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A DISCRETE EULER IDENTITY

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ABSTRACT. A discrete analogue of the weighted Montgomery identity (i.e. Euler identity) for finite sequences of vectors in normed linear space is given as well as a discrete analogue of Ostrowski type inequalities and estimates of difference of two arithmetic means.

Key words and phrases: Discrete Montgomery identity, Discrete Ostrowski inequality.

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1. INTRODUCTION

The following Ostrowski inequality is well known [10]:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty .$$

It holds for every $x \in [a,b]$ whenever $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with derivative $f' : (a,b) \to \mathbb{R}$ bounded on (a,b) i.e.

$$\left\| f' \right\|_\infty = \sup_{t \in (a,b)} \left| f'(t) \right| < +\infty.$$

Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, $f' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$ and $w : [a,b] \to [0,\infty)$ some probability density function, i.e. integrable function satisfying $\int_a^b w(t)\,dt = 1$;

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define $W(t) = \int_a^t w(x)\,dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. The following identity, given by Pečarić in [11], is the weighted Montgomery identity

$$f(x) = \int_a^b w(t) f(t)\,dt + \int_a^b P_w(x,t) f'(t)\,dt,$$

where the weighted Peano kernel is

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1 & x < t \le b. \end{cases}$$

All results in this paper are discrete analogues of results from [1]. The aim of this paper is to prove the discrete analogue of the weighted Euler identity for finite sequences of vectors in normed linear spaces and to use it to obtain some new discrete Ostrowski type inequalities as well as estimates of differences between two (weighted) arithmetic means. In Section 2, a discrete weighted Montgomery (i.e. Euler) identity is presented. In Section 3, Ostrowski's inequality and its generalization are proved. These are the discrete analogues of some results from [6]. In Section 4, estimates of differences between two (weighted) arithmetic means are given and these are the discrete analogues of some results from [2], [3], [4], [5] and [12].

2. DISCRETE WEIGHTED EULER IDENTITY

Let $x_1, x_2, \dots, x_n$ be a finite sequence of vectors in the normed linear space $(X, \|\cdot\|)$ and $w_1, w_2, \dots, w_n$ finite sequence of positive real numbers. If, for $1 \le k \le n$,

$$W_k = \sum_{i=1}^{k} w_i, \quad \overline{W_k} = \sum_{i=k+1}^{n} w_i = W_n - W_k,$$

then we have, see [9],

$$\sum_{i=1}^{n} w_i x_i = x_k W_n + \sum_{i=1}^{k-1} W_i (x_i - x_{i+1}) + \sum_{i=k}^{n-1} \overline{W_i} (x_{i+1} - x_i), \quad 1 \le k \le n. \tag{2.1}$$

The difference operator $\boldsymbol{\Delta}$ is defined by

$$\Delta x_i = x_{i+1} - x_i. \tag{2.2}$$

So using formula (2.1), we get the discrete analogue of weighted Montgomery identity

$$x_k = \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n-1} D_w(k,i)\,\Delta x_i, \tag{2.3}$$

where the discrete Peano kernel is defined by

$$D_w(k,i) = \frac{1}{W_n} \cdot \begin{cases} W_i, & 1 \le i \le k-1, \\ \left(-\overline{W_i}\right), & k \le i \le n. \end{cases} \tag{2.4}$$

If we take $w_i = 1, \; i = 1, \dots, n$, then $W_i = i$ and $\overline{W_i} = n - i$, and (2.3) reduces to the discrete Montgomery identity

$$x_k = \frac{1}{n} \sum_{i=1}^{n} x_i + \sum_{i=1}^{n-1} D_n(k,i)\,\Delta x_i, \tag{2.5}$$

where

$$D_n(k,i) = \begin{cases} \frac{i}{n}, & 1 \le i \le k-1, \\ \frac{i}{n} - 1, & k \le i \le n. \end{cases}$$

If $n \in \mathbb{N}$, Δ^n is inductively defined by

$$\Delta^n x_i = \Delta^{n-1}(\Delta x_i).$$

It is then easy to prove, by induction or directly using the elementary theory of operators,see [8], that

$$\Delta^n x_i = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x_{i+k}.$$

In the next theorem we give the generalization of the identity (2.3).

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_1, x_2, \dots, x_n$ a finite sequence of vectors in X, $w_1, w_2, \dots, w_n$ finite sequence of positive real numbers. Then for all $m \in \{2, 3, \dots, n-1\}$ and $k \in \{1, 2, \dots, n\}$ the following identity is valid:*

$$x_k = \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + \sum_{r=1}^{m-1}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right) \times \left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_r=1}^{n-r} D_w(k,i_1) D_{n-1}(i_1,i_2)\cdots D_{n-r+1}(i_{r-1},i_r)\right) + \sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_m=1}^{n-m} D_w(k,i_1) D_{n-1}(i_1,i_2)\cdots D_{n-m+1}(i_{m-1},i_m)\Delta^m x_{i_m}. \tag{2.6}$$

Proof. We prove our assertion by induction with respect to m. For $m = 2$ we have to prove the identity

$$x_k = \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + \frac{1}{n-1}\left(\sum_{i=1}^{n-1}\Delta x_i\right)\left(\sum_{i=1}^{n-1} D_w(k,i)\right) + \sum_{i=1}^{n-1}\sum_{j=1}^{n-2} D_w(k,i) D_{n-1}(i,j)\Delta^2 x_j.$$

Applying the identity (2.5) for the finite sequence of vectors Δx_i, $i = 1, 2, \dots, n-1$, we obtain

$$\Delta x_i = \frac{1}{n-1}\sum_{i=1}^{n-1}\Delta x_i + \sum_{j=1}^{n-2} D_{n-1}(i,j)\Delta^2 x_j$$

so, again using (2.3), we have

$$\begin{aligned} x_k &= \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n-1} D_w(k,i)\left(\frac{1}{n-1}\sum_{i=1}^{n-1}\Delta x_i + \sum_{j=1}^{n-2} D_{n-1}(i,j)\Delta^2 x_j\right) \\ &= \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + \frac{1}{n-1}\left(\sum_{i=1}^{n-1}\Delta x_i\right)\left(\sum_{i=1}^{n-1} D_w(k,i)\right) \\ &\quad + \sum_{i=1}^{n-1}\sum_{j=1}^{n-2} D_w(k,i) D_{n-1}(i,j)\Delta^2 x_j. \end{aligned}$$

