Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 3, Article 76, 2005

GEOMETRIC INEQUALITIES FOR A SIMPLEX

SHIGUO YANG

DEPARTMENT OF MATHEMATICS
ANHUI INSTITUTE OF EDUCATION
HEFEI, 230061, P.R. CHINA.
sxx@ahieedu.net.cn

Received 18 October, 2004; accepted 11 March, 2005
Communicated by J. Sándor

ABSTRACT. In this paper, we study a problem of geometric inequalities for an n-simplex. Some new geometric inequalities for a simplex are established. As special cases, some known inequalities are deduced.

Key words and phrases: Simplex, Volume, Inradius, Circumradius, Inequality.

2000 *Mathematics Subject Classification.* 52A40, 51K16.

1. INTRODUCTION

Let σ_n be an n-dimensional simplex in the n-dimensional Euclidean space E^n, $\tau = \{A_0, A_1, \dots, A_n\}$ denote the vertex set of σ_n, V the volume of σ_n, R and r the circumradius and inradius of σ_n, respectively. For $i = 0, 1, \dots, n$, let r_i be the radius of ith escribed sphere of σ_n, F_i the area of the ith face $f_i = A_0 \cdots A_{i-1}A_{i+1} \cdots A_n$ of σ_n. Let P be an arbitrary interior point of the simplex σ_n, d_i the distance from the point P to the ith face f_i of σ_n, h_i the altitude of σ_n from vertex A_i for $i = 0, 1, \dots, n$.

Let $a_0,\ a_1$ and a_2 denote the edge-lengths of triangle $A_0A_1A_2$ (2-dimensional simplex). An important inequality for a triangle was established by Janić (see [1]) as follows:

$$\frac{a_0^2}{r_1r_2} + \frac{a_1^2}{r_2r_0} + \frac{a_2^2}{r_0r_1} \geq 4. \tag{1.1}$$

Let P be an arbitrary interior point of the triangle $A_0A_1A_2$. Gerasimov (see [2]) obtained an inequality for the triangle $A_0A_1A_2$ as follows:

$$\frac{d_1d_2}{a_1a_2} + \frac{d_2d_0}{a_2a_0} + \frac{d_0d_1}{a_0a_1} \leq \frac{1}{4}. \tag{1.2}$$

ISSN (electronic): 1443-5756

The author would like to express his thanks to the editor for his kind help and invaluable suggestions in the formatting and writing of this paper.

096-04

2. MAIN RESULTS

We will extend inequalities (1.1) and (1.2) to an n-dimensional simplex. Our main results are contained in the following theorem:

Theorem 2.1. *For the n-dimensional simplex σ_n we have*

$$\sum_{i=0}^{n} \frac{F_i^{n/(n-1)}}{r_0 \cdots r_{i-1} r_{i+1} \cdots r_n} \geq \frac{(n-1)^n n^{3n^2/2(n-1)}}{n^n (n+1)^{(n-2)/2} (n!)^{n/(n-1)}}, \tag{2.1}$$

with equality iff the simplex σ_n is regular.

By letting $n = 2$ in relation (2.1), inequality (1.1) is reobtained.

Theorem 2.2. *Let P be an arbitrary interior point of the simplex σ_n, and let $\theta \in (0, 1]$ be a real number. Then we have*

$$\sum_{i=0}^{n} \frac{d_0 \cdots d_{i-1} d_{i+1} \cdots d_n}{(F_0 \cdots F_{i-1} F_{i+1} \cdots F_n)^{2\theta - 1}} \leq \frac{(n!)^{2\theta}}{(n+1)^{(n-1)(1-\theta)} n^{n(3\theta-1)}} V^{n-2(n-1)\theta}, \tag{2.2}$$

with equality iff the simplex σ_n is regular and the point P is the circumcenter of σ_n.

If we take $\theta = \frac{n}{2(n-1)}$ in inequality (2.2), we obtain the following corollary:

Corollary 2.3. *Let P be an arbitrary interior point of the simplex σ_n. Then we have*

$$\sum_{i=0}^{n} \frac{d_0 \cdots d_{i-1} d_{i+1} \cdots d_n}{(F_0 \cdots F_{i-1} F_{i+1} \cdots F_n)^{1/(n-1)}} \leq \frac{(n!)^{n/(n-1)}}{(n+1)^{(n-2)/2} n^{n(n+2)/2(n-1)}}, \tag{2.3}$$

with equality iff the simplex σ_n is regular and the point P is the circumcenter of σ_n.

