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AN INEQUALITY IMPROVING THE SECOND HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS DEFINED ON LINEAR SPACES AND APPLICATIONS FOR SEMI-INNER PRODUCTS

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Abstract

An inequality for convex functions defined on linear spaces is obtained which contains in a particular case a refinement for the second part of the celebrated Hermite-Hadamard inequality. Applications for semi-inner products on normed linear spaces are also provided.

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1. Introduction

Let X be a real linear space, $a, b \in X$, $a \neq b$ and let $[a,b] := \{(1-\lambda)\,a + \lambda b,\ \lambda \in [0,1]\}$ be the *segment* generated by a and b. We consider the function $f : [a,b] \to \mathbb{R}$ and the attached function $g(a,b) : [0,1] \to \mathbb{R}$, $g(a,b)(t) := f[(1-t)\,a + tb]$, $t \in [0,1]$.

It is well known that f is convex on $[a,b]$ iff $g(a,b)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}(a,b)(s) = (\nabla_{\pm} f[(1-s)\,a + sb])(b-a)$, $s \in (0,1)$

(ii) $g'_{+}(a,b)(0) = (\nabla_{+} f(a))(b-a)$

(iii) $g'_{-}(a,b)(1) = (\nabla_{-} f(b))(b-a)$

where $(\nabla_{\pm} f(x))(y)$ are the *Gâteaux lateral derivatives*, we recall that

$$\begin{aligned}(\nabla_{+} f(x))(y) &:= \lim_{h\to 0+}\left[\frac{f(x+hy)-f(x)}{h}\right],\\ (\nabla_{-} f(x))(y) &:= \lim_{k\to 0-}\left[\frac{f(x+ky)-f(x)}{k}\right],\quad x,y\in X.\end{aligned}$$

The following inequality is the well known Hermite-Hadamard integral inequality for convex functions defined on a segment $[a,b] \subset X$:

$$\text{(HH)}\qquad f\left(\frac{a+b}{2}\right) \le \int_0^1 f[(1-t)\,a + tb]\,dt \le \frac{f(a)+f(b)}{2},$$

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which easily follows by the classical Hermite-Hadamard inequality for the convex function $g(a,b):[0,1]\to\mathbb{R}$

$$g(a,b)\left(\frac{1}{2}\right)\leq\int_0^1 g(a,b)(t)\,dt\leq\frac{g(a,b)(0)+g(a,b)(1)}{2}.$$

For other related results see the monograph on line [1].

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s:=(\triangledown_+ f_0(y))(x)=\lim_{t\to 0+}\left[\frac{\|y+tx\|^2-\|y\|^2}{2t}\right]$;

(v) $\langle x,y\rangle_i:=(\triangledown_- f_0(y))(x)=\lim_{s\to 0-}\left[\frac{\|y+sx\|^2-\|y\|^2}{2s}\right]$;

for any $x,y\in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [2]), assuming that $p,q\in\{s,i\}$ and $p\neq q$:

(a) $\langle x,x\rangle_p=\|x\|^2$ for all $x\in X$;

(aa) $\langle \alpha x,\beta y\rangle_p=\alpha\beta\langle x,y\rangle_p$ if $\alpha,\beta\geq 0$ and $x,y\in X$;

(aaa) $\left|\langle x,y\rangle_p\right|\leq\|x\|\,\|y\|$ for all $x,y\in X$;

(av) $\langle \alpha x+y,x\rangle_p=\alpha\langle x,x\rangle_p+\langle y,x\rangle_p$ if $x,y\in X$ and $\alpha\in\mathbb{R}$;

(v) $\langle -x,y\rangle_p = -\langle x,y\rangle_q$ for all $x,y\in X$;

(va) $\langle x+y,z\rangle_p \leq \|x\|\,\|z\| + \langle y,z\rangle_p$ for all $x,y,z\in X$;

(vaa) The mapping $\langle \cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p=s$ (or $p=i$);

(vaaa) The normed linear space $(X,\|\cdot\|)$ is smooth at the point $x_0\in X\backslash\{0\}$ if and only if $\langle y,x_0\rangle_s = \langle y,x_0\rangle_i$ for all $y\in X$; in general $\langle y,x\rangle_i \leq \langle y,x\rangle_s$ for all $x,y\in X$;

(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y,x\rangle_i = \langle y,x\rangle = \langle y,x\rangle_s$ for all $x,y\in X$.