Hence the identity (2.6) holds for $m=2$.
Now, we assume that it holds for a natural number $m \in \{2,3,\ldots,n-2\}$. Applying the identity (2.5) for the $\Delta^m x_{i_m}$

$$\Delta^m x_{i_m} = \frac{1}{n-m}\sum_{i=1}^{n-m}\Delta^m x_i + \sum_{i_{m+1}=1}^{n-m-1} D_{n-m}\left(i_m, i_{m+1}\right)\Delta^{m+1}x_{i_{m+1}}$$

and using the induction hypothesis, we get

$$\begin{aligned}
x_k &= \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + \sum_{r=1}^{m-1}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right) \\
&\qquad\qquad \times \left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_r=1}^{n-r} D_w\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_{n-r+1}\left(i_{r-1},i_r\right)\right) \\
&\quad + \left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_m=1}^{n-m} D_w\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_{n-m+1}\left(i_{m-1},i_m\right)\right. \\
&\qquad\qquad \left.\times\left(\frac{1}{n-m}\sum_{i=1}^{n-m}\Delta^m x_i + \sum_{i_{m+1}=1}^{n-m-1} D_{n-m}\left(i_m,i_{m+1}\right)\Delta^{m+1}x_{i_{m+1}}\right)\right) \\
&= \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + \sum_{r=1}^{m}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right) \\
&\qquad\qquad \times \left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_r=1}^{n-r} D_w\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_{n-r+1}\left(i_{r-1},i_r\right)\right) \\
&\quad + \sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_{m+1}=1}^{n-(m+1)} D_w\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_{n-m}\left(i_m,i_{m+1}\right)\Delta^{m+1}x_{i_{m+1}}.
\end{aligned}$$

We see that (2.6) is valid for $m+1$ and our assertion is proved. □

Remark 2.2. For $m=n-1$ (2.6) becomes

$$\begin{aligned}
x_k &= \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + \sum_{r=1}^{n-2}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right) \\
&\qquad \times \left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_r=1}^{n-r} D_w\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_{n-r+1}\left(i_{r-1},i_r\right)\right) \\
&\qquad\quad + \sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_{n-1}=1}^{1} D_w\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_2\left(i_{n-2},i_{n-1}\right)\Delta^{n-1}x_{i_{n-1}}.
\end{aligned}$$

Corollary 2.3. *Let $(X,\|\cdot\|)$ be a normed linear space, $x_1,x_2,\ldots,x_n$ a finite sequence of vectors in X. Then for all $m\in\{2,3,\ldots,n-1\}$ and $k\in\{1,2,\ldots,n\}$ the following identity is valid:*

$$x_k = \frac{1}{n}\sum_{i=1}^{n} x_i + \sum_{r=1}^{m-1}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right)$$

$$\times\left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_r=1}^{n-r}D_n(k,i_1)\,D_{n-1}(i_1,i_2)\cdots D_{n-r+1}(i_{r-1},i_r)\right)$$
$$+\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_m=1}^{n-m}D_n(k,i_1)\,D_{n-1}(i_1,i_2)\cdots D_{n-m+1}(i_{m-1},i_m)\,\Delta^m x_{i_m}.$$

Proof. Apply Theorem 2.1 with $w_i = 1$, $i = 1,\dots,n$. □

Remark 2.4. If we apply (2.6) with $n = 2l-1$ and $k = l$ we get

$$x_l = \frac{1}{W_{2l-1}}\sum_{i=1}^{2l-1}w_i x_i + \sum_{r=1}^{m-1}\frac{1}{2l-1-r}\left(\sum_{i=1}^{2l-1-r}\Delta^r x_i\right)$$
$$\times\left(\sum_{i_1=1}^{2l-2}\sum_{i_2=1}^{2l-3}\cdots\sum_{i_r=1}^{2l-1-r}D_w(l,i_1)\,D_{2l-2}(i_1,i_2)\cdots D_{2l-r}(i_{r-1},i_r)\right)$$
$$+\sum_{i_1=1}^{2l-2}\sum_{i_2=1}^{2l-3}\cdots\sum_{i_m=1}^{2l-1-m}D_w(l,i_1)\,D_{2l-2}(i_1,i_2)\cdots D_{2l-m}(i_{m-1},i_m)\,\Delta^m x_{i_m}.$$

We may regard this identity as a generalized midpoint identity since for $m = 1$ it reduces to

$$x_l = \frac{1}{W_{2l-1}}\sum_{i=1}^{2l-1}w_i x_i + \sum_{i=1}^{2l-2}D_w(l,i)\,\Delta x_i \tag{2.7}$$

and further for $w_i = 1, i = 1,2,\dots,2l-1$ to

$$x_l = \frac{1}{2l-1}\sum_{i=1}^{2l-1}x_i + \frac{1}{2l-1}\sum_{i=1}^{l-1}i\left(\Delta x_i - \Delta x_{2l-1-i}\right). \tag{2.8}$$

Similarly, if we apply (2.6) with $k = 1$ and then with $k = n$, then sum these two equalities and divide them by 2, we get

$$\frac{x_1+x_n}{2} = \frac{1}{W_n}\sum_{i=1}^{n}w_i x_i + \sum_{r=1}^{m-1}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right) \tag{2.9}$$
$$\times\left(\sum_{i_1=1}^{n-1}\cdots\sum_{i_r=1}^{n-r}\frac{D_w(1,i_1)+D_w(n,i_1)}{2}D_{n-1}(i_1,i_2)\cdots D_{n-r+1}(i_{r-1},i_r)\right)$$
$$+\sum_{i_1=1}^{n-1}\cdots\sum_{i_m=1}^{n-m}\frac{D_w(1,i_1)+D_w(n,i_1)}{2}D_{n-1}(i_1,i_2)\cdots D_{n-m+1}(i_{m-1},i_m)\,\Delta^m x_{i_m}.$$

We may regard this identity as a generalized trapezoid identity since for $m = 1$ it reduces to

$$\frac{x_1+x_n}{2} = \frac{1}{W_n}\sum_{i=1}^{n}w_i x_i + \sum_{i=1}^{n-1}\frac{D_w(1,i)+D_w(n,i)}{2}\Delta x_i, \tag{2.10}$$

and further for $w_i = 1, i = 1,2,\dots,n$ to

$$\frac{x_1+x_n}{2} = \frac{1}{n}\sum_{i=1}^{n}x_i + \frac{1}{n}\sum_{i=1}^{n-1}\left(i-\frac{n}{2}\right)\Delta x_i. \tag{2.11}$$

(2.8) and (2.11) were obtained by Dragomir in [7].

