

# Journal of Inequalities in Pure and Applied Mathematics

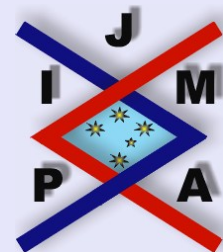
## INEQUALITIES DEFINING CERTAIN SUBCLASSES OF ANALYTIC AND MULTIVALENT FUNCTIONS INVOLVING FRACTIONAL CALCULUS OPERATORS

R.K. RAINA AND I.B. BAPNA

Department Of Mathematics  
M.P. University Of Agri. & Technology  
College Of Technology And Engineering  
Udaipur 313001, Rajasthan, India.  
*EMail:* [rainark\\_7@hotmail.com](mailto:rainark_7@hotmail.com)

Department Of Mathematics,  
Govt. Postgraduate College  
Bhilwara 311001  
Rajasthan, India.  
*EMail:* [bapnain@yahoo.com](mailto:bapnain@yahoo.com)

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## Abstract

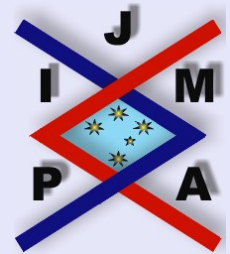
Making use of a certain fractional calculus operator, we introduce two new subclasses  $M_\delta(p; \lambda, \mu, \eta)$  and  $T_\delta(p; \lambda, \mu, \eta)$  of analytic and  $p$ -valent functions in the open unit disk. The results investigated exhibit the sufficiency conditions for a function to belong to each of these classes. Several geometric properties of such multivalent functions follow, and these consequences are also mentioned.

*2000 Mathematics Subject Classification:* 30C45, 26A33.

*Key words:* Analytic functions, Multivalent functions, Starlike functions, Convex functions, Fractional calculus operators.

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# 1. Introduction and Definitions

Let  $\mathcal{A}_p$  denote the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}),$$

which are analytic and  $p$ -valent in the open unit disk  $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ .

A function  $f(z) \in \mathcal{A}_p$  is said to be  $p$ -valently starlike in  $\mathcal{U}$ , if

$$(1.2) \quad \Re \left\{ \frac{z f'(z)}{f(z)} \right\} > 0 \quad (z \in \mathcal{U}),$$

and the function  $f(z) \in \mathcal{A}_p$  is said to be  $p$ -valently convex in  $\mathcal{U}$ , if

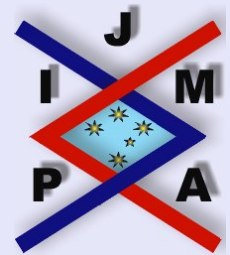
$$(1.3) \quad \Re \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathcal{U}).$$

Further, a function  $f(z) \in \mathcal{A}_p$  is said to be  $p$ -valently close-to-convex in  $\mathcal{U}$ , if

$$(1.4) \quad \Re \left\{ \frac{f'(z)}{z^{p-1}} \right\} > 0 \quad (z \in \mathcal{U}).$$

One may refer to [1], [2] and [9] for above definitions and other related details.

The operator  $J_{0,z}^{\lambda,\mu,\eta}$  occurring in this paper is the Saigo type fractional calculus operator defined as follows ([8]):



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**Definition 1.1.** Let  $0 \leq \lambda < 1$  and  $\mu, \eta \in \mathbb{R}$ , then

$$(1.5) \quad J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz} \left( \frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-t)^{-\lambda} {}_2F_1 \left( \mu - \lambda, 1 - \eta; 1 - \lambda; 1 - \frac{t}{z} \right) f(t) dt \right),$$

where the function  $f(z)$  is analytic in a simply-connected region of the  $z$ -plane containing the origin, with the order

$$f(z) = O(|z|^\varepsilon) \quad (z \rightarrow 0), \quad \text{where } \varepsilon > \max\{0, \mu - \eta\} - 1.$$

It being understood that  $(z-t)^{-\lambda}$  denotes the principal value for  $0 \leq \arg(z-t) < 2\pi$ . The  ${}_2F_1$  function occurring in the right-hand side of (1.5) is the familiar Gaussian hypergeometric function (see [9] for its definition).