Applying inequality (HH) for the convex function $f_0(x) = \frac{1}{2}\|x\|^2$, one may deduce the inequality

$$\left\|\frac{x+y}{2}\right\|^2 \leq \int_0^1 \|(1-t)x+ty\|^2\,dt \leq \frac{\|x\|^2+\|y\|^2}{2} \tag{1.1}$$

for any $x,y\in X$. The same (HH) inequality applied for $f_1(x)=\|x\|$, will give the following refinement of the triangle inequality:

$$\left\|\frac{x+y}{2}\right\| \leq \int_0^1 \|(1-t)x+ty\|\,dt \leq \frac{\|x\|+\|y\|}{2},\quad x,y\in X. \tag{1.2}$$

In this paper we point out an integral inequality for convex functions which is related to the first Hermite-Hadamard inequality in (HH) and investigate its applications for semi-inner products in normed linear spaces.

2. The Results

We start with the following lemma which is also of interest in itself.

Lemma 2.1. *Let* $h : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ *be a convex function on* $[\alpha, \beta]$*. Then for any* $\gamma \in [\alpha, \beta]$ *one has the inequality*

$$\frac{1}{2}\left[(\beta-\gamma)^2 h'_+(\gamma) - (\gamma-\alpha)^2 h'_-(\gamma)\right] \le (\gamma-\alpha)h(\alpha) + (\beta-\gamma)h(\beta) - \int_\alpha^\beta h(t)\,dt \le \frac{1}{2}\left[(\beta-\gamma)^2 h'_-(\beta) - (\gamma-\alpha)^2 h'_+(\alpha)\right]. \tag{2.1}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities.*
The second inequality also holds for $\gamma = \alpha$ *or* $\gamma = \beta$*.*

Proof. It is easy to see that for any locally absolutely continuous function $h : (\alpha, \beta) \to \mathbb{R}$, we have the identity

$$\int_\alpha^\beta (t-\gamma)\,h'(t)\,dt = (\gamma-\alpha)h(\alpha) + (\beta-\gamma)h(\beta) - \int_\alpha^\beta h(t)\,dt \tag{2.2}$$

for any $\gamma \in (\alpha, \beta)$, where h' is the derivative of h which exists a.e. on (α, β).

Since h is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any $\gamma \in (\alpha, \beta)$, we have the inequalities

$$h'(t) \le h'_-(\gamma) \text{ for a.e. } t \in [\alpha, \gamma] \tag{2.3}$$

and

$$h'(t) \geq h'_+(\gamma) \text{ for a.e. } t \in [\gamma, \beta]. \tag{2.4}$$

If we multiply (2.3) by $\gamma - t \geq 0$, $t \in [\alpha, \gamma]$ and integrate on $[\alpha, \gamma]$, we get

$$\int_\alpha^\gamma (\gamma - t) h'(t)\, dt \leq \frac{1}{2} (\gamma - \alpha)^2 h'_-(\gamma) \tag{2.5}$$

and if we multiply (2.4) by $t - \gamma \geq 0$, $t \in [\gamma, \beta]$, and integrate on $[\gamma, \beta]$, we also have

$$\int_\gamma^\beta (t - \gamma) h'(t)\, dt \geq \frac{1}{2} (\beta - \gamma)^2 h'_+(\gamma). \tag{2.6}$$

Now, if we subtract (2.5) from (2.6) and use the representation (2.2), we deduce the first inequality in (2.1).