3. DISCRETE OSTROWSKI TYPE INEQUALITIES

The Bernoulli numbers B_i, $i \geq 0$, are defined by the implicit recurrence relation

$$\sum_{i=0}^{m} \binom{m+1}{i} B_i = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0. \end{cases}$$

If, for $n \in \mathbb{N}$ and $m \in \mathbb{R}$,we write

$$S_m(n) = 1^m + 2^m + 3^m + \cdots + (n-1)^m,$$

it is well known, see [8], that if $m \in \mathbb{N}$

$$S_m(n) = \frac{1}{m+1} \sum_{i=0}^{m} \binom{m+1}{i} B_i \, n^{m+1-i}.$$

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_1, x_2, \ldots, x_n$ a finite sequence of vectors in X, $w_1, w_2, \ldots, w_n$ finite sequence of positive real numbers. Let also (p, q) be a pair of conjugate exponents*[1], $m \in \{2, 3, \ldots, n-1\}$ *and* $k \in \{1, 2, \ldots, n\}$ *the following inequality holds:*

$$\begin{aligned} &\left\| x_k - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i - \sum_{r=1}^{m-1} \frac{1}{n-r} \left(\sum_{i=1}^{n-r} \Delta^r x_i \right) \right. \\ &\quad \left. \times \left(\sum_{i_1=1}^{n-1} \sum_{i_2=1}^{n-2} \cdots \sum_{i_r=1}^{n-r} D_w(k, i_1) D_{n-1}(i_1, i_2) \cdots D_{n-r+1}(i_{r-1}, i_r) \right) \right\| \\ &\quad \leq \left\| \sum_{i_1=1}^{n-1} \sum_{i_2=1}^{n-2} \cdots \sum_{i_{m-1}=1}^{n-m+1} D_w(k, i_1) D_{n-1}(i_1, i_2) \cdots D_{n-m+1}(i_{m-1}, \cdot) \right\|_q \|\Delta^m x\|_p, \end{aligned} \tag{3.1}$$

where

$$\|\Delta^m x\|_p = \begin{cases} \left(\sum\limits_{i=1}^{n-m} \|\Delta^m x_i\|^p \right)^{\frac{1}{p}}, & \textit{if } 1 \leq p < \infty, \\ \max\limits_{1 \leq i \leq n-m} \|\Delta^m x_i\| & \textit{if } p = \infty. \end{cases}$$

Proof. By using the (2.6) and the Hölder inequality. □

Corollary 3.2. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_1, x_2, \ldots, x_n$ a finite sequence of vectors in X, $w_1, w_2, \ldots, w_n$ a finite sequence of positive real numbers. Let also (p, q) be a pair of conjugate exponents. Then for all $k \in \{1, 2, \ldots, n\}$ the following inequalities hold:*

$$\left\| x_k - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right\| \leq \begin{cases} \dfrac{1}{W_n} \left(\sum\limits_{i=1}^{n} |k-i| \, w_i \right) \cdot \|\Delta x\|_\infty, \\ \dfrac{1}{W_n} \left(\sum\limits_{i=1}^{k-1} \left(\sum\limits_{j=1}^{i} w_j \right)^q + \sum\limits_{i=k}^{n-1} \left(\sum\limits_{j=i+1}^{n} w_i \right)^q \right)^{\frac{1}{q}} \cdot \|\Delta x\|_p, \\ \dfrac{1}{W_n} \max\{W_{k-1}, W_n - W_k\} \cdot \|\Delta x\|_1. \end{cases}$$

[1]That is: $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

Proof. By using the discrete analogue of the weighted Montgomery identity (2.3) and applying the Hölder inequality we get

$$\left\| x_k - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right\| \leq \left\| D_w(k, \cdot) \right\|_q \left\| \Delta x \right\|_p .$$

We have

$$\begin{aligned} \left\| D_w(k, \cdot) \right\|_1 &= \frac{1}{W_n} \left(\sum_{i=1}^{k-1} |W_i| + \sum_{i=k}^{n-1} \left| -\overline{W_i} \right| \right) \\ &= \frac{1}{W_n} \left(\sum_{i=1}^{k-1} (k-i) w_i + \sum_{i=1}^{n-k} i w_{k+i} \right) \\ &= \frac{1}{W_n} \sum_{i=1}^{n} |k-i| w_i \end{aligned}$$

and the first inequality is proved.
Since

$$\begin{aligned} \left\| D_w(k, \cdot) \right\|_q &= \frac{1}{W_n} \left(\sum_{i=1}^{k-1} |W_i|^q + \sum_{i=k}^{n-1} \left| -\overline{W_i} \right|^q \right)^{\frac{1}{q}} \\ &= \frac{1}{W_n} \left(\sum_{i=1}^{k-1} \left(\sum_{j=1}^{i} w_j \right)^q + \sum_{i=k}^{n-1} \left(\sum_{j=i+1}^{n} w_i \right)^q \right)^{\frac{1}{q}} \end{aligned}$$

the second inequality is proved.
Finally, for the third

$$\left\| D_w(k, \cdot) \right\|_\infty = \frac{1}{W_n} \max \left\{ W_{k-1}, W_n - W_k \right\} ,$$

which completes the proof. □

The first and the third inequality from Corollary 3.2 and also the following corollary was proved by Dragomir in [7].

Corollary 3.3. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_1, x_2, \ldots, x_n$ a finite sequence of vectors in X, $w_1, w_2, \ldots, w_n$ finite sequence of positive real numbers, and also let (p, q) be a pair of conjugate exponents. Then for all $k \in \{1, 2, \ldots, n\}$ the following inequalities hold:*

$$\left\| x_k - \frac{1}{n} \sum_{i=1}^{n} x_i \right\| \leq \begin{cases} \frac{1}{n} \left(\frac{n^2-1}{4} + \left(k - \frac{n+1}{2} \right)^2 \right) \cdot \left\| \Delta x \right\|_\infty , \\ \frac{1}{n} \left(S_q(k) + S_q(n-k+1) \right)^{\frac{1}{q}} \cdot \left\| \Delta x \right\|_p , \\ \frac{1}{n} \max \left\{ k-1, n-k \right\} \cdot \left\| \Delta x \right\|_1 . \end{cases} \tag{3.2}$$

Proof. If we apply Corollary 3.2 with $w_i = 1$, $i = 1, 2, \ldots, n$, or use the discrete Montgomery identity (2.5), we have

$$\begin{aligned} \left\| x_k - \frac{1}{n} \sum_{i=1}^{n} x_i \right\| &= \left\| \sum_{i=1}^{n-1} D_n(k, i) \Delta x_i \right\| \\ &\leq \left(\sum_{i=1}^{n-1} |D_n(k, i)|^q \right)^{\frac{1}{q}} \left(\sum_{i=1}^{n-1} \left\| \Delta x_i \right\|^p \right)^{\frac{1}{p}} . \end{aligned}$$

Since for $q = 1$

$$\sum_{i=1}^{n-1} |D_n(k,i)| = \frac{1}{n}\left(\frac{n^2-1}{4} + \left(k - \frac{n+1}{2}\right)^2\right),$$

the first inequality follows.
For the second let $1 < q < \infty$

$$\sum_{i=1}^{n-1} |D_n(k,i)|^q = \frac{1}{n^q}\left(\sum_{i=1}^{k-1} i^q + \sum_{i=k}^{n-1} (n-i)^q\right) = \frac{1}{n^q}\left(S_q(k) + S_q(n-k+1)\right)$$

the second inequality follows.
Finally for $q = \infty$ and

$$\max_{1\le i\le n-1} \{|D(k,i)|\} = \frac{1}{n}\max\{k-1, n-k\}$$

implies the last inequality. □

Corollary 3.4. *Assume that all assumptions from Theorem 3.1 hold. Then the following inequality holds*