If $n = 2$ in inequality (2.3), then inequality (1.2) follows from inequality (2.3).
By taking $\theta = \frac{1}{2}$ in inequality (2.2), we obtain a generalization of Gerber's inequality as follows:

Corollary 2.4. *Let P be arbitrary interior point of the simplex σ_n. Then*

$$\sum_{i=0}^{n} d_0 \cdots d_{i-1} d_{i+1} \cdots d_n \leq \frac{n!}{(n+1)^{(n-1)/2} n^{n/2}} V, \tag{2.4}$$

with equality iff the simplex σ_n is regular.

Using inequality (2.4) and the arithmetic-geometric mean inequality we get Gerber's inequality [3] as follows:

$$\prod_{i=0}^{n} d_i \leq \frac{(n!)^{(n+1)/n}}{n^{(n+1)/2} (n+1)^{1/2n}} V^{(n+1)/n}. \tag{2.5}$$

Theorem 2.5. *Let P be an arbitrary interior point of the simplex σ_n. Then we have*

$$\sum_{i=0}^{n} \frac{1}{d_0 \cdots d_{i-1} d_{i+1} \cdots d_n} \geq (n+1) n^{n+1} \cdot \frac{r}{R^{n+1}}, \tag{2.6}$$

with equality iff the simplex σ_n is regular and the point P is the circumcenter of σ_n.

If the point P is the incenter I of the simplex σ_n, i.e. $d_i = r (i = 0, 1, \ldots, n)$, then the following n-dimensional Euler inequality stated in [4] is obtained from (2.6):

$$R \geq nr. \tag{2.7}$$

3. LEMMAS AND PROOFS OF THEOREMS

To prove the theorems stated above, we need some lemmas as follows.

Let m_i $(i = 0, 1, \ldots, n)$ be positive numbers, $V_{i_0 i_1 \cdots i_k}$ denote the k-dimensional volume of the k-dimensional simplex $A_{i_0} A_{i_1} \cdots A_{i_k}$ for $A_{i_0}, A_{i_1}, \ldots, A_{i_k} \in \tau$. Put

$$M_k = \sum_{0 \le i_0 < i_1 < \cdots < i_k \le n} m_{i_0} m_{i_1} \cdots m_{i_k} V^2_{i_0 i_1 \cdots i_k}, \qquad (1 \le k \le n),$$

$$M_0 = \sum_{i=0}^{n} m_i.$$

Lemma 3.1. *For positive numbers m_i $(i = 0, 1, \ldots, n)$ and the n-dimensional simplex σ_n, we have*

$$M_k^l \ge \frac{[(n-l)!(l!)^3]^k}{[(n-k)!(k!)^3]^l} (n! \cdot M_0)^{l-k} M_l^k, \qquad (1 \le k < l \le n), \tag{3.1}$$

with equality iff the simplex σ_n is regular and $m_0 = m_1 = \cdots = m_n$.

Lemma 3.2.

$$\left(\prod_{i=0}^{n} F_i \right)^{\frac{n}{n^2-1}} \ge \frac{1}{(n+1)^{1/2}} \left(\frac{n^{3n}}{n!^2} \right)^{\frac{1}{2(n-1)}} V^n, \tag{3.2}$$

with equality iff the simplex σ_n is regular.

For the proof of Lemmas 3.1 and 3.2, the reader is referred to [5] or [1].

Lemma 3.3.

$$\sum_{i=0}^{n} \frac{h_0 \cdots h_{i-1} h_{i+1} \cdots h_n}{r_0 \cdots r_{i-1} r_{i+1} \cdots r_n} \ge (n+1)(n_1)^n, \tag{3.3}$$

with equality iff the simplex σ_n is regular.

For the proof of Lemma 3.3, see [5].

Lemma 3.4.

$$V \ge \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!} r^n, \tag{3.4}$$

with equality iff the simplex σ_n is regular.

This is also known, see [5] or [1].

