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A RELATION TO HILBERT'S INTEGRAL INEQUALITY AND SOME BASE HILBERT-TYPE INEQUALITIES

BICHENG YANG
DEPARTMENT OF MATHEMATICS
GUANGDONG EDUCATION INSTITUTE
GUANGZHOU, GUANGDONG 510303
P. R. CHINA
bcyang@pub.guangzhou.gd.cn

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ABSTRACT. In this paper, by using the way of weight function and real analysis techniques, a new integral inequality with some parameters and a best constant factor is given, which is a relation to Hilbert's integral inequality and some base Hilbert-type integral inequalities. The equivalent form and the reverse forms are considered.

Key words and phrases: Base Hilbert-type integral inequality; Parameter; Weight function.

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1. INTRODUCTION

If $f,g \geq 0$, $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$ then we have the following Hilbert's integral inequality [1]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible. Under the same condition of (1.1), we also have the following basic Hilbert-type integral inequalities [1, 2, 3]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < 4 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}; \tag{1.2}$$

$$\int_0^\infty \int_0^\infty \frac{|\ln(x/y)|f(x)g(y)}{x+y}dxdy < c_0 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}; \tag{1.3}$$

$$\int_0^\infty \int_0^\infty \frac{|\ln(x/y)|f(x)g(y)}{\max\{x,y\}}dxdy < 8 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}, \tag{1.4}$$

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where the constant factors 4, $c_0\left(=\sum_{k=1}^{\infty}\frac{8(-1)^{k-1}}{(2k-1)^2}=7.3277^+\right)$ and 8 are the best possible. In 2005, Hardy-Riesz gave a best extension of (1.1) by introducing one pair of conjugate exponents (p,q) $(p>1, \frac{1}{p}+\frac{1}{q}=1)$ as [4]

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}, \tag{1.5}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.5) is referred to as Hardy-Hilbert's integral inequality, which is important in analysis and its applications [5]. In 1998, Yang gave a best extension of (1.1) by introducing an independent parameter $\lambda>0$ as [6, 7]

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_0^\infty x^{1-\lambda}f^2(x)dx\int_0^\infty x^{1-\lambda}g^2(x)dx\right\}^{\frac{1}{2}}, \tag{1.6}$$

where the constant factor $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ is the best possible and the Beta function $B(u,v)$ is defined by [8]:

$$B(u,v):=\int_0^\infty \frac{1}{(1+t)^{u+v}}t^{u-1}dt \qquad (u,v>0). \tag{1.7}$$

In 2004-2005, by introducing two pairs of conjugate exponents and an independent parameter, Yang et al. [9, 10] gave two different extensions of (1.1) and (1.5) as: If $p,r>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{r}+\frac{1}{s}=1$, $\lambda>0$, $\phi(x)=x^{p(1-\frac{\lambda}{r})-1}$, $\psi(x)=x^{q(1-\frac{\lambda}{s})-1}$, $f,g\geq 0$,

$$0<||f||_{p,\phi}:=\left\{\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx\right\}^{\frac{1}{p}}<\infty$$

and

$$0<||g||_{q,\psi}:=\left\{\int_0^\infty x^{q(1-\frac{\lambda}{s})-1}g^q(x)dx\right\}^{\frac{1}{q}}<\infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy < \frac{\pi}{\lambda\sin(\frac{\pi}{r})}||f||_{p,\phi}||g||_{q,\psi}; \tag{1.8}$$

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)||f||_{p,\phi}||g||_{q,\psi}, \tag{1.9}$$

where the constant factors $\frac{\pi}{\lambda\sin(\frac{\pi}{r})}$ and $B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)$ are the best possible. Yang [11] also considered the reverse of (1.8) and (1.9).

In this paper, by using weight functions and real analysis techniques, a new integral inequality with the homogeneous kernel of $-\lambda$ degree

$$k_\lambda(x,y)=\frac{|\ln(x/y)|^\beta}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} \qquad (\lambda>0,\alpha\in\mathbb{R},\beta>-1)$$

is given, which is a relation to (1.1) and the above basic Hilbert-type integral inequalities (1.2), (1.3) and (1.4). The equivalent and reverse forms are considered. All the new inequalities possess the best constant factors.