$$\left\| x_l - \frac{1}{W_{2l-1}}\sum_{i=1}^{2l-1} w_i x_i - \sum_{r=1}^{m-1}\frac{1}{2l-1-r}\left(\sum_{i=1}^{2l-1-r}\Delta^r x_i\right) \times \left(\sum_{i_1=1}^{2l-2}\sum_{i_2=1}^{2l-3}\cdots\sum_{i_r=1}^{2l-1-r} D_w(l,i_1)\, D_{2l-2}(i_1,i_2)\cdots D_{2l-r}(i_{r-1},i_r)\right)\right\| \le \left\|\sum_{i_1=1}^{2l-2}\sum_{i_2=1}^{2l-3}\cdots\sum_{i_{m-1}=1}^{2l-m} D_w(l,i_1)\, D_{2l-2}(i_1,i_2)\cdots D_{2l-m}(i_{m-1},\cdot)\right\|_q \|\Delta^m x\|_p;$$

it may be regarded as a generalized midpoint inequality since for $m = 1$ it reduces to

$$\left\| x_l - \frac{1}{W_{2l-1}}\sum_{i=1}^{2l-1} w_i x_i\right\| \le \begin{cases} \dfrac{1}{W_{2l-1}}\left(\sum\limits_{i=1}^{2l-1}|l-i|\, w_i\right)\cdot\|\Delta x\|_\infty, \\ \dfrac{1}{W_{2l-1}}\left(\sum\limits_{i=1}^{l-1}\left(\sum\limits_{j=1}^{i} w_j\right)^q + \sum\limits_{i=l}^{2l-2}\left(\sum\limits_{j=i+1}^{n} w_i\right)^q\right)^{\frac{1}{q}}\cdot\|\Delta x\|_p, \\ \dfrac{1}{W_{2l-1}}\max\{W_{l-1}, W_{2l-1} - W_l\}\cdot\|\Delta x\|_1; \end{cases}$$

if in addition $w_i = 1, i = 1, 2, \ldots, 2l-1$ it further reduces to

$$\left\| x_l - \frac{1}{2l-1}\sum_{i=1}^{2l-1} x_i\right\| \le \begin{cases} \dfrac{l(l-1)}{2l-1}\cdot\|\Delta x\|_\infty, \\ \dfrac{1}{2l-1}\left(2S_q(l)\right)^{\frac{1}{q}}\cdot\|\Delta x\|_p, \\ \dfrac{l-1}{2l-1}\cdot\|\Delta x\|_1. \end{cases} \tag{3.3}$$

Proof. Apply (3.1) with $n = 2l - 1$ and $k = l$ to get the first inequality.
For the second, taking $m = 1$, or applying Hölder's inequality to (2.7), gives

$$\left\| x_l - \frac{1}{W_{2l-1}} \sum_{i=1}^{2l-1} w_i x_i \right\| = \left\| \sum_{i=1}^{2l-2} D_w(l,i)\,\Delta x_i \right\| \le \|D_w(l,\cdot)\|_q \|\Delta x\|_p .$$

Now

$$\|D_w(l,\cdot)\|_1 = \frac{1}{W_{2l-1}} \left(\sum_{i=1}^{l-1} |W_i| + \sum_{i=l}^{2l-1} \left|-\overline{W_i}\right| \right) = \frac{1}{W_{2l-1}} \left(\sum_{i=1}^{2l-1} |l-i|\, w_i \right),$$

$$\begin{aligned}\|D_w(l,\cdot)\|_q &= \frac{1}{W_{2l-1}} \left(\sum_{i=1}^{l-1} |W_i|^q + \sum_{i=l}^{2l-1} \left|-\overline{W_i}\right|^q \right)^{\frac{1}{q}} \\ &= \frac{1}{W_{2l-1}} \left(\sum_{i=1}^{l-1} \left(\sum_{j=1}^{i} w_j \right)^q + \sum_{i=l}^{2l-2} \left(\sum_{j=i+1}^{n} w_i \right)^q \right)^{\frac{1}{q}},\end{aligned}$$

$$\|D_w(l,\cdot)\|_\infty = \frac{1}{W_{2l-1}} \max\{W_{l-1}, W_{2l-1} - W_l\}$$

and the second inequality is proved.
Now if we take $w_i = 1, i = 1, 2, \ldots, 2l-1$, or apply inequality (3.2) with $n = 2l-1$ and $k = l$,

$$\|D_{2l-1}(l,\cdot)\|_1 = \frac{1}{2l-1} \sum_{i=1}^{2l-1} |l-i| = \frac{l(l-1)}{2l-1},$$

$$\|D_{2l-1}(l,\cdot)\|_q = \frac{1}{2l-1} \left(\sum_{i=1}^{2l-1} |l-i|^q \right)^{\frac{1}{q}} = \frac{1}{2l-1} \left(2S_q(l)\right)^{\frac{1}{q}},$$

$$\|D_{2l-1}(l,\cdot)\|_\infty = \frac{1}{2l-1} \max\{l-1, 2l-1-l\} = \frac{l-1}{2l-1},$$

and thus the third inequality is proved. □

Corollary 3.5. *Let all the assumptions from Theorem 3.1 hold. Then the following inequality holds:*

$$\begin{aligned}&\left\| \frac{x_1 + x_n}{2} - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i - \sum_{r=1}^{m-1} \frac{1}{n-r} \left(\sum_{i=1}^{n-r} \Delta^r x_i \right) \right. \\ &\qquad \left. \times \left(\sum_{i_1=1}^{n-1} \cdots \sum_{i_r=1}^{n-r} \frac{D_w(1,i_1) + D_w(n,i_1)}{2} D_{n-1}(i_1,i_2) \cdots D_{n-r+1}(i_{r-1},i_r) \right) \right\| \\ &\quad \le \left\| \sum_{i_1=1}^{n-1} \cdots \sum_{i_{m-1}=1}^{n-m+1} \frac{D_w(1,i_1) + D_w(n,i_1)}{2} D_{n-1}(i_1,i_2) \cdots D_{n-m+1}(i_{m-1},\cdot) \right\|_q \|\Delta^m x\|_p ;\end{aligned}$$

this may regarded as a generalized trapezoid inequality since for $m=1$ it reduces to

$$\left\| \frac{x_1+x_n}{2} - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \right\| \leq \begin{cases} \sum_{i=1}^{n-1} \left| \frac{W_i}{W_n} - \frac{1}{2} \right| \cdot \|\Delta x\|_\infty , \\ \left(\sum_{i=1}^{n-1} \left| \frac{W_i}{W_n} - \frac{1}{2} \right|^q \right)^{\frac{1}{q}} \cdot \|\Delta x\|_p , \\ \max \left\{ \left| \frac{w_1}{W_n} - \frac{1}{2} \right| , \left| \frac{w_n}{W_n} - \frac{1}{2} \right| \right\} \cdot \|\Delta x\|_1 . \end{cases}$$

and if in addition, $w_i = 1, i = 1,2,\ldots,n$ it further reduces to

$$(3.4)\quad \left\| \frac{x_1+x_n}{2} - \frac{1}{n}\sum_{i=1}^{n} x_i \right\| \leq \begin{cases} \frac{1}{n}\left(n-1-\left\lfloor \frac{n}{2}\right\rfloor\right)\left(\left\lfloor \frac{n}{2}\right\rfloor\right) \cdot \|\Delta x\|_\infty , \\ \begin{cases} \frac{1}{n}\left(2S_q\left(\frac{n}{2}\right)\right)^{\frac{1}{q}} \cdot \|\Delta x\|_p , & \textit{if } n \textit{ is even,} \\ \frac{1}{n}\left(\frac{S_q(n-1)}{2^{q-1}} - 2S_q\left(\frac{n-1}{2}\right)\right)^{\frac{1}{q}} \cdot \|\Delta x\|_p , & \textit{if } n \textit{ is odd,} \end{cases} \\ \frac{n-2}{2n} \cdot \|\Delta x\|_1 . \end{cases}$$