If we assume that the first inequality (2.1) holds with a constant $C > 0$ instead of $\frac{1}{2}$, i.e.,

$$\begin{aligned} C\left[(\beta - \gamma)^2 h'_+(\gamma) - (\gamma - \alpha)^2 h'_-(\gamma)\right] \\ \leq (\gamma - \alpha) h(\alpha) + (\beta - \gamma) h(\beta) - \int_\alpha^\beta h(t)\, dt \end{aligned} \tag{2.7}$$

and take the convex function $h_0(t) := k\left|t - \frac{\alpha+\beta}{2}\right|$, $k > 0$, $t \in [\alpha, \beta]$, then

$$\begin{aligned} h'_{0+}\left(\frac{\alpha + \beta}{2}\right) &= k, \\ h'_{0-}\left(\frac{\alpha + \beta}{2}\right) &= -k, \end{aligned}$$

$$
\begin{aligned}
h_0(\alpha) &= \frac{k(\beta-\alpha)}{2} = h_0(\beta), \\
\int_{\alpha}^{\beta} h_0(t)\,dt &= \frac{1}{4}k(\beta-\alpha)^2,
\end{aligned}
$$

and the inequality (2.7) becomes, for $\gamma = \frac{\alpha+\beta}{2}$,

$$
C\left[\frac{1}{4}(\beta-\alpha)^2 k + \frac{1}{4}(\beta-\alpha)^2 k\right] \le \frac{1}{4}k(\beta-\alpha)^2,
$$

giving $C \le \frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$ in the first inequality in (2.1).

If either $h'_+(\alpha) = -\infty$ or $h'_-(\beta) = -\infty$, then the second inequality in (2.1) holds true.

Assume that $h'_+(\alpha)$ and $h'_-(\beta)$ are finite. Since h is convex on $[\alpha,\beta]$, we have

$$
h'(t) \ge h'_+(\alpha) \text{ for a.e. } t \in [\alpha,\gamma] \quad (\gamma \text{ may be equal to } \beta) \tag{2.8}
$$

and

$$
h'(t) \le h'_-(\beta) \text{ for a.e. } t \in [\gamma,\beta] \quad (\gamma \text{ may be equal to } \alpha). \tag{2.9}
$$

If we multiply (2.8) by $\gamma - t \ge 0$, $t \in [\alpha,\gamma]$ and integrate on $[\alpha,\gamma]$, then we deduce

$$
\int_{\alpha}^{\gamma} (\gamma - t)\,h'(t)\,dt \ge \frac{1}{2}(\gamma-\alpha)^2 h'_+(\alpha) \tag{2.10}
$$

and if we multiply (2.9) by $t-\gamma \geq 0$, $t \in [\gamma, \beta]$, and integrate on $[\gamma, \beta]$, then we also have

$$\int_{\gamma}^{\beta} (t-\gamma)\, h'(t)\, dt \leq \frac{1}{2} (\beta-\gamma)^2\, h'_{-}(\beta)\,. \tag{2.11}$$

Finally, if we subtract (2.10) from (2.11) and use the representation (2.2), we deduce the second part of (2.1).

Now, assume that the second inequality in (2.1) holds with a constant $D > 0$ instead of $\frac{1}{2}$, i.e.,

$$\begin{aligned} (\gamma-\alpha)\, f(\alpha) + (\beta-\gamma)\, f(\beta) - \int_{\alpha}^{\beta} f(t)\, dt \\ \geq D\left[(\beta-\gamma)^2 f'_{-}(\beta) - (\gamma-\alpha)^2 f'_{+}(\alpha)\right]. \end{aligned} \tag{2.12}$$

If we consider the convex function h_0 given above, then we have $h'_{-}(\beta) = k$, $h'_{+}(\alpha) = -k$ and by (2.12) applied for h_0 and $x = \frac{\alpha+\beta}{2}$ we get

$$\frac{1}{4} k (\beta-\alpha)^2 \leq D\left[\frac{1}{4} k (\beta-\alpha)^2 + \frac{1}{4} k (\beta-\alpha)^2\right],$$

giving $D \geq \frac{1}{2}$, and the sharpness of the constant $\frac{1}{2}$ is proved. □

Corollary 2.2. *With the assumptions of Lemma 2.1 and if $\gamma \in (\alpha, \beta)$ is a point of differentiability for h, then*

$$\left(\frac{\alpha+\beta}{2} - \gamma\right) h'(\gamma) \tag{2.13}$$

$$\leq \left(\frac{\gamma-\alpha}{\beta-\alpha}\right) h(\alpha)+\left(\frac{\beta-\gamma}{\beta-\alpha}\right) h(\beta)-\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} h(t)\, dt .$$