2. SOME LEMMAS

We introduce the following Gamma function [8]:

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt \qquad (s > 0). \tag{2.1}$$

Lemma 2.1. *For $a, b > 0$, it follows that*

$$\int_0^1 x^{a-1}(-\ln x)^{b-1}dx = \frac{1}{a^b}\Gamma(b) = \int_1^\infty y^{-a-1}(\ln y)^{b-1}dy. \tag{2.2}$$

Proof. Setting $x = e^{-t/a}$ in first integral of (2.2), by (2.1), we find the first equation of (2.2). Setting $y = 1/x$ in the first integral of (2.2), we obtain the second equation of (2.2). The lemma is hence proved. □

Lemma 2.2. *If $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\lambda > 0$, $\alpha \in \mathbb{R}$ and $\beta > -1$, define the weight function as*

$$\varpi_\lambda(s, x) := x^{\frac{\lambda}{r}} \int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta y^{\frac{\lambda}{s}-1}}{(x+y)^{\lambda-\alpha}(\max\{x, y\})^\alpha}dy \qquad (x \in (0, \infty)). \tag{2.3}$$

Then we have

$$\varpi_\lambda(s, x) = k_\lambda(r) := \int_0^\infty \frac{|\ln u|^\beta u^{\frac{\lambda}{r}-1}}{(u+1)^{\lambda-\alpha}(\max\{u, 1\})^\alpha}du, \tag{2.4}$$

where $k_\lambda(r)$ is a positive number and

$$k_\lambda(r) = \Gamma(\beta + 1)\sum_{k=0}^\infty \binom{\alpha - \lambda}{k}\left[\frac{1}{\left(k + \frac{\lambda}{r}\right)^{\beta+1}} + \frac{1}{\left(k + \frac{\lambda}{s}\right)^{\beta+1}}\right]. \tag{2.5}$$

Proof. Setting $u = x/y$ in (2.3), by simplification, we obtain (2.4). In view of (2.2), we obtain

$$\begin{aligned}
0 < k_\lambda(r) &= \int_0^1 \frac{(-\ln u)^\beta u^{\frac{\lambda}{r}-1}}{(u+1)^{\lambda-\alpha}}du + \int_1^\infty \frac{(\ln u)^\beta u^{\frac{\lambda}{r}-\alpha-1}}{(u+1)^{\lambda-\alpha}}du \\
&\le 2^{|\alpha-\lambda|}\left[\int_0^1 (-\ln u)^{(\beta+1)-1}u^{\frac{\lambda}{r}-1}du + \int_1^\infty (\ln u)^{(\beta+1)-1}u^{\frac{-\lambda}{s}-1}du\right] \\
&= 2^{|\alpha-\lambda|}\left[\left(\frac{r}{\lambda}\right)^{\beta+1} + \left(\frac{s}{\lambda}\right)^{\beta+1}\right]\Gamma(\beta+1) < \infty.
\end{aligned}$$

Hence $k_\lambda(r)$ is a positive number. Using the property of power series, we find

$$\begin{aligned}
k_\lambda(r) &= \int_0^1 \frac{(-\ln u)^\beta u^{\frac{\lambda}{r}-1}}{(u+1)^{\lambda-\alpha}}du + \int_1^\infty \frac{(\ln u)^\beta u^{\frac{-\lambda}{s}-1}}{(1+u^{-1})^{\lambda-\alpha}}du \\
&= \int_0^1 \sum_{k=0}^\infty \binom{\alpha-\lambda}{k}(-\ln u)^\beta u^{\frac{\lambda}{r}+k-1}du + \int_1^\infty \sum_{k=0}^\infty \binom{\alpha-\lambda}{k}(\ln u)^\beta u^{\frac{-\lambda}{s}-k-1}du \\
&= \sum_{k=0}^\infty \binom{\alpha-\lambda}{k}\left[\int_0^1 (-\ln u)^\beta u^{\frac{\lambda}{r}+k-1}du + \int_1^\infty (\ln u)^\beta u^{\frac{-\lambda}{s}-k-1}du\right].
\end{aligned}$$