**Definition 1.2.** Under the hypotheses of Definition 1.1, a fractional calculus operator  $J_{0,z}^{\lambda+m,\mu+m,\eta+m}$  is defined by ([7])

$$(1.6) \quad J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z) \quad (z \in \mathcal{U}; m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}).$$

We observe that

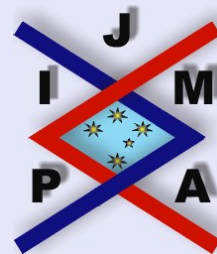
$$(1.7) \quad D_z^\lambda f(z) = J_{0,z}^{\lambda,\lambda,\eta} f(z) \quad (0 \leq \lambda < 1),$$

and

$$(1.8) \quad D_z^{\lambda+m} f(z) = J_{0,z}^{\lambda+m,\lambda+m,\eta+m} f(z) \quad (0 \leq \lambda < 1; m \in \mathbb{N}_0),$$

where  $D_z^{\lambda+m}$  is the well known fractional derivative operator ([6], [9]).

We introduce here two subclasses of functions  $\mathcal{M}_\delta(p; \lambda, \mu, \eta)$  and  $\mathcal{I}_\delta(p; \lambda, \mu, \eta)$  which are defined as follows.



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**Definition 1.3.** Let  $\delta \in \mathbb{R} \setminus \{0\}$ ,  $p \in \mathbb{N}$ ,  $0 \leq \lambda < 1$ ,  $\mu < 1$ , and  $\eta > \max(\lambda, \mu) - p - 1$ . Then the function  $f(z) \in \mathcal{A}_p$  is said to belong to  $\mathcal{M}_\delta(p; \lambda, \mu, \eta)$  if it satisfies the inequality

$$(1.9) \quad \left| \left( \frac{z J_{0,z}^{\lambda+1, \mu+1, \eta+1} f(z)}{J_{0,z}^{\lambda, \mu, \eta} f(z)} \right)^\delta - (p - \mu)^\delta \right| < (p - \mu)^\delta \quad (z \in \mathcal{U}),$$

where the value of  $\left( z J_{0,z}^{\lambda+1, \mu+1, \eta+1} f(z) / J_{0,z}^{\lambda, \mu, \eta} f(z) \right)^\delta$  is taken as its principal value.

**Definition 1.4.** Under the hypotheses of Definition 1.3, the function  $f(z) \in \mathcal{A}_p$  is said to belong to  $\mathcal{T}_\delta(p; \lambda, \mu, \eta)$  if it satisfies the inequality

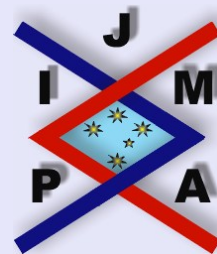
$$(1.10) \quad \left| \left( z^{\mu-p} J_{0,z}^{\lambda, \mu, \eta} f(z) \right)^\delta - \left( \frac{\Gamma(p+1)\Gamma(p+\eta-\mu+1)}{\Gamma(p-\mu+1)\Gamma(p+\eta-\lambda+1)} \right)^\delta \right| < \left( \frac{\Gamma(p+1)\Gamma(p+\eta-\mu+1)}{\Gamma(p-\mu+1)\Gamma(p+\eta-\lambda+1)} \right)^\delta \quad (z \in \mathcal{U}),$$

where the value of  $\left( z^{\mu-p} J_{0,z}^{\lambda, \mu, \eta} f(z) \right)^\delta$  is considered to be its principal value. For  $\lambda = \mu$ , we have

$$(1.11) \quad \mathcal{M}_\delta(p; \mu, \mu, \eta) = \mathcal{M}_\delta(p; \mu),$$

and

$$(1.12) \quad \mathcal{T}_\delta(p; \mu, \mu, \eta) = \mathcal{T}_\delta(p; \mu).$$



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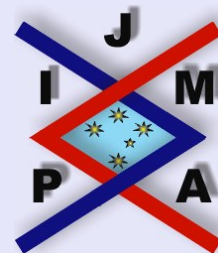
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The classes  $\mathcal{M}_\delta(p; \mu)$  and  $\mathcal{T}_\delta(p; \mu)$  were studied recently in [4]. In view of the operational relation (1.8), it may be noted that the functions in  $\mathcal{M}_1(p; 0)$  are  $p$ -valently starlike in  $\mathcal{U}$ , whereas, the functions in  $\mathcal{T}_1(p; 1)$  are  $p$ -valently close-to-convex in  $\mathcal{U}$ .

In this paper we investigate characterization properties giving sufficiency conditions for functions of the form (1.1) to belong to the classes  $\mathcal{M}_\delta(p; \lambda, \mu, \eta)$  and  $\mathcal{T}_\delta(p; \lambda, \mu, \eta)$  involving the fractional calculus operator (1.6). Several consequences of the main results and their relevance to known results are also pointed out.




