\beta(0)) + \|\phi\|_{L^1} + T\max(\sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|).$$

Using the Gronwall's Lemma ([11, p. 36]) we get for each $t \in J$

$$\begin{aligned} |y(t)| &\le z_0 + z_0\int_0^t e^{t-s}ds \\ &\le z_0 + z_0(e^t - 1). \end{aligned}$$

Thus

$$\|y\|_\infty \le z_0 + z_0(e^T - 1).$$

This shows that M is bounded.

Hence, Lemma 2.2 applies and N has a fixed point which is a solution to problem (3.1) – (3.2).

Step 6: *We shall show that the solution y of (3.1)-(3.2) satisfies*

$$\alpha(t) \le y(t) \le \beta(t) \ \textit{ for all } \ t \in J.$$

Let y be a solution to (3.1) – (3.2). We prove that

$$\alpha(t) \le y(t) \ \text{ for all } \ t \in J.$$

Suppose not. Then there exist $t_1, t_2 \in J, \ t_1 < t_2$ such that $\alpha(t_1) = y(t_1)$ and

$$\alpha(t) > y(t) \ \text{ for all } \ t \in (t_1,t_2).$$

In view of the definition of τ one has

$$y'(t) + y(t) \in F(t,\alpha(t)) + \alpha(t) \quad \text{a.e. on} \quad (t_1,t_2).$$

Thus there exists $v(t) \in F(t,\alpha(t))$ a.e. on J with $v(t) \ge v_1(t)$ a.e. on (t_1,t_2) such that

$$y'(t) + y(t) = v(t) + \alpha(t) \quad \text{a.e. on} \quad (t_1,t_2).$$

An integration on $(t_1,t]$, with $t \in (t_1,t_2)$ yields

$$\begin{aligned} y(t) - y(t_1) &= \int_{t_1}^t [v(s) + (\alpha - y)(s)]ds \\ &> \int_{t_1}^t v(s)ds. \end{aligned}$$

Since α is a lower solution to (1.1) – (1.2), then

$$\alpha(t) - \alpha(t_1) \leq \int_{t_1}^{t} v_1(s)ds, \ \ t \in (t_1, t_2).$$

It follows from the facts $y(t_1) = \alpha(t_1)$, $v(t) \geq v_1(t)$ that

$$\alpha(t) < y(t) \ \text{ for all } \ t \in (t_1, t_2)$$

which is a contradiction, since $y(t) < \alpha(t)$ for all $t \in (t_1, t_2)$. Consequently

$$\alpha(t) \leq y(t) \ \text{ for all } \ t \in J.$$

Analogously, we can prove that

$$y(t) \leq \beta(t) \ \text{ for all } \ t \in J.$$

This shows that the problem (3.1) – (3.2) has a solution in the interval $[\alpha, \beta]$.

Finally, we prove that every solution of (3.1) – (3.2) is also a solution to (1.1) – (1.2). We only need to show that

$$\alpha(0) \leq y(0) - L(\overline{y}(0), \overline{y}(T)) \leq \beta(0).$$

Notice first that we can prove that

$$\alpha(T) \leq y(T) \leq \beta(T).$$

Suppose now that $y(0) - L(\overline{y}(0), \overline{y}(T)) < \alpha(0)$. Then $y(0) = \alpha(0)$ and

$$y(0) - L(\alpha(0), \overline{y}(T)) < \alpha(0).$$

Since L is nonincreasing in y, we have

$$\alpha(0) \leq \alpha(0) - L(\alpha(0), \alpha(T)) \leq \alpha(0) - L(\alpha(0), \overline{y}(T)) < \alpha(0)$$

which is a contradiction.

Analogously we can prove that

$$y(0) - L(\tau(0), \tau(T)) \leq \beta(0).$$

Then y is a solution to (1.1) – (1.2). □

Remark 3.4. Observe that if $L(x, y) = ax - by - c$, we obtain that Theorem 3.1 gives an existence result for the problem

$$y'(t) \in F(t, y(t)), \ \ \text{for a.e. } \ t \in J = [0, T]$$

$$ay(0) - by(T) = c$$

with $a, \ b \geq 0, \ a + b > 0$ which includes the periodic case $(a = b = 1, \ c = 0)$ and the initial and the terminal problem.

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