Then in view of (2.2), we have (2.5). The lemma is proved. □

Lemma 2.3. *If* $p > 0$ $(p \neq 1)$, $r > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\lambda > 0$, $\alpha \in \mathbb{R}$, $\beta > -1$, $n \in \mathbb{N}$, $n > \frac{r}{|q|\lambda}$, *then for* $n \to \infty$, *we have*

$$I_n := \frac{1}{n} \int_1^\infty \int_1^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta x^{\frac{\lambda}{r} - \frac{1}{np} - 1} y^{\frac{\lambda}{s} - \frac{1}{nq} - 1}}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} dxdy = k_\lambda(r) + o(1). \tag{2.6}$$

Proof. Setting $u = y/x$, by Fubini's theorem [12], we obtain

$$\begin{aligned}
I_n &= \frac{1}{n} \int_1^\infty \left[\int_1^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta x^{\frac{\lambda}{r} - \frac{1}{np} - 1} y^{\frac{\lambda}{s} - \frac{1}{nq} - 1}}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} dx \right] dy \\
&= \frac{1}{n} \int_1^\infty y^{-\frac{1}{n}-1} \left[\int_0^y \frac{|\ln u|^\beta u^{\frac{\lambda}{s} + \frac{1}{np} - 1}}{(1+u)^{\lambda-\alpha}(\max\{1,u\})^\alpha} du \right] dy \\
&= \frac{1}{n} \int_1^\infty y^{-\frac{1}{n}-1} \left[\int_0^1 \frac{(-\ln u)^\beta u^{\frac{\lambda}{s} + \frac{1}{np} - 1}}{(1+u)^{\lambda-\alpha}} du + \int_1^y \frac{(\ln u)^\beta u^{\frac{\lambda}{s} + \frac{1}{np} - 1}}{(1+u)^{\lambda-\alpha} u^\alpha} du \right] dy \\
&= \int_0^1 \frac{(-\ln u)^\beta u^{\frac{\lambda}{s} + \frac{1}{np} - 1}}{(1+u)^{\lambda-\alpha}} du + \frac{1}{n} \int_1^\infty y^{-\frac{1}{n}-1} \left[\int_1^y \frac{(\ln u)^\beta u^{\frac{\lambda}{s} + \frac{1}{np} - 1}}{(1+u)^{\lambda-\alpha} u^\alpha} du \right] dy \\
&= \int_0^1 \frac{(-\ln u)^\beta u^{\frac{\lambda}{s} + \frac{1}{np} - 1}}{(1+u)^{\lambda-\alpha}} du + \frac{1}{n} \int_1^\infty \left(\int_u^\infty y^{-\frac{1}{n}-1} dy \right) \frac{(\ln u)^\beta u^{\frac{\lambda}{s} + \frac{1}{np} - 1}}{(1+u)^{\lambda-\alpha} u^\alpha} du \\
&= \int_0^1 \frac{(-\ln u)^\beta u^{\frac{\lambda}{s} + \frac{1}{np} - 1}}{(1+u)^{\lambda-\alpha}} du + \int_1^\infty \frac{(\ln u)^\beta u^{\frac{\lambda}{s} - \frac{1}{nq} - 1}}{(1+u)^{\lambda-\alpha} u^\alpha} du. \qquad (2.7)
\end{aligned}$$

(i) If $p > 0$ $(p \neq 1)$ and $q > 0$, then by Levi's theorem [12], we find

$$\begin{aligned}
\int_0^1 \frac{(-\ln u)^\beta u^{\frac{\lambda}{s} + \frac{1}{np} - 1}}{(1+u)^{\lambda-\alpha}} du &= \int_0^1 \frac{(-\ln u)^\beta u^{\frac{\lambda}{s} - 1}}{(1+u)^{\lambda-\alpha}} du + o_1(1), \\
\int_1^\infty \frac{(\ln u)^\beta u^{\frac{\lambda}{s} - \frac{1}{nq} - 1}}{(1+u)^{\lambda-\alpha} u^\alpha} du &= \int_1^\infty \frac{(\ln u)^\beta u^{\frac{\lambda}{s} - 1}}{(1+u)^{\lambda-\alpha} u^\alpha} du + o_2(1) \quad (n \to \infty);
\end{aligned}$$