Proof of Theorem 2.1. Without loss of generality, let $F_0 \le F_1 \le \cdots \le F_n$. By the known formula ([1])

$$r_i = \frac{nV}{\sum_{j=0}^{n} F_j - 2F_i}, \qquad (i = 0, 1, \ldots, n), \tag{3.5}$$

it follows that $r_0 \le r_1 \le \cdots \le r_n$ and

$$\frac{1}{\prod\limits_{j=1} F_j r_j} \le \frac{1}{\prod\limits_{\substack{j=0 \\ j \ne i}} F_j r_j} \le \cdots \le \frac{1}{\prod\limits_{\substack{j=0 \\ j \ne n}} F_j r_j}.$$

Using the Chebyshev inequality, we have

$$\sum_{i=0}^{n}\frac{F_i^{n/(n-1)}}{\prod\limits_{\substack{j=0\\ j\neq i}} r_j}=\left(\prod_{i=0}^{n}F_i\right)\sum_{i=0}^{n}\frac{F_i^{1/(n-1)}}{\prod\limits_{\substack{j=0\\ j\neq i}} F_j r_j}\ge\frac{1}{n+1}\left(\prod_{i=0}^{n}F_i\right)\left(\sum_{i=0}^{n}F_i^{1/(n-1)}\right)\left(\sum_{i=0}^{n}\frac{1}{\prod\limits_{\substack{j=0\\ j\neq i}} F_j r_j}\right). \tag{3.6}$$

Substituting $F_j=\frac{nV}{h_j}$ $(j=0,1,\dots,n)$ into the right side of inequality (3.6) and using the arithmetic-geometric mean inequality we get

$$\sum_{i=0}^{n}\frac{F_i^{n/(n-1)}}{\prod\limits_{\substack{j=0\\ j\neq i}} r_j}\ge\frac{1}{n+1}\left(\prod_{i=0}^{n}F_i\right)\left(\sum_{i=0}^{n}F_i^{1/(n-1)}\right)\cdot\frac{1}{(nV)^n}\sum_{i=0}^{n}\frac{h_0\cdots h_{i-1}h_{i+1}\cdots h_n}{r_0\cdots r_{i-1}r_{i+1}\cdots r_n}\ge\frac{\left(\prod\limits_{i=0}^{n}F_i\right)^{\frac{n^2}{(n^2-1)}}}{(nV)^n}\sum_{i=0}^{n}\frac{h_0\cdots h_{i-1}h_{i+1}\cdots h_n}{r_0\cdots r_{i-1}r_{i+1}\cdots r_n}. \tag{3.7}$$

By inequalities (3.7), (3.2) and (3.3) we obtain relation (2.1). It is easy to see that equality in (2.1) holds iff the simplex σ_n is regular. The proof of Theorem 2.1 is thus complete. □

Proof of Theorem 2.2. Taking $k=n-1,\ l=n$ in inequality (3.1), we can write

$$\left(\sum_{i=0}^{n}m_0\cdots m_{i-1}m_{i+1}\cdots m_nF_i^2\right)^n\ge\frac{n^{3n}}{n!^2}\left(\sum_{i=0}^{n}m_i\right)\left(\prod_{i=0}^{n}m_i\right)^{n-1}V^{2(n-1)}. \tag{3.8}$$

By putting $m_0\cdots m_{i-1}m_{i+1}\cdots m_n=\lambda_iF_i^{-2}$ $(i=0,1,\dots,n)$ in equality (3.8), we get

$$\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n\left(\prod_{i=0}^{n}F_i^2\right)\ge\frac{(nV)^{2(n-1)}}{(n-1)!^2}\left(\prod_{i=0}^{n}\lambda_i\right)\left(\sum_{i=0}^{n}\frac{F_i^2}{\lambda_i}\right). \tag{3.9}$$

We now prove that the following inequality (3.10) is valid for any number $\theta\in(0,1]$:

$$\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n\prod_{i=0}^{n}F_i^{2\theta}\ge\left(\sum_{i=0}^{n}\lambda_i\right)^n\left(\sum_{i=0}^{n}\frac{F_i^{2\theta}}{\lambda_i}\right)\frac{(n+1)^{2(n-1)\theta}}{n^{n(1-\theta)}}\cdot\frac{(nV)^{2(n-1)\theta}}{(n-1)!^{2\theta}}. \tag{3.10}$$

When $\theta=1$, inequalities (3.10) and (3.9) are the same, so inequality (3.10) is valid for $\theta=1$. For $\theta\in(0,1)$, using inequality (3.9) we have

$$\begin{aligned}\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n\prod_{i=0}^{n}F_i^{2\theta}&=\left[\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n\prod_{i=0}^{n}F_i^2\right]^\theta\cdot\left[\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n\right]^{1-\theta}\\&\ge\left[\frac{(nV)^{2(n-1)}}{(n-1)!^2}\left(\prod_{i=0}^{n}\lambda_i\right)\left(\sum_{i=0}^{n}\frac{F_i^2}{\lambda_i}\right)\right]^\theta\cdot\left[\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n\right]^{1-\theta}.\end{aligned} \tag{3.11}$$