Proof. To obtain the first inequality take (2.9) and apply Hölder's inequality.
For the second we take $m=1$ or apply Hölder's inequality to (2.10),

$$\left\| \frac{x_1+x_n}{2} - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \right\| = \left\| \sum_{i=1}^{n-1} \frac{D_w(1,i)+D_w(n,i)}{2}\Delta x_i \right\| \leq \left\| \frac{D_w(1,\cdot)+D_w(n,\cdot)}{2} \right\|_q \|\Delta x\|_p .$$

Now

$$\left\| \frac{D_w(1,\cdot)+D_w(n,\cdot)}{2} \right\|_1 = \sum_{i=1}^{n-1} \left| \frac{W_i - \overline{W_i}}{2W_n} \right| = \sum_{i=1}^{n-1} \left| \frac{W_i}{W_n} - \frac{1}{2} \right| ,$$

$$\left\| \frac{D_w(1,\cdot)+D_w(n,\cdot)}{2} \right\|_q = \left(\sum_{i=1}^{n-1} \left| \frac{W_i}{W_n} - \frac{1}{2} \right|^q \right)^{\frac{1}{q}} ,$$

$$\begin{aligned} \left\| \frac{D_w(1,\cdot)+D_w(n,\cdot)}{2} \right\|_\infty &= \max_{1\leq i\leq n-1} \left\{ \left| \frac{W_i}{W_n} - \frac{1}{2} \right| \right\} \\ &= \max \left\{ \left| \frac{W_1}{W_n} - \frac{1}{2} \right| , \left| \frac{W_{n-1}}{W_n} - \frac{1}{2} \right| \right\} \\ &= \max \left\{ \left| \frac{w_1}{W_n} - \frac{1}{2} \right| , \left| \frac{w_n}{W_n} - \frac{1}{2} \right| \right\} \end{aligned}$$

and the second inequality is proved.
Now if we take $w_i = 1, i = 1,2,\ldots,n$, or use (2.11) and apply Hölder's inequality, we get

$$\left\| \frac{x_1+x_n}{2} - \frac{1}{n}\sum_{i=1}^{n} x_i \right\| \leq \left\| \frac{i}{n} - \frac{1}{2} \right\|_q \|\Delta x\|_p .$$

For $q=1$

$$\left\|\frac{i}{n}-\frac{1}{2}\right\|_1=\frac{1}{n}\sum_{i=1}^{n-1}\left|i-\frac{n}{2}\right|=\frac{1}{n}\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)\left(\left\lfloor\frac{n}{2}\right\rfloor\right);$$

for $1<q<\infty$

$$\left\|\frac{i}{n}-\frac{1}{2}\right\|_q=\frac{1}{n}\left(\sum_{i=1}^{n-1}\left|i-\frac{n}{2}\right|^q\right)^{\frac{1}{q}}$$

$$=\begin{cases}\frac{1}{n}\left(2S_q\left(\frac{n}{2}\right)\right)^{\frac{1}{q}}, & \text{if } n \text{ is even,}\\ \frac{1}{n}\left(\frac{S_q(n-1)}{2^{q-1}}-2S_q\left(\frac{n-1}{2}\right)\right)^{\frac{1}{q}}, & \text{if } n \text{ is odd;}\end{cases}$$

and for $q=\infty$

$$\left\|\frac{i}{n}-\frac{1}{2}\right\|_\infty=\max_{1\le i\le n-1}\left\{\frac{i}{n}-\frac{1}{2}\right\}=\frac{n-2}{2n}.$$

□

Remark 3.6. The first inequality from (3.3) was obtained by Dragomir in [7] and also an incorrect version of the first inequality from (3.4), viz.:

$$\left\|\frac{x_1+x_n}{2}-\frac{1}{n}\sum_{i=1}^{n}x_i\right\|\le\begin{cases}\frac{k-1}{2}\left\|\Delta x\right\|_\infty, & \text{if } n=2k,\\ \frac{2k^2+2k+1}{2(2k+1)}\left\|\Delta x\right\|_\infty, & \text{if } n=2k+1.\end{cases}$$

The second coefficient $\frac{2k^2+2k+1}{2(2k+1)}$ should be $\frac{k^2}{2k+1}$ since

$$\frac{1}{2k+1}\left((2k+1)-1-\left\lfloor\frac{2k+1}{2}\right\rfloor\right)\left(\left\lfloor\frac{2k+1}{2}\right\rfloor\right)=\frac{k^2}{2k+1}.$$

4. ESTIMATES OF THE DIFFERENCES BETWEEN TWO WEIGHTED ARITHMETIC MEANS

In this section we will give the estimates of the differences between two weighted arithmetic means using the discrete weighted Montgomery (Euler) identity. We suppose $l,m,n\in\mathbb{N}$. The first method is by subtracting two weighted Montgomery identities. The second is by summing the discrete weighted Montgomery identity. Both methods are possible for both the case $1\le l\le m\le n$, i.e. $[l,m]\subseteq[1,n]$ and the case $1\le l\le n\le m$, i.e. $[1,n]\cap[l,m]=[l,n]$.

