ARGUMENT ESTIMATES FOR CERTAIN ANALYTIC FUNCTIONS ASSOCIATED WITH THE CONVOLUTION STRUCTURE

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Abstract: The purpose of the present paper is to investigate some argument properties for certain analytic functions in the open unit disk associated with the convolution structure. Some interesting applications are also considered as special cases of main results presented here.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z : |z| < 1\}$.

If $f \in \mathcal{A}$ is given by (1.1) and $g \in \mathcal{A}$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.2}$$

We observe that several known operators are deducible from the convolution. That is, for various choices of g in (1.2), we obtain some interesting operators studied by many authors. For example, for functions $f \in \mathcal{A}$ and the function defined by

$$g(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \cdots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_s)_{n-1} (n-1)!} z^n \tag{1.3}$$

$$(\alpha_i \in \mathbb{C}, \beta_j \in \mathbb{C} \backslash Z_0^-; Z_0^- = \{0, -1, -2, \dots\}; \quad i = 1, \dots, q; \quad j = 1, \dots, s;$$
$$q \leq s + 1; \quad q, s \in N_0 = N \cup \{0\}; z \in U),$$

the convolution (1.2) with the function g defined by (1.3) gives the operator studied by Dziok and Srivastava ([5], see also [4, 6]):

$$(g * f)(z) := H(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z). \tag{1.4}$$

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We note that the linear operator $H(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)$ includes various other linear operators which were introduced and studied various researchers in the literature.

Next, if we define the function g by

$$g(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n \quad (\lambda \geq 0;\ k \in \mathbb{Z}), \tag{1.5}$$

then for functions $f \in \mathcal{A}$, the convolution (1.2) with the function g defined by (1.5) reduces to the multiplier transformation studied by Cho and Srivastava [2]:

$$(g * f)(z) := I_\lambda^k f(z). \tag{1.6}$$

For arbitrary fixed real numbers A and B $(-1 \leq B < A \leq 1)$, we denote by $P(A, B)$ the class of functions of the form

$$q(z) = 1 + c_1 z + \cdots,$$

which are analytic in the unit disk $\mathbb{U}$ and satisfies the condition

$$q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{1.7}$$

where the symbol $\prec$ stands for usual subordination. We note that the class $P(A, B)$ was introduced and studied by Janowski [9].

We also observe from (1.7) (see, also [11]) that a function $q(z) \in P(A, B)$ if and only if

$$\left| q(z) - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \quad (B \neq -1;\ z \in \mathbb{U}) \tag{1.8}$$

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and

$$\mathrm{Re}\{q(z)\} > \frac{1-A}{2} \quad (B = -1;\ z \in \mathbb{U}). \tag{1.9}$$

In the present paper, we obtain some argument properties for certain analytic functions in $\mathcal{A}$ associated with the convolution structure by using the techniques involving the principle of differential subordination. Relevant connections of the results, which are presented in this paper, with various known operators are also considered.

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2. Main Results

Theorem 2.1. *Let* $f, g \in \mathcal{A}$ *and* $\beta \geq 0,\ 0 < \eta \leq 1$. *Suppose also that*

$$\frac{z(g*h)'(z)}{(g*h)(z)} \prec \frac{1+Az}{1+Bz} \quad (h \in \mathcal{A};\ -1 \leq B < A \leq 1;\ z \in \mathbb{U}). \tag{2.1}$$

If

$$\left|\arg\left\{\beta\frac{(g*f)'(z)}{(g*h)'(z)} + (1-\beta)\frac{(g*f)(z)}{(g*h)(z)}\right\}\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}),$$

then

$$\left|\arg\left\{\frac{(g*f)(z)}{(g*h)(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}),$$

where α $(0 < \alpha \leq 1)$ *is the solution of the equation given by*

$$\eta = \begin{cases} \alpha + \frac{2}{\pi}\tan^{-1}\frac{\beta\alpha\sin\frac{\pi}{2}(1-t(A,B))}{\frac{1+A}{1+B}+\beta\alpha\cos\frac{\pi}{2}(1-t(A,B))} & \text{for } B \neq -1, \\ \alpha & \text{for } B = -1, \end{cases} \tag{2.2}$$

and

$$t(A,B) = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{1-AB}\right). \tag{2.3}$$

Proof. Let

$$p(z) = \frac{(g*f)(z)}{(g*h)(z)} \quad \text{and} \quad q(z) = \frac{z(g*h)'(z)}{(g*h)(z)}.$$

Then by a simple calculation, we have

$$\beta\frac{(g*f)'(z)}{(g*h)'(z)} + (1-\beta)\frac{(g*f)(z)}{(g*h)(z)} = p(z) + \frac{\beta z p'(z)}{q(z)}.$$

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While, from the assumption (2.1) with (1.8) and (1.9), we obtain

$$q(z) = \rho e^{\frac{\pi\theta}{2} i},$$

where

$$\begin{cases} \frac{1-A}{1-B} < \rho < \frac{1+A}{1+B} \\ -t(A,B) < \theta < t(A,B) \quad \text{for } B \neq -1, \end{cases}$$

when $t(A,B)$ is given by (2.3) and

$$\begin{cases} \frac{1-A}{2} < \rho < \infty \\ -1 < \theta < 1 \qquad \text{for } B = -1. \end{cases}$$

The remaining part of the proof of the Theorem 2.1 follows by known results due to Miller and Mocanu [9] and Nunokawa [10] and applying a method similar to that of Cho et al. [3, Proof of Theorem 2.3], so we omit the details. □

In particular, if we put $g(z) = z/(1-z)$ in Theorem 2.1, we have the following result.