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## 2. Results Required

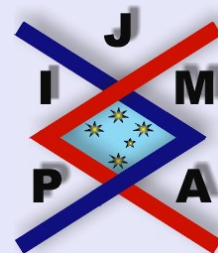
We mention the following results which are used in the sequel:

**Lemma 2.1.** ([8]). *If  $0 \leq \lambda < 1$ ;  $\mu, \eta \in \mathbb{R}$  and  $k > \max\{0, \mu - \eta\} - 1$ , then*

$$(2.1) \quad J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(1+k)\Gamma(1-\mu+\eta+k)}{\Gamma(1-\mu+k)\Gamma(1-\lambda+\eta+k)} z^{k-\mu}.$$

**Lemma 2.2.** ([5]). *Let  $w(z)$  be an analytic function in the unit disk  $\mathcal{U}$  with  $w(0) = 0$ , and let  $0 < r < 1$ . If  $|w(z)|$  attains at  $z_0$  its maximum value on the circle  $|z| = r$ , then*

$$(2.2) \quad z_0 w'(z_0) = k w(z_0) \quad (k \geq 1).$$



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### 3. Main Results

We begin by proving

**Theorem 3.1.** Let  $\delta \in \mathbb{R} \setminus \{0\}$ ,  $p \in \mathbb{N}$ ,  $0 \leq \lambda < 1$ ,  $\mu < 1$ ,  $\eta > \max(\lambda, \mu) - p - 1$ , and  $a > 0, b \geq 0$ , such that  $a + 2b \leq 1$ . If a function  $f(z) \in \mathcal{A}_p$  satisfies the inequality

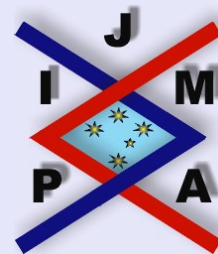
$$(3.1) \quad \Re \left[ 1 + z \left( \frac{J_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} - \frac{J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{J_{0,z}^{\lambda,\mu,\eta} f(z)} \right) \right] \begin{cases} < \frac{a+b}{\delta(1+a)(1-b)} & (\delta > 0) \\ > \frac{a+b}{\delta(1+a)(1-b)} & (\delta < 0) \end{cases} \quad (z \in \mathcal{U}),$$

then  $f(z) \in \mathcal{M}_\delta(p; \lambda, \mu, \eta)$ .

*Proof.* Let  $f(z) \in \mathcal{A}_p$ , and define a function  $w(z)$  by

$$(3.2) \quad \left( \frac{z J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{J_{0,z}^{\lambda,\mu,\eta} f(z)} \right)^\delta = (p - \mu)^\delta \left( \frac{1 + aw(z)}{1 - bw(z)} \right) \quad (z \in \mathcal{U}).$$

Then it follows from (2.1) that  $w(z)$  is analytic function in  $\mathcal{U}$ , and  $w(0) = 0$ .



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Differentiation of (3.2) gives

$$\begin{aligned}
 (3.3) \quad & \left\{ 1 + z \left( \frac{J_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} - \frac{J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{J_{0,z}^{\lambda,\mu,\eta} f(z)} \right) \right\} \\
 &= \frac{1}{\delta} \left( \frac{(a+b)zw'(z)}{(1+aw(z))(1-bw(z))} \right) \\
 &= \phi(z) \text{ (say).}
 \end{aligned}$$

Assume that there exists a point  $z_0 \in \mathcal{U}$  such that

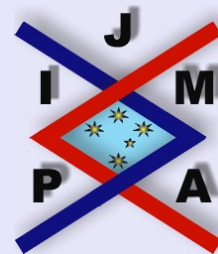
$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, applying Lemma 2.2, we can write

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1),$$

and  $w(z_0) = e^{i\theta}$  ( $\theta \in [0, 2\pi)$ ), so that from (3.3) we have

$$\begin{aligned}
 \Re\{\phi(z_0)\} &= \frac{k(a+b)}{\delta} \Re \left\{ \frac{w(z_0)}{(1+aw(z_0))(1-bw(z_0))} \right\} \\
 &= \frac{k}{\delta} \Re \left\{ \frac{1}{1-bw(z_0)} - \frac{1}{1+aw(z_0)} \right\} \\
 &= \frac{k}{\delta} \Re \left\{ \frac{1-be^{-i\theta}}{1+b^2-2b\cos\theta} - \frac{1+ae^{-i\theta}}{1+a^2+2a\cos\theta} \right\} \\
 &= \frac{k}{\delta} \left\{ \frac{1}{2+\frac{b^2-1}{1-b\cos\theta}} - \frac{1}{2+\frac{a^2-1}{1+a\cos\theta}} \right\} = \frac{k\Delta}{\delta},
 \end{aligned}$$



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where  $\theta \neq \cos^{-1}(-1/a)$  and  $\theta \neq \cos^{-1}(-1/b)$ .

