

# Journal of Inequalities in Pure and Applied Mathematics

## AN INEQUALITY IMPROVING THE SECOND HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS DEFINED ON LINEAR SPACES AND APPLICATIONS FOR SEMI-INNER PRODUCTS

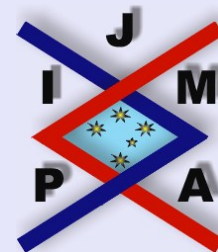
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## Abstract

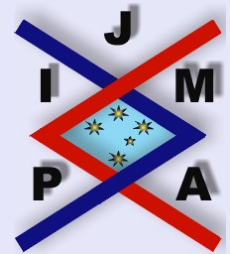
An inequality for convex functions defined on linear spaces is obtained which contains in a particular case a refinement for the second part of the celebrated Hermite-Hadamard inequality. Applications for semi-inner products on normed linear spaces are also provided.

*2000 Mathematics Subject Classification:* Primary 26D15, 26D10; Secondary 46B10.

*Key words:* Hermite-Hadamard integral inequality, Convex functions, Semi-Inner products.

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# 1. Introduction

Let  $X$  be a real linear space,  $a, b \in X$ ,  $a \neq b$  and let  $[a, b] := \{(1 - \lambda)a + \lambda b, \lambda \in [0, 1]\}$  be the *segment* generated by  $a$  and  $b$ . We consider the function  $f : [a, b] \rightarrow \mathbb{R}$  and the attached function  $g(a, b) : [0, 1] \rightarrow \mathbb{R}$ ,  $g(a, b)(t) := f[(1 - t)a + tb]$ ,  $t \in [0, 1]$ .

It is well known that  $f$  is convex on  $[a, b]$  iff  $g(a, b)$  is convex on  $[0, 1]$ , and the following lateral derivatives exist and satisfy

- (i)  $g'_\pm(a, b)(s) = (\nabla_\pm f[(1 - s)a + sb])(b - a)$ ,  $s \in (0, 1)$
- (ii)  $g'_+(a, b)(0) = (\nabla_+ f(a))(b - a)$
- (iii)  $g'_-(a, b)(1) = (\nabla_- f(b))(b - a)$

where  $(\nabla_\pm f(x))(y)$  are the *Gâteaux lateral derivatives*, we recall that

$$\begin{aligned}
 (\nabla_+ f(x))(y) &: = \lim_{h \rightarrow 0^+} \left[ \frac{f(x + hy) - f(x)}{h} \right], \\
 (\nabla_- f(x))(y) &: = \lim_{k \rightarrow 0^-} \left[ \frac{f(x + ky) - f(x)}{k} \right], \quad x, y \in X.
 \end{aligned}$$

The following inequality is the well known Hermite-Hadamard integral inequality for convex functions defined on a segment  $[a, b] \subset X$ :

$$\text{(HH)} \quad f\left(\frac{a+b}{2}\right) \leq \int_0^1 f[(1-t)a + tb] dt \leq \frac{f(a) + f(b)}{2},$$



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which easily follows by the classical Hermite-Hadamard inequality for the convex function  $g(a, b) : [0, 1] \rightarrow \mathbb{R}$

$$g(a, b) \left( \frac{1}{2} \right) \leq \int_0^1 g(a, b)(t) dt \leq \frac{g(a, b)(0) + g(a, b)(1)}{2}.$$

For other related results see the monograph on line [1].

Now, assume that  $(X, \|\cdot\|)$  is a normed linear space. The function  $f_0(s) = \frac{1}{2} \|x\|^2$ ,  $x \in X$  is convex and thus the following limits exist

$$(iv) \langle x, y \rangle_s := (\nabla_+ f_0(y))(x) = \lim_{t \rightarrow 0^+} \left[ \frac{\|y+tx\|^2 - \|y\|^2}{2t} \right];$$

$$(v) \langle x, y \rangle_i := (\nabla_- f_0(y))(x) = \lim_{s \rightarrow 0^-} \left[ \frac{\|y+sx\|^2 - \|y\|^2}{2s} \right];$$

for any  $x, y \in X$ . They are called the *lower* and *upper semi-inner* products associated to the norm  $\|\cdot\|$ .