Corollary 2.2. *Let $f \in \mathcal{A}$ and $\beta > 0,\ 0 < \eta \leq 1$. If*

$$\left| \arg \left\{ \beta f'(z) + (1-\beta) \frac{f(z)}{z} \right\} \right| < \frac{\pi}{2}\eta,$$

then

$$\left| \arg \left\{ \frac{f(z)}{z} \right\} \right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha < 1)$ is the solution of the equation given by

$$\eta = \alpha + \frac{2}{\pi} \tan^{-1}(\alpha\beta). \tag{2.4}$$

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Theorem 2.3. *Let $f,g,h \in \mathcal{A}$ and $\mu > 0, 0 < \eta < 1$. If*

$$\left|\arg\left[\left(\frac{(g*h)(z)}{(g*f)(z)}\right)^{\mu}\left\{1+\frac{z(g*f)'(z)}{(g*f)(z)}-\frac{z(g*h)'(z)}{(g*h)(z)}\right\}\right]\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}) \tag{2.5}$$

then

$$\left|\arg\left\{\frac{(g*f)(z)}{(g*h)(z)}\right\}^{\mu}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}),$$

where α $(0 < \alpha < 1)$ is the solution of the equation given by

$$\eta = -\alpha + \frac{2}{\pi}\tan^{-1}\frac{\alpha}{\mu}. \tag{2.6}$$

Proof. Let

$$p(z) = \left\{\frac{(g*f)(z)}{(g*h)(z)}\right\}^{\mu} \quad (\mu > 0; z \in \mathbb{U}). \tag{2.7}$$

By differentiating both sides of (2.7) logarithmically and simplifying, we get

$$\left\{\frac{(g*h)(z)}{(g*f)(z)}\right\}^{\mu}\left\{1+\frac{z(g*f)'(z)}{(g*f)(z)}-\frac{z(g*h)'(z)}{(g*h)(z)}\right\} = \frac{1}{p(z)}\left(1+\frac{zp'(z)}{\mu p(z)}\right).$$

Now by using a lemma due to Nunokawa [10] and a method similar to the proof of Theorem 2.1, we get Theorem 2.3. □

Setting $(g*h)(z) = z$ and $g(z) = z/(1-z)$ in Theorem 2.3, we obtain Corollary 2.4 below which is comparable to the result studied by Lashin [8].

Corollary 2.4. *Let $f,g \in \mathcal{A}$ and $0 < \mu, \eta < 1$. If*

$$\left|\arg\left[\left\{\frac{z}{(g*f)(z)}\right\}^{\mu+1}(g*f)'(z)\right]\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}),$$

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then

$$\left|\arg\left\{\frac{(g*f)(z)}{z}\right\}^{\mu}\right| < \frac{\pi}{2}\alpha \quad (z\in\mathbb{U}),$$

where α $(0<\alpha<1)$ *is the solution of the equation given by (2.6).*

Theorem 2.5. *Let* $f,g\in\mathcal{A}$ *and* $\beta>0,\ 0<\eta\le 1$. *If*

$$\left|\arg\left[\left(\frac{z(g*f)'(z)}{\varphi[(g*f)(z)]}\right)\left\{1+\beta\frac{z(g*f)''(z)}{(g*f)'(z)}-\beta\frac{z\varphi'[(g*f)(z)]}{\varphi[(g*f)(z)]}\right\}\right]\right| < \frac{\pi}{2}\eta, \tag{2.8}$$

where $\varphi[w]$ *is analytic in* $(g*f)(U)$, $\varphi[0]=\varphi'[0]-1=0$ *and* $\varphi[w]\neq 0$ *in* $(g*f)(U)\backslash\{0\}$, *then*

$$\left|\arg\left\{\frac{z(g*f)'(z)}{\varphi(g*f)(z)}\right\}\right| < \frac{\pi}{2}\alpha,$$

where α $(0<\alpha<1)$ *is the solution of the equation given by (2.4).*

Proof. Our proof of Theorem 2.5 is much akin to that of Theorem 2.3. Indeed in place (2.7) we define $p(z)$ by

$$p(z)=\left\{\frac{(g*f)(z)}{(g*h)(z)}\right\}^{\mu} \quad (\mu>0; z\in\mathbb{U}). \tag{2.9}$$

We choose to skip the detailed involved. □

By setting $\varphi[(g*f)(z)]=(g*f)(z)$ and $g(z)=z/(1-z)$, we have the following result.

Corollary 2.6. *Let* $f\in\mathcal{A}$ *and* $\beta>0,\ 0<\eta\le 1$. *If*

$$\left|\arg\left[\left(\frac{zf'(z)}{f(z)}\right)\left\{1+\beta\frac{zf''(z)}{f'(z)}-\beta\frac{zf'(z)}{f(z)}\right\}\right]\right| < \frac{\pi}{2}\eta \quad (z\in\mathbb{U})$$

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then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}),$$

where α $(0 < \alpha < 1)$ *is the solution of the equation given by (2.4).*

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3. Some Remarks and Observations

Using the Hadamard product (or convolution) defined by (1.2) and applying the differential subordination techniques, we obtained some argument properties of normalized analytic functions in the open unit disk $\mathbb{U}$. If we replace g in Theorems 2.1, 2.3 and 2.5 by the function $H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)$ defined by (1.4) or the multiplier transformation I_λ^k defined by (1.5), then we have the corresponding results to the Theorems 2.1, 2.3 and 2.5. Moreover, we note that, if we suitably choose φ introduced in Theorem 2.5 (which is called the φ-like function [1]), then we can obtain various interesting applications.

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