Simple calculations (under the constraints mentioned with the hypotheses for the parameters  $a$  and  $b$ ) yield that  $\Delta \geq \frac{(a+b)}{(1+a)(1-b)}$ , and since  $k \geq 1$ , it follows that

$$(3.4) \quad \Re\{\phi(z_0)\} = \frac{k\Delta}{\delta} \begin{cases} > \frac{(a+b)}{\delta(1+a)(1-b)} & (\delta > 0), \\ < \frac{(a+b)}{\delta(1+a)(1-b)} & (\delta < 0). \end{cases}$$

This contradicts our condition (3.1), and we conclude from (3.2) that

$$\begin{aligned} \left| \left( \frac{z J_{0,z}^{\lambda+1, \mu+1, \eta+1} f(z)}{J_{0,z}^{\lambda, \mu, \eta}} \right)^\delta - (p - \mu)^\delta \right| &= (p - \mu)^\delta \left| \frac{(a + b)w(z)}{1 - bw(z)} \right| \\ &< (p - \mu)^\delta \left( \frac{a + b}{1 - b} \right) \leq (p - \mu)^\delta. \end{aligned}$$

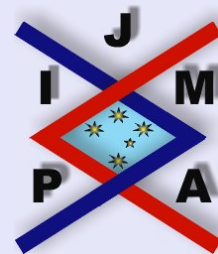
This completes the proof of Theorem 3.1.  $\square$

Next we prove

**Theorem 3.2.** Let  $\delta \in \mathbb{R} \setminus \{0\}$ ,  $p \in \mathbb{N}$ ,  $0 \leq \lambda < 1$ ,  $\mu < 1$ ,  $\eta > \max(\lambda, \mu) - p - 1$ , and  $a > 0$ ,  $b \geq 0$  such that  $a + 2b \leq 1$ . If a function  $f(z) \in \mathcal{A}_p$  satisfies the inequality

$$(3.5) \quad \Re \left( \frac{z J_{0,z}^{\lambda+1, \mu+1, \eta+1} f(z)}{J_{0,z}^{\lambda, \mu, \eta}} \right) \begin{cases} < p - \mu + \frac{a+b}{\delta(1+a)(1-b)} & (\delta > 0) \\ > p - \mu + \frac{a+b}{\delta(1+a)(1-b)} & (\delta < 0) \end{cases} \quad (z \in \mathcal{U}),$$

then  $f(z) \in T_\delta(p; \lambda, \mu, \eta)$ .



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*Proof.* Consider

$$(3.6) \quad \left( z^{\mu-p} J_{0,z}^{\lambda,\mu,\eta} f(z) \right)^\delta \\ = \left( \frac{\Gamma(1+p)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)} \right)^\delta \left( \frac{1+aw(z)}{1-bw(z)} \right) \quad (z \in \mathcal{U}).$$

Using the same method as elucidated in the proof of Theorem 3.1, we arrive at the desired result.  $\square$

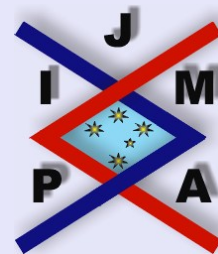
**Remark 3.1.** *If we set  $\lambda = \mu, a = 1, b = 0$ , then Theorems 3.1 and 3.2 by appealing to the operational relation (1.8) correspond to the recently established results due to Irmak et al. [4, pp. 271–272].*

Theorems 3.1 and 3.2 would also yield various results involving analytic and multivalent functions by suitably choosing the values of  $a, b, \delta, \mu$  and  $p$ . Setting  $\delta = 1$  in Theorems 3.1 and 3.2, we have

**Corollary 3.3.** *Let  $p \in \mathbb{N}, 0 \leq \lambda < 1, \mu < 1, \eta > \max(\lambda, \mu) - p - 1$ , and  $a > 0, b \geq 0$  such that  $a + 2b \leq 1$ . If a function  $f(z) \in \mathcal{A}_p$  satisfies the inequality*

$$(3.7) \quad \Re \left\{ 1 + z \left( \frac{J_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} - \frac{J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{J_{0,z}^{\lambda,\mu,\eta} f(z)} \right) \right\} \\ < \frac{a+b}{(1+a)(1-b)} \quad (z \in \mathcal{U}),$$

then  $f(z) \in \mathcal{M}_1(p; \lambda, \mu, \eta)$ .