Now, recall that the following inequality, which is well known in the literature as the Hermite-Hadamard inequality for convex functions, holds

$$h\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} h(t)\, dt \leq \frac{h(\alpha)+h(\beta)}{2}. \tag{2.14}$$

The following corollary provides some bounds for the difference

$$\frac{h(\alpha)+h(\beta)}{2}-\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} h(t)\, dt .$$

Corollary 2.3. *Let $h:[\alpha, \beta] \rightarrow \mathbb{R}$ be a convex function on $[\alpha, \beta]$. Then we have the inequality*

$$\begin{aligned} 0 & \leq \frac{1}{8}\left[h_{+}^{\prime}\left(\frac{\alpha+\beta}{2}\right)-h_{-}^{\prime}\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \\ & \leq \frac{h(\alpha)+h(\beta)}{2}-\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} h(t)\, dt \\ & \leq \frac{1}{8}\left[h_{-}^{\prime}(\beta)-h_{+}^{\prime}(\alpha)\right](\beta-\alpha) . \end{aligned} \tag{2.15}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

We are now able to state the corresponding result for convex functions defined on linear spaces.

Theorem 2.4. *Let X be a linear space, $a, b \in X$, $a \neq b$ and $f : [a, b] \subset X \to \mathbb{R}$ be a convex function on the segment $[a, b]$. Then for any $s \in (0, 1)$ one has the inequality*

$$\begin{aligned} (2.16) \quad & \frac{1}{2}\left[(1-s)^2\left(\nabla_+ f\left[(1-s)a+sb\right]\right)(b-a) \right. \\ & \qquad \left. -s^2\left(\nabla_- f\left[(1-s)a+sb\right]\right)(b-a)\right] \\ & \leq (1-s)f(a)+sf(b)-\int_0^1 f\left[(1-t)a+tb\right]dt \\ & \leq \frac{1}{2}\left[(1-s)^2\left(\nabla_- f(b)\right)(b-a)-s^2\left(\nabla_+ f(a)\right)(b-a)\right]. \end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.
The second inequality also holds for $s = 0$ or $s = 1$.

Proof. Follows by Lemma 2.1 applied for the convex function

$$h(t) = g(a, b)(t) = f\left[(1-t)a+tb\right], \quad t \in [0, 1],$$

and for the choices $\alpha = 0$, $\beta = 1$, and $\gamma = s$. □

Corollary 2.5. *If $f : [a, b] \to \mathbb{R}$ is as in Theorem 2.4 and Gâteaux differentiable in $c := (1-\lambda)a + \lambda b$, $\lambda \in (0, 1)$ along the direction $(b - a)$, then we have the inequality:*

$$\begin{aligned} (2.17) \quad & \left(\frac{1}{2}-\lambda\right)\left(\nabla f(c)\right)(b-a) \\ & \leq (1-\lambda)f(a)+\lambda f(b)-\int_0^1 f\left[(1-t)a+tb\right]dt. \end{aligned}$$

The following result related to the second Hermite-Hadamard inequality for functions defined on linear spaces also holds.

Corollary 2.6. *If f is as in Theorem 2.4, then*

$$\frac{1}{8}\left[\nabla_{+}f\left(\frac{a+b}{2}\right)(b-a)-\nabla_{-}f\left(\frac{a+b}{2}\right)(b-a)\right] \tag{2.18}$$
$$\leq \frac{f(a)+f(b)}{2}-\int_{0}^{1}f\left[(1-t)a+tb\right]dt$$
$$\leq \frac{1}{8}\left[\left(\nabla_{-}f(b)\right)(b-a)-\left(\nabla_{+}f(a)\right)(b-a)\right].$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

Now, let $\Omega\subset\mathbb{R}^{n}$ be an open convex set in $\mathbb{R}^{n}$.

If $F:\Omega\to\mathbb{R}$ is a differentiable convex function on Ω, then, obviously, for any $\bar{c}\in\Omega$ we have

$$\nabla F(\bar{c})(\bar{y})=\sum_{i=1}^{n}\frac{\partial F(\bar{c})}{\partial x_{i}}\cdot y_{i},\quad \bar{y}\in\mathbb{R}^{n},$$

where $\frac{\partial F}{\partial x_i}$ are the partial derivatives of F with respect to the variable x_i $(i=1,\dots,n)$.