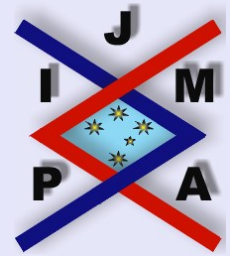
For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [2]), assuming that  $p, q \in \{s, i\}$  and  $p \neq q$ :

$$(a) \langle x, x \rangle_p = \|x\|^2 \text{ for all } x \in X;$$

$$(aa) \langle \alpha x, \beta y \rangle_p = \alpha\beta \langle x, y \rangle_p \text{ if } \alpha, \beta \geq 0 \text{ and } x, y \in X;$$

$$(aaa) \left| \langle x, y \rangle_p \right| \leq \|x\| \|y\| \text{ for all } x, y \in X;$$

$$(av) \langle \alpha x + y, x \rangle_p = \alpha \langle x, x \rangle_p + \langle y, x \rangle_p \text{ if } x, y \in X \text{ and } \alpha \in \mathbb{R};$$



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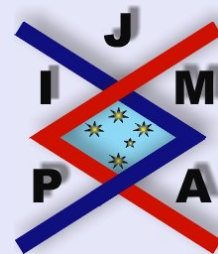


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(v)  $\langle -x, y \rangle_p = -\langle x, y \rangle_q$  for all  $x, y \in X$ ;

(va)  $\langle x + y, z \rangle_p \leq \|x\| \|z\| + \langle y, z \rangle_p$  for all  $x, y, z \in X$ ;

(vaa) The mapping  $\langle \cdot, \cdot \rangle_p$  is continuous and subadditive (superadditive) in the first variable for  $p = s$  (or  $p = i$ );

(vaaa) The normed linear space  $(X, \|\cdot\|)$  is smooth at the point  $x_0 \in X \setminus \{0\}$  if and only if  $\langle y, x_0 \rangle_s = \langle y, x_0 \rangle_i$  for all  $y \in X$ ; in general  $\langle y, x \rangle_i \leq \langle y, x \rangle_s$  for all  $x, y \in X$ ;

(ax) If the norm  $\|\cdot\|$  is induced by an inner product  $\langle \cdot, \cdot \rangle$ , then  $\langle y, x \rangle_i = \langle y, x \rangle = \langle y, x \rangle_s$  for all  $x, y \in X$ .

Applying inequality (HH) for the convex function  $f_0(x) = \frac{1}{2} \|x\|^2$ , one may deduce the inequality

$$(1.1) \quad \left\| \frac{x+y}{2} \right\|^2 \leq \int_0^1 \|(1-t)x + ty\|^2 dt \leq \frac{\|x\|^2 + \|y\|^2}{2}$$

for any  $x, y \in X$ . The same (HH) inequality applied for  $f_1(x) = \|x\|$ , will give the following refinement of the triangle inequality:

$$(1.2) \quad \left\| \frac{x+y}{2} \right\| \leq \int_0^1 \|(1-t)x + ty\| dt \leq \frac{\|x\| + \|y\|}{2}, \quad x, y \in X.$$

In this paper we point out an integral inequality for convex functions which is related to the first Hermite-Hadamard inequality in (HH) and investigate its applications for semi-inner products in normed linear spaces.

## 2. The Results

We start with the following lemma which is also of interest in itself.

**Lemma 2.1.** *Let  $h : [\alpha, \beta] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a convex function on  $[\alpha, \beta]$ . Then for any  $\gamma \in [\alpha, \beta]$  one has the inequality*

$$(2.1) \quad \frac{1}{2} [(\beta - \gamma)^2 h'_+(\gamma) - (\gamma - \alpha)^2 h'_-(\gamma)] \\ \leq (\gamma - \alpha) h(\alpha) + (\beta - \gamma) h(\beta) - \int_{\alpha}^{\beta} h(t) dt \\ \leq \frac{1}{2} [(\beta - \gamma)^2 h'_-(\beta) - (\gamma - \alpha)^2 h'_+(\alpha)].$$

*The constant  $\frac{1}{2}$  is sharp in both inequalities.*

*The second inequality also holds for  $\gamma = \alpha$  or  $\gamma = \beta$ .*

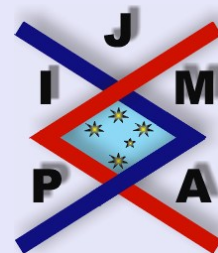
*Proof.* It is easy to see that for any locally absolutely continuous function  $h : (\alpha, \beta) \rightarrow \mathbb{R}$ , we have the identity

$$(2.2) \quad \int_{\alpha}^{\beta} (t - \gamma) h'(t) dt = (\gamma - \alpha) h(\alpha) + (\beta - \gamma) h(\beta) - \int_{\alpha}^{\beta} h(t) dt$$

for any  $\gamma \in (\alpha, \beta)$ , where  $h'$  is the derivative of  $h$  which exists a.e. on  $(\alpha, \beta)$ .