Theorem 4.1. *Let $(X,\|\cdot\|)$ be a normed linear space, $x_1,x_2,\dots,x_{\max\{m,n\}}$ a finite sequence of vectors in X, $l,m,n\in\mathbb{N}$, $w_1,w_2,\dots,w_n$ and $u_l,u_{l+1},\dots,u_m$, two finite sequences of positive real numbers. Let also $W=\sum_{i=1}^n w_i$, $U=\sum_{i=l}^m u_i$ and for $k\in\mathbb{N}$*

$$W_k=\begin{cases}\sum_{i=1}^{k}w_i, & 1\le k\le n,\\ W, & k>n,\end{cases}$$

$$U_k=\begin{cases}0, & k<l,\\ \sum_{i=l}^{k}u_i & l\le k\le m,\\ U, & k>m.\end{cases}\tag{4.1}$$

If $[1,n] \cap [l,m] \neq \emptyset$*, then, for both cases* $[l,m] \subseteq [1,n]$ *and* $[1,n] \cap [l,m] = [l,n]$*, the next formula is valid*

$$\frac{1}{W}\sum_{i=1}^{n} w_i x_i - \frac{1}{U}\sum_{i=l}^{m} u_i x_i = \sum_{i=1}^{\max\{m,n\}} K(i)\,\Delta x_i, \tag{4.2}$$

where

$$K(i) = \frac{U_i}{U} - \frac{W_i}{W}, \quad 1 \leq i \leq \max\{m,n\}.$$

Proof. For $k \in ([1,n] \cap [l,m]) \cap \mathbb{N}$, we subtract the identities

$$x_k = \frac{1}{W}\sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n-1} D_w(k,i)\,\Delta x_i,$$

and

$$x_k = \frac{1}{U}\sum_{i=l}^{m} u_i x_i + \sum_{i=l}^{m-1} D_u(k,i)\,\Delta x_i.$$

Then put

$$K(k,i) = D_u(k,i) - D_w(k,i).$$

As $K(k,i)$ does not depend on k we write simply $K(i)$:

$$K(i) = \begin{cases} -\frac{W_i}{W}, & 1 \leq i \leq l-1, \\ \frac{U_i}{U} - \frac{W_i}{W}, & l \leq i \leq m, \\ 1 - \frac{W_i}{W}, & m+1 \leq i \leq n, \end{cases} \quad \text{if} \quad [l,m] \subseteq [1,n], \tag{4.3}$$

$$K(i) = \begin{cases} -\frac{W_i}{W}, & 1 \leq i \leq l-1, \\ \frac{U_i}{U} - \frac{W_i}{W}, & l \leq i \leq n, \\ \frac{U_i}{U} - 1, & n+1 \leq i \leq m, \end{cases} \quad \text{if} \quad [1,n] \cap [l,m] = [l,n]. \tag{4.4}$$

□

Theorem 4.2. *Let all assumptions from Theorem 4.1 hold and* (p,q) *be a pair of conjugate exponents. Then we have*

$$\left\| \frac{1}{W}\sum_{i=1}^{n} w_i x_i - \frac{1}{U}\sum_{i=l}^{m} u_i x_i \right\| \leq \|K\|_q \|\Delta x\|_p.$$

The constant $\|K\|_q$ *is sharp for* $1 \leq p \leq \infty$*.*

Proof. We use the identity (4.2) and apply the Hölder inequality to obtain

$$\left| \frac{1}{W}\sum_{i=1}^{n} w_i x_i - \frac{1}{U}\sum_{i=l}^{m} u_i x_i \right| = \left| \sum_{i=1}^{\max\{m,n\}} K(i)\,\Delta x_i \right| \leq \|K\|_q \|\Delta x\|_p.$$

For the proof of the sharpness of the constant $\|K\|_q$, we will find x, a finite sequence of vectors in X such that

$$\left|\sum_{i=1}^{\max\{m,n\}} K(i)\,\Delta x_i\right| = \left(\sum_{i=1}^{\max\{m,n\}} |K(i)|^q\right)^{\frac{1}{q}} \|\Delta x\|_p .$$

For $1 < p < \infty$ take x to be such that

$$\Delta x_i = \operatorname{sgn} K(i) \cdot |K(i)|^{\frac{1}{p-1}} .$$

For $p = \infty$ take

$$\Delta x_i = \operatorname{sgn} K(i) .$$

For $p = 1$ we will find a finite sequence of vectors x such that

$$\left|\sum_{i=1}^{\max\{m,n\}} K(i)\,\Delta x_i\right| = \max_{1\le i\le \max\{m,n\}} |K(i)| \left(\sum_{i=1}^{\max\{m,n\}} |\Delta x_i|\right) .$$

Suppose that $|K(i)|$ attains its maximum at $i_0 \in ([1,n] \cup [l,m]) \cap \mathbb{N}$. First we assume that $K(i_0) > 0$. Define x such that $\Delta x_{i_0} = 1$ and $\Delta x_i = 0$, $i \neq i_0$, i.e.

$$x_i = \begin{cases} 0, & 1 \le i \le i_0, \\ 1, & i_0 + 1 < i \le \max\{m,n\} . \end{cases}$$

Then,

$$\left|\sum_{i=1}^{\max\{m,n\}} K(i)\,\Delta x_i\right| = |K(i_0)| = \max_{1\le i\le \max\{m,n\}} |K(i)| \left(\sum_{i=1}^{\max\{m,n\}} |\Delta x_i|\right),$$

and the statement follows. In the case $K(i_0) < 0$, we take x such that $\Delta x_{i_0} = -1$ and $\Delta x_i = 0$, $i \neq i_0$, i.e.

$$x_i = \begin{cases} 1, & 1 \le i \le i_0, \\ 0, & i_0 + 1 \le i \le \max\{m,n\} , \end{cases}$$

and the rest of proof is the same as above. □

Corollary 4.3. *Assume all assumptions from the Theorem 4.2 hold and additionally assume* $1 \le l < m \le n$. *Then we have*

$$\left\|\frac{1}{W}\sum_{i=1}^{n} w_i x_i - \frac{1}{U}\sum_{i=l}^{m} u_i x_i\right\|$$
$$\le \begin{cases} \left[\sum_{i=1}^{l-1}\left|\frac{W_i}{W}\right| + \sum_{i=l}^{m}\left|\frac{Ui}{U} - \frac{W_i}{W}\right| + \sum_{i=m+1}^{n}\left|1 - \frac{W_i}{W}\right|\right] \|\Delta x\|_\infty , \\ \left[\sum_{i=1}^{l-1}\left|\frac{W_i}{W}\right|^q + \sum_{i=l}^{m}\left|\frac{Ui}{U} - \frac{W_i}{W}\right|^q + \sum_{i=m+1}^{n}\left|1 - \frac{W_i}{W}\right|^q\right]^{\frac{1}{q}} \|\Delta x\|_p , \\ \max\left\{\frac{W_{l-1}}{W}, 1 - \frac{W_{m+1}}{W}, \max_{l\le i\le m}\left|\frac{Ui}{U} - \frac{W_i}{W}\right|\right\} \|\Delta x\|_1 , \end{cases}$$

and for $1 \le l < n \le m$

$$\left|\frac{1}{W}\sum_{i=1}^{n} w_i x_i - \frac{1}{U}\sum_{i=l}^{m} u_i x_i\right| \le \begin{cases} \left[\sum_{i=1}^{l-1}\left|\frac{W_i}{W}\right| + \sum_{i=l}^{n}\left|\frac{Ui}{U} - \frac{W_i}{W}\right| + \sum_{i=n+1}^{m}\left|\frac{U_i}{U} - 1\right|\right] \|\Delta x\|_\infty, \\ \left[\sum_{i=1}^{l-1}\left|\frac{W_i}{W}\right|^q + \sum_{i=l}^{n}\left|\frac{Ui}{U} - \frac{W_i}{W}\right|^q + \sum_{i=n+1}^{m}\left|\frac{U_i}{U} - 1\right|^q\right]^{\frac{1}{q}} \|\Delta x\|_p, \\ \max\left\{\frac{W_{l-1}}{W}, 1 - \frac{U_{n+1}}{U}, \max_{l \le i \le n}\left|\frac{Ui}{U} - \frac{W_i}{W}\right|\right\} \|\Delta x\|_1. \end{cases}$$

Proof. Directly from the Theorem 4.2. □

Remark 4.4. If we suppose $n = m$ in both of the cases $1 \le l < m \le n$ and $1 \le l < n \le m$,then the analogous results coincides.