By Maclaurin's inequality ([1]) we have

$$\left(\frac{1}{n+1}\sum_{i=0}^{n}\lambda_0\cdots\lambda_{i-1}\lambda_{i+1}\cdots\lambda_n\right)^{\frac{1}{n}} \le \frac{1}{n+1}\sum_{i=0}^{n}\lambda_i,$$

i.e.

$$\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^{n} \ge \frac{(n+1)^{n-1}}{n^n}\left(\prod_{i=0}^{n}\lambda_i\right)\left(\sum_{i=0}^{n}\frac{1}{\lambda_i}\right). \tag{3.12}$$

From (3.11) and (3.12) we can write

$$\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^{n}\prod_{i=0}^{n}F_i^{2\theta} \ge \left(\sum_{i=0}^{n}\lambda_i\right)\left[\sum_{i=0}^{n}\left(\frac{F_i^{2\theta}}{\lambda_i^{\theta}}\right)^{\frac{1}{\theta}}\right]^{\theta}\cdot\left[\sum_{i=0}^{n}\left(\frac{1}{\lambda_i^{1-\theta}}\right)^{\frac{1}{1-\theta}}\right]^{1-\theta} \times\left[\frac{(n+1)^{n-1}}{n^n}\right]^{1-\theta}\left[\frac{(nV)^{2(n-1)}}{(n-1)!^2}\right]^{\theta}. \tag{3.13}$$

By Hölder's inequality ([1]) we have

$$\left[\sum_{i=0}^{n}\left(\frac{F_i^{2\theta}}{\lambda_i^{\theta}}\right)^{\frac{1}{\theta}}\right]^{\theta}\cdot\left[\sum_{i=0}^{n}\left(\frac{1}{\lambda_i^{1-\theta}}\right)^{\frac{1}{1-\theta}}\right]^{1-\theta} \ge \sum_{i=0}^{n}\frac{F_i^{2\theta}}{\lambda_i}. \tag{3.14}$$

Using (3.13) and (3.14) we get relation (3.9).

Taking $\lambda_i = d_iF_i\ (i=0,1,\dots,n)$ in equality (3.9) and noting the fact that $\sum_{i=0}^{n} d_iF_i = nV$, we get inequality (2.2). It is easy to prove that equality in (2.2) holds iff the simplex σ_n is regular and the point P is the circumcenter of σ_n. The proof of Theorem 2.2 is thus complete. □

Proof of Theorem 2.5. Inequality (3.9) can be written also as

$$\frac{n^{3n}}{n!^2}V^{2(n-1)}\sum_{i=0}^{n}\lambda_0\cdots\lambda_{i-1}\lambda_{i+1}\cdots\lambda_nF_i^2 \le \left(\sum_{i=0}^{n}\lambda_i\right)^{n}\prod_{i=0}^{n}F_i^2. \tag{3.15}$$

Let V' denote the volume of the n-dimensional simplex $\sigma_n' = A_0'A_1'\cdots A_n',\ F_i'$ being the area of the ith face f_i' of σ_n'. By Cauchy's inequality and inequality (3.15), we have

$$\begin{aligned}\frac{n^{3n}}{n!^2}&V^{n-1}(V')^{n-1}\sum_{i=0}^{n}\lambda_0\cdots\lambda_{i-1}\lambda_{i+1}\cdots\lambda_nF_iF_i'\\ &\le\left[\frac{n^{3n}}{n!^2}V^{2(n-1)}\sum_{i=0}^{n}\lambda_0\cdots\lambda_{i-1}\lambda_{i+1}\cdots\lambda_nF_i^2\right]^{\frac{1}{2}}\\ &\qquad\times\left[\frac{n^{3n}}{n!^2}(V')^{2(n-1)}\sum_{i=0}^{n}\lambda_0\cdots\lambda_{i-1}\lambda_{i+1}\cdots\lambda_n(F_i')^2\right]^{\frac{1}{2}}\\ &\le\left(\sum_{i=0}^{n}\lambda_i\right)^{n}\left(\prod_{i=0}^{n}F_i\right)\left(\prod_{i=0}^{n}F_i'\right).\end{aligned} \tag{3.16}$$