(ii) If $q < 0$, setting $n_0 \in \mathbb{N}$, $n_0 > \frac{r}{|q|\lambda}$, $\frac{1}{s'} = \frac{1}{s} - \frac{1}{n_0 q \lambda}$, $\frac{1}{r'} = \frac{1}{r} + \frac{1}{n_0 q \lambda}$, then for $n \geq n_0$, we find

$$\int_1^\infty \frac{(\ln u)^\beta u^{\frac{\lambda}{s} - \frac{1}{nq} - 1}}{(1+u)^{\lambda-\alpha} u^\alpha} du \leq \int_1^\infty \frac{(\ln u)^\beta u^{\frac{\lambda}{s} - \frac{1}{n_0 q} - 1}}{(1+u)^{\lambda-\alpha} u^\alpha} du \leq k_\lambda(r'),$$

and by Lebesgue's control convergence theorem, we have

$$\int_1^\infty \frac{(\ln u)^\beta u^{\frac{\lambda}{s} - \frac{1}{nq} - 1}}{(1+u)^{\lambda-\alpha} u^\alpha} du = \int_1^\infty \frac{(\ln u)^\beta u^{\frac{\lambda}{s} - 1}}{(1+u)^{\lambda-\alpha} u^\alpha} du + o_3(1) \quad (n \to \infty).$$

Hence by the above results and (2.7), we obtain (2.6). The lemma is proved. □

3. MAIN RESULTS

Theorem 3.1. *Assume that* $p > 0$ $(p \neq 1)$, $r > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\lambda > 0$, $\alpha \in \mathbb{R}$, $\beta > -1$, $\phi(x) = x^{p(1-\frac{\lambda}{r})-1}$, $\psi(x) = x^{q(1-\frac{\lambda}{s})-1}$ $(x \in (0,\infty))$, $f, g \geq 0$,

$$0 < ||f||_{p,\phi} = \left\{ \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx \right\}^{\frac{1}{p}} < \infty, \qquad 0 < ||g||_{q,\psi} < \infty.$$

(i) *For* $p > 1$, *we have the following inequality:*

$$I := \int_0^\infty \int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta f(x)g(y)}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} dxdy < k_\lambda(r)||f||_{p,\phi}||g||_{q,\psi}; \tag{3.1}$$

(ii) *For* $0 < p < 1$, *we have the reverse of (3.1), where the constant factor* $k_\lambda(r)$ *expressed by (2.5) in (3.1) and its reverse is the best possible.*

Proof. (i) By Hölder's inequality with weight [13], in view of (2.3), we find

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} \left[\frac{x^{(1-\frac{\lambda}{r})/q}}{y^{(1-\frac{\lambda}{s})/p}} f(x)\right] \left[\frac{y^{(1-\frac{\lambda}{s})/p}}{x^{(1-\frac{\lambda}{r})/q}} g(y)\right] dxdy \\ &\leq \left\{\int_0^\infty \int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} \cdot \frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}} f^p(x)dxdy\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\int_0^\infty \int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} \cdot \frac{y^{(1-\frac{\lambda}{s})(q-1)}}{x^{1-\frac{\lambda}{r}}} g^q(y)dxdy\right\}^{\frac{1}{q}} \\ &= \left\{\int_0^\infty \varpi_\lambda(s,x)\phi(x)f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty \varpi_\lambda(r,y)\psi(y)g^q(y)dy\right\}^{\frac{1}{q}}. \qquad (3.2) \end{aligned}$$

We confirm that the middle of (3.2) keeps the form of strict inequality. Otherwise, there exist constants A and B, such that they are not all zero and [13]

$$A\frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}} f^p(x) = B\frac{y^{(1-\frac{\lambda}{s})(q-1)}}{x^{1-\frac{\lambda}{r}}} g^q(y) \quad a.e. \text{ in } (0,\infty)\times(0,\infty).$$

It follows that $Ax^{p(1-\frac{\lambda}{r})}f^p(x) = By^{q(1-\frac{\lambda}{s})}g^q(y)$ $a.e.$ in $(0,\infty)\times(0,\infty)$. Assuming that $A \neq 0$, there exists $y > 0$, such that $x^{p(1-\frac{\lambda}{r})-1}f^p(x) = \left[By^{q(1-\frac{\lambda}{s})}g^q(y)\right]\frac{1}{Ax}$ $a.e.$ in $x \in (0,\infty)$. This contradicts the fact that $0 < ||f||_{p,\phi} < \infty$. Then inequality (3.1) is valid by using (2.4) and (2.5).