Since  $h$  is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any  $\gamma \in (\alpha, \beta)$ , we have the inequalities

$$(2.3) \quad h'(t) \leq h'_-(\gamma) \text{ for a.e. } t \in [\alpha, \gamma]$$



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and

$$(2.4) \quad h'(t) \geq h'_+(\gamma) \text{ for a.e. } t \in [\gamma, \beta].$$

If we multiply (2.3) by  $\gamma - t \geq 0$ ,  $t \in [\alpha, \gamma]$  and integrate on  $[\alpha, \gamma]$ , we get

$$(2.5) \quad \int_{\alpha}^{\gamma} (\gamma - t) h'(t) dt \leq \frac{1}{2} (\gamma - \alpha)^2 h'_-(\gamma)$$

and if we multiply (2.4) by  $t - \gamma \geq 0$ ,  $t \in [\gamma, \beta]$ , and integrate on  $[\gamma, \beta]$ , we also have

$$(2.6) \quad \int_{\gamma}^{\beta} (t - \gamma) h'(t) dt \geq \frac{1}{2} (\beta - \gamma)^2 h'_+(\gamma).$$

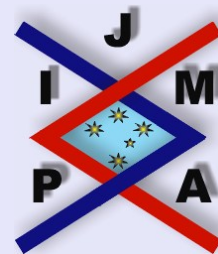
Now, if we subtract (2.5) from (2.6) and use the representation (2.2), we deduce the first inequality in (2.1).

If we assume that the first inequality (2.1) holds with a constant  $C > 0$  instead of  $\frac{1}{2}$ , i.e.,

$$(2.7) \quad C [(\beta - \gamma)^2 h'_+(\gamma) - (\gamma - \alpha)^2 h'_-(\gamma)] \\ \leq (\gamma - \alpha) h(\alpha) + (\beta - \gamma) h(\beta) - \int_{\alpha}^{\beta} h(t) dt$$

and take the convex function  $h_0(t) := k \left| t - \frac{\alpha + \beta}{2} \right|$ ,  $k > 0$ ,  $t \in [\alpha, \beta]$ , then

$$h'_{0+} \left( \frac{\alpha + \beta}{2} \right) = k, \\ h'_{0-} \left( \frac{\alpha + \beta}{2} \right) = -k,$$



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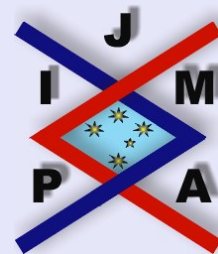


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$$h_0(\alpha) = \frac{k(\beta - \alpha)}{2} = h_0(\beta),$$

$$\int_{\alpha}^{\beta} h_0(t) dt = \frac{1}{4}k(\beta - \alpha)^2,$$

and the inequality (2.7) becomes, for  $\gamma = \frac{\alpha + \beta}{2}$ ,

$$C \left[ \frac{1}{4}(\beta - \alpha)^2 k + \frac{1}{4}(\beta - \alpha)^2 k \right] \leq \frac{1}{4}k(\beta - \alpha)^2,$$

giving  $C \leq \frac{1}{2}$ , which proves the sharpness of the constant  $\frac{1}{2}$  in the first inequality in (2.1).

If either  $h'_+(\alpha) = -\infty$  or  $h'_-(\beta) = -\infty$ , then the second inequality in (2.1) holds true.

Assume that  $h'_+(\alpha)$  and  $h'_-(\beta)$  are finite. Since  $h$  is convex on  $[\alpha, \beta]$ , we have

$$(2.8) \quad h'(t) \geq h'_+(\alpha) \text{ for a.e. } t \in [\alpha, \gamma] \quad (\gamma \text{ may be equal to } \beta)$$

and

$$(2.9) \quad h'(t) \leq h'_-(\beta) \text{ for a.e. } t \in [\gamma, \beta] \quad (\gamma \text{ may be equal to } \alpha).$$