Remark 4.5. By setting $l = m = k$ and $u_k = 1$ in the first inequality from Corollary 4.3 we get the weighted Ostrowski inequality from Corollary 3.2.

Corollary 4.6. *If all assumptions from Theorem 4.2 hold and, in addition, assume* $1 \le k \le m$*, then we have*

$$\left|\frac{1}{W}\sum_{i=1}^{k} w_i x_i - \frac{1}{U}\sum_{i=k}^{m} u_i x_i\right| \le \begin{cases} \left[\sum_{i=1}^{k-1}\left|\frac{W_i}{W}\right| + \left|\frac{U_k}{U} - \frac{W_k}{W}\right| + \sum_{i=k+1}^{m}\left|\frac{U_i}{U} - 1\right|\right] \|\Delta x\|_\infty, \\ \left[\sum_{i=1}^{k-1}\left|\frac{W_i}{W}\right|^q + \left|\frac{U_k}{U} - \frac{W_k}{W}\right|^q + \sum_{i=k+1}^{m}\left|\frac{U_i}{U} - 1\right|^q\right]^{\frac{1}{q}} \|\Delta x\|_p, \\ \max\left\{\frac{W_{k-1}}{W}, 1 - \frac{U_{k+1}}{U}, \left|\frac{U_k}{U} - \frac{W_k}{W}\right|\right\} \|\Delta x\|_1. \end{cases}$$

Proof. By setting $n = l = k$ in the second inequality from the Corollary 4.3. □

The second method of giving an estimate of the difference between the two weighted arithmetic means is by summing the weighted Montgomery identity. In this way we also get formula (4.2). For the case $1 \le l \le m \le n$ let $j \in \{l, l+1, \dots, m\}$, from (2.3) we have

$$u_j x_j = u_j \frac{1}{W}\sum_{i=1}^{n} w_i x_i + u_j \sum_{i=1}^{n-1} D_w(j,i)\,\Delta x_i,$$

so

$$\frac{1}{U}\sum_{j=l}^{m} u_j x_j = \left(\frac{1}{U}\sum_{j=l}^{m} u_j\right)\frac{1}{W}\sum_{i=1}^{n} w_i x_i + \frac{1}{U}\sum_{j=l}^{m} u_j \sum_{i=1}^{n-1} D_w(j,i)\,\Delta x_i.$$

By interchange of the order of summation we get

$$\begin{aligned}
&\frac{1}{U}\sum_{j=l}^{m}u_jx_j-\frac{1}{W}\sum_{i=1}^{n}w_ix_i\\
&\qquad=\frac{1}{U}\sum_{j=l}^{m}u_j\sum_{i=1}^{j-1}\frac{W_i}{W}\Delta x_i+\frac{1}{U}\sum_{j=l}^{m}u_j\sum_{i=j}^{n-1}\left(\frac{W_i}{W}-1\right)\Delta x_i\\
&\qquad=\frac{1}{U}\sum_{i=1}^{l-1}\sum_{j=l}^{m}u_j\frac{W_i}{W}\Delta x_i+\frac{1}{U}\sum_{i=l}^{m-1}\sum_{j=i+1}^{m}u_j\frac{W_i}{W}\Delta x_i\\
&\qquad\qquad+\frac{1}{U}\sum_{i=l}^{m-1}\sum_{j=l}^{i}u_j\left(\frac{W_i}{W}-1\right)\Delta x_i+\frac{1}{U}\sum_{i=m}^{n-1}\sum_{j=l}^{m}u_j\left(\frac{W_i}{W}-1\right)\Delta x_i\\
&\qquad=\sum_{i=1}^{l-1}\frac{W_i}{W}\Delta x_i+\sum_{i=l}^{m-1}\left(1-\frac{U_i}{U}\right)\frac{W_i}{W}\Delta x_i\\
&\qquad\qquad+\sum_{i=l}^{m-1}\frac{U_i}{U}\left(\frac{W_i}{W}-1\right)\Delta x_i+\sum_{i=m}^{n-1}\left(\frac{W_i}{W}-1\right)\Delta x_i\\
&\qquad=\sum_{i=1}^{l-1}\frac{W_i}{W}\Delta x_i+\sum_{i=l}^{m}\left(\frac{W_i}{W}-\frac{U_i}{U}\right)\Delta x_i+\sum_{i=m+1}^{n-1}\left(\frac{W_i}{W}-1\right)\Delta x_i.
\end{aligned}$$

This identity is equivalent to (4.2) with (4.3).

For case $1\le l\ \le n\le m$, let $j\in\{l,l+1,\dots,n\}$, and again from (2.3) we have

$$u_jx_j=u_j\frac{1}{W}\sum_{i=1}^{n}w_ix_i+u_j\sum_{i=1}^{n-1}D_w\left(j,i\right)\Delta x_i,$$

so

$$\sum_{j=l}^{n}u_jx_j=\sum_{j=l}^{n}u_j\frac{1}{W}\sum_{i=1}^{n}w_ix_i+\sum_{j=l}^{n}u_j\sum_{i=1}^{n-1}D_w\left(j,i\right)\Delta x_i$$

and

$$\begin{aligned}
&\frac{1}{U}\sum_{j=l}^{m}u_jx_j-\frac{1}{U}\sum_{j=n+1}^{m}u_jx_j\\
&\qquad=\left(\frac{1}{U}\sum_{j=l}^{m}u_j\right)\frac{1}{W}\sum_{i=1}^{n}w_ix_i-\left(\frac{1}{U}\sum_{j=n+1}^{m}u_j\right)\frac{1}{W}\sum_{i=1}^{n}w_ix_i\\
&\qquad\qquad+\left(\frac{1}{U}\sum_{j=l}^{n}u_j\right)\sum_{i=1}^{n-1}D_w\left(j,i\right)\Delta x_i.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{U}\sum_{j=l}^{m}u_jx_j-\frac{1}{W}\sum_{i=1}^{n}w_ix_i\\
&\qquad=\frac{1}{U}\sum_{j=n+1}^{m}u_j\left(x_j-\frac{1}{W}\sum_{i=1}^{n}w_ix_i\right)+\frac{1}{U}\sum_{j=l}^{n}u_j\sum_{i=1}^{n-1}D_w\left(j,i\right)\Delta x_i.
\end{aligned}$$