For $n \in \mathbb{N}$, $n > \frac{r}{|q|\lambda}$, setting f_n, g_n as

$$f_n(x) := \begin{cases} 0, & 0 < x \leq 1; \\ x^{\frac{\lambda}{r}-\frac{1}{np}-1}, & x > 1; \end{cases} \qquad g_n(x) := \begin{cases} 0, & 0 < x \leq 1; \\ x^{\frac{\lambda}{r}-\frac{1}{nq}-1}, & x > 1; \end{cases}$$

if there exists a constant factor $0 < k \leq k_\lambda(r)$, such that (3.1) is still valid if we replace $k_\lambda(r)$ by k, then by (2.6), we have

$$\begin{aligned} k_\lambda(r) + o(1) = I_n &= \frac{1}{n}\int_0^\infty \int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta f_n(x)g_n(y)}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} dxdy \\ &< \frac{1}{n}k||f_n||_{p,\phi}||g_n||_{q,\psi} = k, \end{aligned}$$

and $k_\lambda(r) \leq k$ $(n \to \infty)$. Hence $k = k_\lambda(r)$ is the best constant factor of (3.1).

(ii) For $0 < p < 1$, by the reverse Hölder's inequality with weight [13], in view of (2.3), we find the reverse of (3.2), which still keeps the strict form. Then by (2.4) and (2.5), we have the reverse of (3.1). By using (2.6) and the same manner as mentioned above , we can show that the constant factor in the reverse of (3.1) is still the best possible. The theorem is proved. □

Theorem 3.2. *Assume that* $p > 0$ $(p \neq 1)$, $r > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\lambda > 0$, $\alpha \in \mathbb{R}$, $\beta > -1$, $\phi(x) = x^{p(1-\frac{\lambda}{r})-1}$, $\psi(x) = x^{q(1-\frac{\lambda}{s})-1}$ $(x \in (0, \infty))$, $f \geq 0$, $0 < ||f||_{p,\phi} < \infty$.

(i) *For* $p > 1$, *we have the following inequality, which is equivalent to (3.1) and with the best constant factor* $k^p_\lambda(r)$*:*

$$J := \int_0^\infty y^{\frac{p\lambda}{s}-1} \left[\int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta f(x)}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} dx \right]^p dy < k^p_\lambda(r)||f||^p_{p,\phi}; \tag{3.3}$$

(ii) *For* $0 < p < 1$, *we have the reverse of (3.3), which is equivalent to the reverse of (3.1), with the best constant factor* $k^p_\lambda(r)$.

Proof. (i) For $p > 1$, $x > 0$, setting a bounded measurable function as

$$[f(x)]_n := \min\{f(x), n\} = \begin{cases} f(x), & \text{for } f(x) < n; \\ n, & \text{for } f(x) \geq n, \end{cases}$$

since $||f||_{p,\phi} > 0$, there exists $n_0 \in \mathbb{N}$, such that $\int_{\frac{1}{n}}^n \phi(x)[f(x)]^p_n dx > 0$ $(n \geq n_0)$. Setting $\widetilde{g}_n(y)$ $(y \in (\frac{1}{n}, n); n \geq n_0)$ as

$$\widetilde{g}_n(y) := y^{\frac{p\lambda}{s}-1} \left[\int_{\frac{1}{n}}^n \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} [f(x)]_n dx \right]^{p-1}, \tag{3.4}$$