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**Corollary 3.4.** Let  $p \in \mathbb{N}$ ,  $0 \leq \lambda < 1$ ,  $\mu < 1$ ,  $\eta > \max(\lambda, \mu) - p - 1$ , and  $a > 0, b \geq 0$  such that  $a + 2b \leq 1$ . If a function  $f(z) \in \mathcal{A}_p$  satisfies the inequality

$$(3.8) \quad \Re \left( \frac{z J_{0,z}^{\lambda+1, \mu+1, \eta+1} f(z)}{J_{0,z}^{\lambda, \mu, \eta} f(z)} \right) < p - \mu + \frac{a+b}{(1+a)(1-b)} \quad (z \in \mathcal{U}),$$

then  $f(z) \in \mathcal{T}_1(p; \lambda, \mu, \eta)$ .

Corollaries 3.3 and 3.4 on putting  $\lambda = \mu = 0$ , and using (1.8) give the following results:

**Corollary 3.5.** Let  $p \in \mathbb{N}$ ,  $a > 0, b \geq 0$  such that  $a + 2b \leq 1$ . If a function  $f(z) \in \mathcal{A}_p$  satisfies the inequality

$$(3.9) \quad \Re \left\{ 1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} \right\} < \frac{a+b}{(1+a)(1-b)} \quad (z \in \mathcal{U}),$$

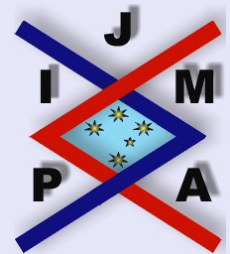
then  $f(z)$  is  $p$ -valently starlike in  $\mathcal{U}$ .

**Corollary 3.6.** Let  $p \in \mathbb{N}$ ,  $a > 0, b \geq 0$  such that  $a + 2b \leq 1$ . If a function  $f(z) \in \mathcal{A}_p$  satisfies the inequality

$$(3.10) \quad \Re \left\{ \frac{z f'(z)}{f(z)} \right\} < p + \frac{a+b}{(1+a)(1-b)} \quad (z \in \mathcal{U}),$$

then  $\Re \left\{ \frac{f(z)}{z^p} \right\} > 0, (z \in \mathcal{U})$ .

Lastly, Corollaries 3.3 and 3.4 on putting  $\lambda = \mu = 1$ , and using (1.8) give



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**Corollary 3.7.** Let  $p \in \mathbb{N}$ ,  $a > 0$ ,  $b \geq 0$  such that  $a + 2b \leq 1$ . If a function  $f(z) \in \mathcal{A}_p$  satisfies the inequality

$$(3.11) \quad \Re \left\{ 1 + \frac{zf'''(z)}{f''(z)} - \frac{zf''(z)}{f'(z)} \right\} < \frac{a+b}{(1+a)(1-b)} \quad (z \in \mathcal{U}),$$

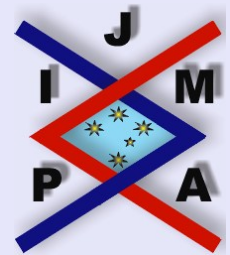
then  $f(z)$  is  $p$ -valently convex in  $\mathcal{U}$ .

**Corollary 3.8.** Let  $p \in \mathbb{N}$ ,  $a > 0$ ,  $b \geq 0$  such that  $a + 2b \leq 1$ . If a function  $f(z) \in \mathcal{A}_p$  satisfies the inequality

$$(3.12) \quad \Re \left\{ \frac{zf''(z)}{f'(z)} \right\} < p - 1 + \frac{a+b}{(1+a)(1-b)} \quad (z \in \mathcal{U}),$$

then  $f(z)$  is  $p$ -valently close-to-convex in  $\mathcal{U}$ .

**Remark 3.2.** When  $a = 1$ ,  $b = 0$ , then the Corollaries 3.5 – 3.8 correspond to the known results [3, pp. 457–458] involving inequalities on  $p$ -valent functions.




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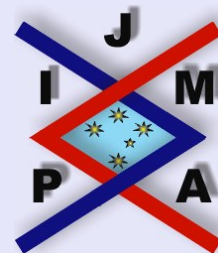
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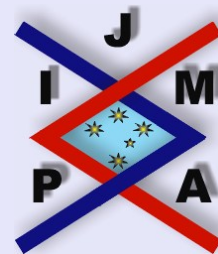
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