If we multiply (2.8) by  $\gamma - t \geq 0$ ,  $t \in [\alpha, \gamma]$  and integrate on  $[\alpha, \gamma]$ , then we deduce

$$(2.10) \quad \int_{\alpha}^{\gamma} (\gamma - t) h'(t) dt \geq \frac{1}{2}(\gamma - \alpha)^2 h'_+(\alpha)$$



and if we multiply (2.9) by  $t - \gamma \geq 0$ ,  $t \in [\gamma, \beta]$ , and integrate on  $[\gamma, \beta]$ , then we also have

$$(2.11) \quad \int_{\gamma}^{\beta} (t - \gamma) h'(t) dt \leq \frac{1}{2} (\beta - \gamma)^2 h'_-(\beta).$$

Finally, if we subtract (2.10) from (2.11) and use the representation (2.2), we deduce the second part of (2.1).

Now, assume that the second inequality in (2.1) holds with a constant  $D > 0$  instead of  $\frac{1}{2}$ , i.e.,

$$(2.12) \quad (\gamma - \alpha) f(\alpha) + (\beta - \gamma) f(\beta) - \int_{\alpha}^{\beta} f(t) dt \geq D [(\beta - \gamma)^2 f'_-(\beta) - (\gamma - \alpha)^2 f'_+(\alpha)].$$

If we consider the convex function  $h_0$  given above, then we have  $h'_-(\beta) = k$ ,  $h'_+(\alpha) = -k$  and by (2.12) applied for  $h_0$  and  $x = \frac{\alpha + \beta}{2}$  we get

$$\frac{1}{4} k (\beta - \alpha)^2 \leq D \left[ \frac{1}{4} k (\beta - \alpha)^2 + \frac{1}{4} k (\beta - \alpha)^2 \right],$$

giving  $D \geq \frac{1}{2}$ , and the sharpness of the constant  $\frac{1}{2}$  is proved.  $\square$

**Corollary 2.2.** *With the assumptions of Lemma 2.1 and if  $\gamma \in (\alpha, \beta)$  is a point of differentiability for  $h$ , then*

$$(2.13) \quad \left( \frac{\alpha + \beta}{2} - \gamma \right) h'(\gamma)$$



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$$\leq \left( \frac{\gamma - \alpha}{\beta - \alpha} \right) h(\alpha) + \left( \frac{\beta - \gamma}{\beta - \alpha} \right) h(\beta) - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t) dt.$$

Now, recall that the following inequality, which is well known in the literature as the Hermite-Hadamard inequality for convex functions, holds

$$(2.14) \quad h\left(\frac{\alpha + \beta}{2}\right) \leq \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t) dt \leq \frac{h(\alpha) + h(\beta)}{2}.$$

The following corollary provides some bounds for the difference

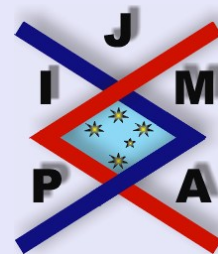
$$\frac{h(\alpha) + h(\beta)}{2} - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t) dt.$$

**Corollary 2.3.** *Let  $h : [\alpha, \beta] \rightarrow \mathbb{R}$  be a convex function on  $[\alpha, \beta]$ . Then we have the inequality*

$$(2.15) \quad \begin{aligned} 0 &\leq \frac{1}{8} \left[ h'_+ \left( \frac{\alpha + \beta}{2} \right) - h'_- \left( \frac{\alpha + \beta}{2} \right) \right] (\beta - \alpha) \\ &\leq \frac{h(\alpha) + h(\beta)}{2} - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t) dt \\ &\leq \frac{1}{8} [h'_-(\beta) - h'_+(\alpha)] (\beta - \alpha). \end{aligned}$$

The constant  $\frac{1}{8}$  is sharp in both inequalities.

We are now able to state the corresponding result for convex functions defined on linear spaces.



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**Theorem 2.4.** Let  $X$  be a linear space,  $a, b \in X$ ,  $a \neq b$  and  $f : [a, b] \subset X \rightarrow \mathbb{R}$  be a convex function on the segment  $[a, b]$ . Then for any  $s \in (0, 1)$  one has the inequality

$$\begin{aligned}
 (2.16) \quad & \frac{1}{2} \left[ (1-s)^2 (\nabla_+ f [(1-s)a + sb]) (b-a) \right. \\
 & \quad \left. - s^2 (\nabla_- f [(1-s)a + sb]) (b-a) \right] \\
 & \leq (1-s) f(a) + s f(b) - \int_0^1 f[(1-t)a + tb] dt \\
 & \leq \frac{1}{2} \left[ (1-s)^2 (\nabla_- f(b)) (b-a) - s^2 (\nabla_+ f(a)) (b-a) \right].
 \end{aligned}$$

The constant  $\frac{1}{2}$  is sharp in both inequalities.