Using (2.16), we may state that

$$\left(\frac{1}{2}-\lambda\right)\sum_{i=1}^{n}\frac{\partial F\left(\lambda\bar{a}+(1-\lambda)\bar{b}\right)}{\partial x_{i}}\cdot(b_{i}-a_{i}) \tag{2.19}$$

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$$\leq (1-\lambda) F(\bar{a}) + \lambda F(\bar{b}) - \int_0^1 F[(1-t)\bar{a} + t\bar{b}]\, dt$$

$$\leq \frac{1}{2}\left[(1-\lambda)^2 \sum_{i=1}^{n} \frac{\partial F(\bar{b})}{\partial x_i} \cdot (b_i - a_i) - \lambda^2 \sum_{i=1}^{n} \frac{\partial F(\bar{a})}{\partial x_i} \cdot (b_i - a_i)\right]$$

for any $\bar{a}, \bar{b} \in \Omega$ and $\lambda \in (0,1)$.

In particular, for $\lambda = \frac{1}{2}$, we get

$$0 \leq \frac{F(\bar{a}) + F(\bar{b})}{2} - \int_0^1 F[(1-t)\bar{a} + t\bar{b}]\, dt \leq \frac{1}{8} \sum_{i=1}^{n} \left(\frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i}\right) \cdot (b_i - a_i). \tag{2.20}$$

In (2.20) the constant $\frac{1}{8}$ is sharp.

3. Applications for Semi-Inner Products

Let $(X, \|\cdot\|)$ be a real normed linear space. We may state the following results for the semi-inner products $\langle \cdot, \cdot \rangle_i$ and $\langle \cdot, \cdot \rangle_s$.

Proposition 3.1. *For any $x, y \in X$ and $\sigma \in (0,1)$ we have the inequalities:*

$$(1-\sigma)^2 \langle y-x, (1-\sigma)x+\sigma y \rangle_s - \sigma^2 \langle y-x, (1-\sigma)x+\sigma y \rangle_i \leq (1-\sigma)\|x\|^2 + \sigma \|y\|^2 - \int_0^1 \|(1-t)x+ty\|^2 dt \leq (1-\sigma)^2 \langle y-x, y \rangle_i - \sigma^2 \langle y-x, y \rangle_s . \tag{3.1}$$

The second inequality in (3.1) also holds for $\sigma = 0$ or $\sigma = 1$.

The proof is obvious by Theorem 2.4 applied for the convex function $f(x) = \frac{1}{2}\|x\|^2$, $x \in X$.

If the space is *smooth*, then we may put $[x, y] = \langle x, y \rangle_i = \langle x, y \rangle_s$ for each $x, y \in X$ and the first inequality in (3.1) becomes

$$(1-2\sigma)[y-x, (1-\sigma)x+\sigma y] \leq (1-\sigma)\|x\|^2 + \sigma\|y\|^2 - \int_0^1 \|(1-t)x+ty\|^2 dt. \tag{3.2}$$

An interesting particular case one can get from (3.1) is the one for $\sigma = \frac{1}{2}$,

$$0 \leq \frac{1}{8}\left[\langle y-x, y+x \rangle_s - \langle y-x, y+x \rangle_i\right] \leq \frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1-t)x+ty\|^2 dt \tag{3.3}$$

$$\leq \frac{1}{4}\left[\langle y-x,y\rangle_i - \langle y-x,x\rangle_s\right].$$

The inequality (3.3) provides a refinement and a counterpart for the second inequality in (1.1).

If we consider now two linearly independent vectors $x, y \in X$ and apply Theorem 2.4 for $f(x) = \|x\|$, $x \in X$, then we get

Proposition 3.2. *For any linearly independent vectors $x, y \in X$ and $\sigma \in (0,1)$, one has the inequalities:*

$$\frac{1}{2}\left[(1-\sigma)^2 \frac{\langle y-x,(1-\sigma)x+\sigma y\rangle_s}{\|(1-\sigma)x+\sigma y\|} - \sigma^2 \frac{\langle y-x,(1-\sigma)x+\sigma y\rangle_i}{\|(1-\sigma)x+\sigma y\|}\right] \tag{3.4}$$
$$\leq (1-\sigma)\|x\| + \sigma\|y\| - \int_0^1 \|(1-t)x+ty\|\,dt$$
$$\leq \frac{1}{2}\left[(1-\sigma)^2 \frac{\langle y-x,y\rangle_i}{\|y\|} - \sigma^2 \frac{\langle y-x,x\rangle_s}{\|x\|}\right].$$

The second inequality also holds for $\sigma = 0$ or $\sigma = 1$.