Since for $j \in \{n+1, n+2, \dots, m\}$

$$x_j - \frac{1}{W}\sum_{i=1}^{n} w_i x_i = \sum_{i=n}^{j-1} \Delta x_i + x_n - \frac{1}{W}\sum_{i=1}^{n} w_i x_i$$
$$= \sum_{i=n}^{j-1} \Delta x_i + \sum_{i=1}^{n-1} D_w(n,i)\,\Delta x_i,$$

we have

$$\frac{1}{U}\sum_{j=l}^{m} u_j x_j - \frac{1}{W}\sum_{i=1}^{n} w_i x_i$$
$$= \frac{1}{U}\sum_{j=l}^{n} u_j \sum_{i=1}^{n-1} D_w(j,i)\,\Delta x_i + \frac{1}{U}\sum_{j=n+1}^{m} u_j \left(\sum_{i=1}^{n-1} D_w(n,i)\,\Delta x_i + \sum_{i=n}^{j-1}\Delta x_i\right).$$

By interchange of the order of summation we get

$$\frac{1}{U}\sum_{j=l}^{m} u_j x_j - \frac{1}{W}\sum_{i=1}^{n} w_i x_i$$
$$= \frac{1}{U}\sum_{j=l}^{n} u_j \sum_{i=1}^{j-1} \frac{W_i}{W}\Delta x_i + \frac{1}{U}\sum_{j=l}^{n} u_j \sum_{i=j}^{n-1}\left(\frac{W_i}{W}-1\right)\Delta x_i$$
$$+ \frac{1}{U}\sum_{j=n+1}^{m} u_j \sum_{i=1}^{n-1} D_w(n,i)\,\Delta x_i + \frac{1}{U}\sum_{j=n+1}^{m} u_j \sum_{i=n}^{j-1}\Delta x_i$$
$$= \frac{1}{U}\sum_{i=1}^{l-1}\sum_{j=l}^{n} u_j \frac{W_i}{W}\Delta x_i + \frac{1}{U}\sum_{i=l}^{n-1}\sum_{j=i+1}^{n} u_j \frac{W_i}{W}\Delta x_i + \frac{1}{U}\sum_{i=l}^{n-1}\sum_{j=l}^{i} u_j\left(\frac{W_i}{W}-1\right)\Delta x_i$$
$$+ \frac{1}{U}\sum_{i=1}^{n-1}\sum_{j=n+1}^{m} u_j D_w(n,i)\,\Delta x_i + \frac{1}{U}\sum_{i=n+1}^{m-1}\sum_{j=i+1}^{m} u_j \Delta x_i$$
$$= \frac{U_n}{U}\sum_{i=1}^{l-1}\frac{W_i}{W}\Delta x_i + \sum_{i=l}^{n-1}\left(\frac{U_n}{U}-\frac{U_i}{U}\right)\frac{W_i}{W}\Delta x_i + \sum_{i=l}^{n-1}\frac{U_i}{U}\left(\frac{W_i}{W}-1\right)\Delta x_i$$
$$+ \left(1-\frac{U_n}{U}\right)\sum_{i=1}^{n-1}\frac{W_i}{W}\Delta x_i + \sum_{i=n+1}^{m-1}\left(1-\frac{U_i}{U}\right)\Delta x_i$$
$$= \sum_{i=1}^{l-1}\frac{W_i}{W}\Delta x_i + \sum_{i=l}^{n}\left(\frac{W_i}{W}-\frac{U_i}{U}\right)\Delta x_i + \sum_{i=n+1}^{m-1}\left(1-\frac{U_i}{U}\right)\Delta x_i.$$

This identity is equivalent to (4.2) with (4.4)

The next theorem is the generalization of Theorem 4.2.

Theorem 4.7. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_1, x_2, \dots, x_{\max\{m,n\}}$ a finite sequence of vectors in X, $w_1, w_2, \dots, w_n$ and $u_l, u_{l+1}, \dots, u_m$ finite sequences of positive real numbers and (p,q) a pair of conjugate exponents. Then for all $s \in \{2, 3, \dots, n-1\}$ and $k \in$*

$([1,n]\cap[l,m])\cap\mathbb{N}$ *the following inequality is valid*

$$\left\| \frac{1}{W}\sum_{i=1}^{n} w_i x_i - \frac{1}{U}\sum_{i=l}^{m} w_i x_i \right.$$
$$+ \sum_{r=1}^{s-1} \frac{\left(\sum_{i=1}^{n-r}\Delta^r x_i\right)}{n-r} \left(\sum_{i_1=1}^{n-1}\cdots\sum_{i_r=1}^{n-r} D_w(k,i_1) D_{n-1}(i_1,i_2)\cdots D_{n-r+1}(i_{r-1},i_r) \right)$$
$$\left. - \sum_{r=1}^{s-1} \frac{\left(\sum_{i=l}^{m-r}\Delta^r x_i\right)}{m-r} \left(\sum_{i_1=l}^{m-1}\cdots\sum_{i_r=l}^{m-r} D_u(k,i_1) D_{m-l}(i_1,i_2)\cdots D_{m-l-r+2}(i_{r-1},i_r) \right) \right\|$$
$$\leq \left\|\mathbf{K}(k,\cdot)\right\|_q \left\|\Delta^s x\right\|_p ,$$

where

$$\mathbf{K}(k,i_m) = \sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_{s-1}=1}^{n-s+1} D_w(k,i_1) D_{n-1}(i_1,i_2)\cdots D_{n-s+1}(i_{s-1},i_s)$$
$$- \sum_{i_1=l}^{m-1}\sum_{i_2=l}^{m-2}\cdots\sum_{i_{s-1}=l}^{m-s+1} D_u(k,i_1) D_{m-l}(i_1,i_2)\cdots D_{m-l-s+2}(i_{s-1},i_s)$$

and we suppose that

$$\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_{s-1}=1}^{n-s+1} D_w(k,i_1) D_{n-1}(i_1,i_2)\cdots D_{n-s+1}(i_{s-1},i_s) = 0, \quad \text{for } i_s \notin [1,n]\cap\mathbb{N}$$

and

$$\sum_{i_1=l}^{m-1}\sum_{i_2=l}^{m-2}\cdots\sum_{i_{s-1}=l}^{m-s+1} D_u(k,i_1) D_{m-l}(i_1,i_2)\cdots D_{m-l-s+2}(i_{s-1},i_s) = 0, \quad \text{for } i_s \notin [l,m]\cap\mathbb{N}.$$

The constant $\left\|\mathbf{K}(k,\cdot)\right\|_q$ *is sharp for* $1\leq p\leq\infty$

Proof. As in Theorem 4.2, we subtract two weighted Montgomery identities, one for the interval $[1,n]\cap\mathbb{N}$ and the other for $[l,m]\cap\mathbb{N}$. After that, our inequality follows by applying Hölder's inequality. The proof for the sharpness of the constant $\left\|\mathbf{K}(k,\cdot)\right\|_q$ is similar to the proof of Theorem 4.2 (with $\mathbf{K}(k,\cdot)$ instead of K and $\Delta^s x$ instead of Δx). □

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