then by (3.1), we find

$$\begin{aligned} 0 &< \int_{\frac{1}{n}}^n \psi(y)\widetilde{g}^q_n(y)dy \\ &= \int_{\frac{1}{n}}^n y^{\frac{p\lambda}{s}-1} \left[\int_{\frac{1}{n}}^n \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta [f(x)]_n dx}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} \right]^p dy \\ &= \int_{\frac{1}{n}}^n \int_{\frac{1}{n}}^n \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta [f(x)]_n \widetilde{g}_n(y)}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} dxdy \\ &< k_\lambda(r) \left\{ \int_{\frac{1}{n}}^n \phi(x)[f(x)]^p_n dx \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n}}^n \psi(y)\widetilde{g}^q_n(y)dy \right\}^{\frac{1}{q}} < \infty; \end{aligned} \tag{3.5}$$

$$0 < \int_{\frac{1}{n}}^n \psi(y)\widetilde{g}^q_n(y)dy < k^p_\lambda(r) \int_0^\infty \phi(x)f^p(x)dx < \infty. \tag{3.6}$$

It follows $0 < ||g||_{q,\psi} < \infty$. For $n \to \infty$, by (3.1), both (3.5) and (3.6) still keep the forms of strict inequality. Hence we have (3.3). On the other-hand, suppose (3.3) is valid. By Hölder's inequality, we have

$$I = \int_0^\infty \left[y^{\frac{-1}{p}+\frac{\lambda}{s}} \int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta f(x)dx}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} \right] \left[y^{\frac{1}{p}-\frac{\lambda}{s}} g(y) \right] dy \leq J^{\frac{1}{p}} ||g||_{q,\psi}. \tag{3.7}$$

In view of (3.3), we have (3.1), which is equivalent to (3.3). We confirm that the constant factor in (3.3) is the best possible. Otherwise, we may get a contradiction by (3.7) that the constant factor in (3.1) is not the best possible.

(ii) For $0 < p < 1$, since $||f||_{p,\phi} > 0$, we confirm that $J > 0$. If $J = \infty$, then the reverse of (3.3) is naturally valid. Suppose $0 < J < \infty$. Setting

$$g(y) := y^{\frac{p\lambda}{s}-1}\left[\int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta}{(x+y)^{\lambda-\alpha}(\max\{x,y\})^\alpha} f(x)dx\right]^{p-1},$$

by the reverse of (3.1), we obtain

$$\infty > ||g||_{q,\psi}^q = J = I > k_\lambda(r)||f||_{p,\phi}||g||_{q,\psi} > 0;$$
$$J^{\frac{1}{p}} = ||g||_{q,\psi}^{q-1} > k_\lambda(r)||f||_{p,\phi}.$$

Hence we have the reverse of (3.3). On the other-hand, suppose the reverse of (3.3) is valid. By the reverse Hölder's inequality, we can get the reverse of (3.7). Hence in view of the reverse of (3.3), we obtain the reverse of (3.1), which is equivalent to the reverse of (3.3). We confirm that the constant factor in the reverse of (3.3) is the best possible. Otherwise, we may get a contradiction by the reverse of (3.7) that the constant factor in the reverse of (3.1) is not the best possible. The theorem is proved. □

Remark 1. For $p = r = 2$ in (3.1), setting $\alpha = \beta = 0, \lambda = 1$, we obtain (1.1); setting $\alpha = 0, \beta = \lambda = 1$, we obtain (1.3). For $\alpha = \lambda > 0, \beta > -1$ in (3.1), we have

$$\text{(3.8)} \qquad \int_0^\infty \int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\beta f(x)g(y)}{(\max\{x,y\})^\lambda}dxdy < \frac{r^{\beta+1}+s^{\beta+1}}{\lambda^{\beta+1}}\Gamma(\beta+1)||f||_{p,\phi}||g||_{q,\psi},$$

where the constant factor $\frac{1}{\lambda^{\beta+1}}(r^{\beta+1}+s^{\beta+1})\Gamma(\beta+1)$ is the best possible. For $p = r = 2$ in (3.8), setting $\lambda = 1, \beta = 0$, we obtain (1.2); setting $\lambda = 1, \beta = 1$, we obtain (1.4). Hence inequality (3.1) is a relation to (1.1), (1.2), (1.3) and (1.4).

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