We note that if the space is smooth, then we have

$$\left(\frac{1}{2}-\sigma\right)\cdot\frac{[y-x,(1-\sigma)x+\sigma y]}{\|(1-\sigma)x+\sigma y\|} \tag{3.5}$$
$$\leq (1-\sigma)\|x\| + \sigma\|y\| - \int_0^1 \|(1-t)x+ty\|\,dt$$

and for $\sigma = \frac{1}{2}$, (3.4) will give the simple inequality

$$0 \le \frac{1}{8}\left[\left\langle y-x, \frac{\frac{x+y}{2}}{\left\|\frac{x+y}{2}\right\|}\right\rangle_s - \left\langle y-x, \frac{\frac{x+y}{2}}{\left\|\frac{x+y}{2}\right\|}\right\rangle_i\right] \le \frac{\|x\|+\|y\|}{2} - \int_0^1 \|(1-t)\,x+ty\|\,dt \le \frac{1}{8}\left[\left\langle y-x, \frac{y}{\|y\|}\right\rangle_i - \left\langle y-x, \frac{x}{\|x\|}\right\rangle_s\right]. \tag{3.6}$$

The inequality (3.6) provides a refinement and a couterpart of the second inequality in (1.2).

Moreover, if we assume that $(H, \langle\cdot,\cdot\rangle)$ is an inner product space, then by (3.6) we get for any $x, y \in H$ with $\|x\| = \|y\| = 1$ that

$$0 \le 1 - \int_0^1 \|(1-t)\,x+ty\|\,dt \le \frac{1}{8}\|y-x\|^2. \tag{3.7}$$

The constant $\frac{1}{8}$ is sharp.

Indeed, if we choose $H = \mathbb{R}$, $\langle a, b\rangle = a \cdot b$, $x = -1$, $y = 1$, then we get equality in (3.7).

We give now some examples.

1. Let $\ell^2(\mathbb{K})$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$; be the Hilbert space of sequences $x = (x_i)_{i\in\mathbb{N}}$ with

$\sum_{i=0}^{\infty}|x_i|^2<\infty$. Then, by (3.7), we have the inequalities

$$0 \leq 1-\int_0^1\left(\sum_{i=0}^{\infty}|(1-t)\,x_i+ty_i|^2\right)^{\frac{1}{2}}dt \leq \frac{1}{8}\cdot\sum_{i=0}^{\infty}|y_i-x_i|^2, \tag{3.8}$$

for any $x,y\in\ell^2(\mathbb{K})$ provided $\sum_{i=0}^{\infty}|x_i|^2=\sum_{i=0}^{\infty}|y_i|^2=1$.

2. Let μ be a positive measure, $L_2(\Omega)$ the Hilbert space of $\mu-$measurable functions on Ω with complex values that are $2-$integrable on Ω, i.e., $f\in L_2(\Omega)$ iff $\int_\Omega|f(t)|^2\,d\mu(t)<\infty$. Then, by (3.7), we have the inequalities

$$0 \leq 1-\int_0^1\left(\int_\Omega|(1-\lambda)\,f(t)+\lambda g(t)|^2\,d\mu(t)\right)^{\frac{1}{2}}d\lambda \leq \frac{1}{8}\cdot\int_\Omega|f(t)-g(t)|^2\,d\mu(t) \tag{3.9}$$

for any $f,g\in L_2(\Omega)$ provided $\int_\Omega|f(t)|^2\,d\mu(t)=\int_\Omega|g(t)|^2\,d\mu(t)=1$.

References

[1] I. CIORANESCU, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, 1990.

[2] S.S. DRAGOMIR and C.E.M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. (ONLINE: `http://rgmia.vu.edu.au/monographs`)