The second inequality also holds for  $s = 0$  or  $s = 1$ .

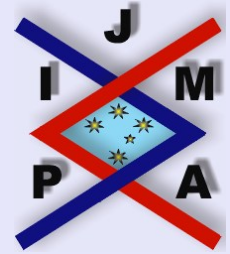
*Proof.* Follows by Lemma 2.1 applied for the convex function

$$h(t) = g(a, b)(t) = f[(1-t)a + tb], \quad t \in [0, 1],$$

and for the choices  $\alpha = 0$ ,  $\beta = 1$ , and  $\gamma = 1$ . □

**Corollary 2.5.** If  $f : [a, b] \rightarrow \mathbb{R}$  is as in Theorem 2.4 and Gâteaux differentiable in  $c := (1-\lambda)a + \lambda b$ ,  $\lambda \in (0, 1)$  along the direction  $(b-a)$ , then we have the inequality:

$$\begin{aligned}
 (2.17) \quad & \left( \frac{1}{2} - \lambda \right) (\nabla f(c)) (b-a) \\
 & \leq (1-\lambda) f(a) + \lambda f(b) - \int_0^1 f[(1-t)a + tb] dt.
 \end{aligned}$$



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The following result related to the second Hermite-Hadamard inequality for functions defined on linear spaces also holds.

**Corollary 2.6.** *If  $f$  is as in Theorem 2.4, then*

$$\begin{aligned}
 (2.18) \quad \frac{1}{8} \left[ \nabla_+ f \left( \frac{a+b}{2} \right) (b-a) - \nabla_- f \left( \frac{a+b}{2} \right) (b-a) \right] \\
 \leq \frac{f(a) + f(b)}{2} - \int_0^1 f[(1-t)a + tb] dt \\
 \leq \frac{1}{8} [(\nabla_- f(b))(b-a) - (\nabla_+ f(a))(b-a)].
 \end{aligned}$$

The constant  $\frac{1}{8}$  is sharp in both inequalities.

Now, let  $\Omega \subset \mathbb{R}^n$  be an open convex set in  $\mathbb{R}^n$ .

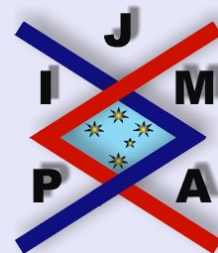
If  $F : \Omega \rightarrow \mathbb{R}$  is a differentiable convex function on  $\Omega$ , then, obviously, for any  $\bar{c} \in \Omega$  we have

$$\nabla F(\bar{c})(\bar{y}) = \sum_{i=1}^n \frac{\partial F(\bar{c})}{\partial x_i} \cdot y_i, \quad \bar{y} \in \mathbb{R}^n,$$

where  $\frac{\partial F}{\partial x_i}$  are the partial derivatives of  $F$  with respect to the variable  $x_i$  ( $i = 1, \dots, n$ ).

Using (2.16), we may state that

$$(2.19) \quad \left( \frac{1}{2} - \lambda \right) \sum_{i=1}^n \frac{\partial F(\lambda \bar{a} + (1-\lambda)\bar{b})}{\partial x_i} \cdot (b_i - a_i)$$



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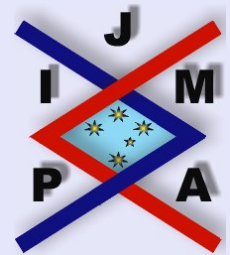
$$\begin{aligned} &\leq (1 - \lambda) F(\bar{a}) + \lambda F(\bar{b}) - \int_0^1 F[(1 - t)\bar{a} + t\bar{b}] dt \\ &\leq \frac{1}{2} \left[ (1 - \lambda)^2 \sum_{i=1}^n \frac{\partial F(\bar{b})}{\partial x_i} \cdot (b_i - a_i) - \lambda^2 \sum_{i=1}^n \frac{\partial F(\bar{a})}{\partial x_i} \cdot (b_i - a_i) \right] \end{aligned}$$

for any  $\bar{a}, \bar{b} \in \Omega$  and  $\lambda \in (0, 1)$ .