If we suppose that σ_n' is a regular simplex with $F_0' = F_1' = \cdots = F_n' = 1$. then

$$V' = (n+1)^{1/2}\left(\frac{n!^2}{n^{3n}}\right)^{\frac{1}{2(n-1)}},$$

so inequality (3.16) becomes

$$\frac{(n+1)^{(n-1)/2}n^{3n/2}}{n!}V^{n-1}\sum_{i=0}^{n}\lambda_0\cdots\lambda_{i-1}\lambda_{i+1}\cdots\lambda_n F_i \le \left(\sum_{i=0}^{n}\lambda_i\right)^n \prod_{i=0}^{n} F_i. \tag{3.17}$$

By letting $\lambda_0 = \lambda_1 = \cdots = \lambda_n = 1$ in inequality (3.17), we get

$$\frac{1}{V} \ge \frac{n^{3n/2(n-1)}}{n!^{1/(n-1)}(n+1)^{(n+1)/2(n-1)}}\left(\sum_{i=0}^{n}\frac{1}{F_0\cdots F_{i-1}F_{i+1}\cdots F_n}\right)^{\frac{1}{(n-1)}}. \tag{3.18}$$

Now by Cauchy's inequality we have

$$\left(\sum_{i=0}^{n} d_0\cdots d_{i-1}d_{i+1}\cdots d_n\right)\left(\sum_{i=0}^{n}\frac{1}{d_0\cdots d_{i-1}d_{i+1}\cdots d_n}\right) \ge (n+1)^2,$$

i.e.

$$\sum_{i=0}^{n}\frac{1}{d_0\cdots d_{i-1}d_{i+1}\cdots d_n} \ge \frac{(n+1)^2}{\sum\limits_{i=0}^{n} d_0\cdots d_{i-1}d_{i+1}\cdots d_n}. \tag{3.19}$$

Using (3.19), (2.4) and (3.18), we get

$$\begin{aligned}\sum_{i=0}^{n}\frac{1}{d_0\cdots d_{i-1}d_{i+1}\cdots d_n} &\ge \frac{(n+1)^{(n+3)/2}n^{n/2}}{n!}\cdot\frac{1}{V}\\ &\ge \frac{(n+1)^{(n^2+n-4)/2(n-1)}n^{n^2/2(n-1)}}{(n-1)!^{n/(n-1)}}\left(\sum_{i=0}^{n}F_i\right)^{\frac{1}{(n-1)}}\left(\frac{1}{\prod_{i=0}^{n}F_i}\right)^{\frac{1}{(n-1)}}.\end{aligned} \tag{3.20}$$

By inequality (3.20), formula $\sum_{i=0}^{n} F_i = \frac{nV}{r}$ and the known inequality ([1]):

$$\prod_{i=0}^{n} F_i \le \frac{(n+1)^{(n^2-1)/2}}{n!^{n+1}n^{(n^2-3n-4)/2}}R^{n^2-1}, \tag{3.21}$$

we get

$$\begin{aligned}\sum_{i=0}^{n}\frac{1}{d_0\cdots d_{i-1}d_{i+1}\cdots d_n}\\ \ge (n+1)^{(n-3)/2(n-1)}n^{(2n^2-n-2)/2(n-1)}\cdot n!^{1/(n-1)}\left(\frac{V}{r}\right)^{\frac{1}{(n-1)}}\cdot\frac{1}{R^{n+1}}.\end{aligned} \tag{3.22}$$

Relations (3.22) and (3.4) imply inequality (2.6). It is easy to prove that equality in (2.6) holds iff the simplex σ_n is regular and the point P is the circumcenter of σ_n. The proof of Theorem 2.5 is thus complete. □

REFERENCES

[1] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND V. VOLENEC, *Recent Advances in Geometric Inequalities*, Kluwer Acad. Publ., Dordrecht, Boston, London, 1989, 434–547.

[2] Ju.I. GERASIMOV, Problem 848, *Mat. v Škole*, **4** (1971), 86.

[3] L. GERBER, The orthecentric simplex as an extreme simplex, *Pacific. J. Math.*, **56** (1975), 97–111.

[4] M.S. KLAMKIN, Problem 85–26, *SIAM. Rev.*, **27**(4) (1985), 576.

[5] J.Zh. ZHANG AND L. YANG, A class of geometric inequalities concerning the masspoint system, *J. China Univ. Sci. Technol.*, **11**(2) (1981), 1–8.