In particular, for  $\lambda = \frac{1}{2}$ , we get

$$\begin{aligned} (2.20) \quad 0 &\leq \frac{F(\bar{a}) + F(\bar{b})}{2} - \int_0^1 F[(1 - t)\bar{a} + t\bar{b}] dt \\ &\leq \frac{1}{8} \sum_{i=1}^n \left( \frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i} \right) \cdot (b_i - a_i). \end{aligned}$$

In (2.20) the constant  $\frac{1}{8}$  is sharp.



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### 3. Applications for Semi-Inner Products

Let  $(X, \|\cdot\|)$  be a real normed linear space. We may state the following results for the semi-inner products  $\langle \cdot, \cdot \rangle_i$  and  $\langle \cdot, \cdot \rangle_s$ .

**Proposition 3.1.** *For any  $x, y \in X$  and  $\sigma \in (0, 1)$  we have the inequalities:*

$$(3.1) \quad \begin{aligned} (1 - \sigma)^2 \langle y - x, (1 - \sigma)x + \sigma y \rangle_s - \sigma^2 \langle y - x, (1 - \sigma)x + \sigma y \rangle_i \\ \leq (1 - \sigma) \|x\|^2 + \sigma \|y\|^2 - \int_0^1 \|(1 - t)x + ty\|^2 dt \\ \leq (1 - \sigma)^2 \langle y - x, y \rangle_i - \sigma^2 \langle y - x, y \rangle_s. \end{aligned}$$

The second inequality in (3.1) also holds for  $\sigma = 0$  or  $\sigma = 1$ .

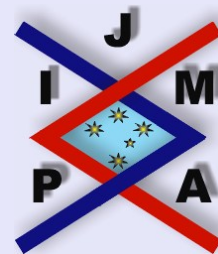
The proof is obvious by Theorem 2.4 applied for the convex function  $f(x) = \frac{1}{2} \|x\|^2$ ,  $x \in X$ .

If the space is *smooth*, then we may put  $[x, y] = \langle x, y \rangle_i = \langle x, y \rangle_s$  for each  $x, y \in X$  and the first inequality in (3.1) becomes

$$(3.2) \quad \begin{aligned} (1 - 2\sigma) [y - x, (1 - \sigma)x + \sigma y] \\ \leq (1 - \sigma) \|x\|^2 + \sigma \|y\|^2 - \int_0^1 \|(1 - t)x + ty\|^2 dt. \end{aligned}$$

An interesting particular case one can get from (3.1) is the one for  $\sigma = \frac{1}{2}$ ,

$$(3.3) \quad \begin{aligned} 0 \leq \frac{1}{8} [\langle y - x, y + x \rangle_s - \langle y - x, y + x \rangle_i] \\ \leq \frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1 - t)x + ty\|^2 dt \end{aligned}$$



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$$\leq \frac{1}{4} [\langle y - x, y \rangle_i - \langle y - x, x \rangle_s].$$

The inequality (3.3) provides a refinement and a counterpart for the second inequality in (1.1).

If we consider now two linearly independent vectors  $x, y \in X$  and apply Theorem 2.4 for  $f(x) = \|x\|$ ,  $x \in X$ , then we get

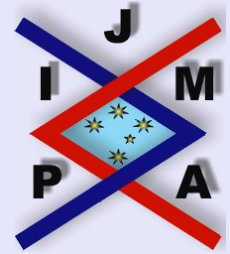
**Proposition 3.2.** *For any linearly independent vectors  $x, y \in X$  and  $\sigma \in (0, 1)$ , one has the inequalities:*

$$(3.4) \quad \frac{1}{2} \left[ (1 - \sigma)^2 \frac{\langle y - x, (1 - \sigma)x + \sigma y \rangle_s}{\|(1 - \sigma)x + \sigma y\|} - \sigma^2 \frac{\langle y - x, (1 - \sigma)x + \sigma y \rangle_i}{\|(1 - \sigma)x + \sigma y\|} \right] \\ \leq (1 - \sigma) \|x\| + \sigma \|y\| - \int_0^1 \|(1 - t)x + ty\| dt \\ \leq \frac{1}{2} \left[ (1 - \sigma)^2 \frac{\langle y - x, y \rangle_i}{\|y\|} - \sigma^2 \frac{\langle y - x, x \rangle_s}{\|x\|} \right].$$

The second inequality also holds for  $\sigma = 0$  or  $\sigma = 1$ .

We note that if the space is smooth, then we have

$$(3.5) \quad \left( \frac{1}{2} - \sigma \right) \cdot \frac{[y - x, (1 - \sigma)x + \sigma y]}{\|(1 - \sigma)x + \sigma y\|} \\ \leq (1 - \sigma) \|x\| + \sigma \|y\| - \int_0^1 \|(1 - t)x + ty\| dt$$



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and for  $\sigma = \frac{1}{2}$ , (3.4) will give the simple inequality

$$\begin{aligned}
 (3.6) \quad 0 &\leq \frac{1}{8} \left[ \left\langle y - x, \frac{\frac{x+y}{2}}{\left\| \frac{x+y}{2} \right\|} \right\rangle_s - \left\langle y - x, \frac{\frac{x+y}{2}}{\left\| \frac{x+y}{2} \right\|} \right\rangle_i \right] \\
 &\leq \frac{\|x\| + \|y\|}{2} - \int_0^1 \|(1-t)x + ty\| dt \\
 &\leq \frac{1}{8} \left[ \left\langle y - x, \frac{y}{\|y\|} \right\rangle_i - \left\langle y - x, \frac{x}{\|x\|} \right\rangle_s \right].
 \end{aligned}$$

The inequality (3.6) provides a refinement and a counterpart of the second inequality in (1.2).

Moreover, if we assume that  $(H, \langle \cdot, \cdot \rangle)$  is an inner product space, then by (3.6) we get for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$  that

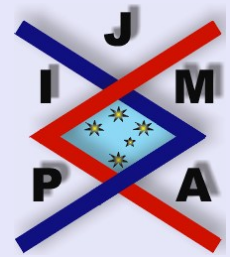
$$(3.7) \quad 0 \leq 1 - \int_0^1 \|(1-t)x + ty\| dt \leq \frac{1}{8} \|y - x\|^2.$$

The constant  $\frac{1}{8}$  is sharp.

Indeed, if we choose  $H = \mathbb{R}$ ,  $\langle a, b \rangle = a \cdot b$ ,  $x = -1$ ,  $y = 1$ , then we get equality in (3.7).

We give now some examples.

1. Let  $\ell^2(\mathbb{K})$ ,  $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ; be the Hilbert space of sequences  $x = (x_i)_{i \in \mathbb{N}}$  with



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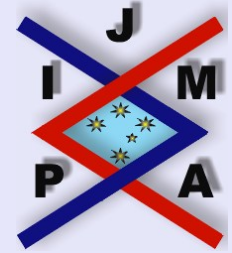
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$\sum_{i=0}^{\infty} |x_i|^2 < \infty$ . Then, by (3.7), we have the inequalities

$$(3.8) \quad 0 \leq 1 - \int_0^1 \left( \sum_{i=0}^{\infty} |(1-t)x_i + ty_i|^2 \right)^{\frac{1}{2}} dt \\ \leq \frac{1}{8} \cdot \sum_{i=0}^{\infty} |y_i - x_i|^2,$$

for any  $x, y \in \ell^2(\mathbb{K})$  provided  $\sum_{i=0}^{\infty} |x_i|^2 = \sum_{i=0}^{\infty} |y_i|^2 = 1$ .

2. Let  $\mu$  be a positive measure,  $L_2(\Omega)$  the Hilbert space of  $\mu$ -measurable functions on  $\Omega$  with complex values that are 2-integrable on  $\Omega$ , i.e.,  $f \in L_2(\Omega)$  iff  $\int_{\Omega} |f(t)|^2 d\mu(t) < \infty$ . Then, by (3.7), we have the inequalities

$$(3.9) \quad 0 \leq 1 - \int_0^1 \left( \int_{\Omega} |(1-\lambda)f(t) + \lambda g(t)|^2 d\mu(t) \right)^{\frac{1}{2}} d\lambda \\ \leq \frac{1}{8} \cdot \int_{\Omega} |f(t) - g(t)|^2 d\mu(t)$$

for any  $f, g \in L_2(\Omega)$  provided  $\int_{\Omega} |f(t)|^2 d\mu(t) = \int_{\Omega} |g(t)|^2 d\mu(t) = 1$ .

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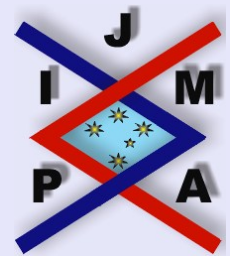
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