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A CONVOLUTION APPROACH ON PARTIAL SUMS OF CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

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ABSTRACT. In this paper, we determine sharp lower bounds for $\operatorname{Re}\left\{\frac{f(z)*\psi(z)}{f_n(z)*\psi(z)}\right\}$ and $\operatorname{Re}\left\{\frac{f_n(z)*\psi(z)}{f(z)*\psi(z)}\right\}$. We extend the results of ([1] – [5]) and correct the conditions for the results of Frasin [2, Theorem 2.7], [1, Theorem 2], Rosy et al. [4, Theorems 4.2 and 4.3], as well as Raina and Bansal [3, Theorem 6.2].

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1. INTRODUCTION

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. A function $f(z)$ in S is said to be starlike of order α $(0 \leq \alpha < 1)$, denoted by $S^*(\alpha)$, if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U),$$

and is said to be convex of order α $(0 \leq \alpha < 1)$, denoted by $K(\alpha)$, if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U).$$

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Let $T^{*}(\alpha)$ and $C(\alpha)$ be subclasses of $S^{*}(\alpha)$ and $K(\alpha)$, respectively, whose functions are of the form

$$f(z)=z-\sum_{k=2}^{\infty}a_kz^k,\quad a_k\geq 0. \tag{1.2}$$

A sufficient condition for a function of the form (1.1) to be in $S^{*}(\alpha)$ is that

$$\sum_{k=2}^{\infty}(k-\alpha)\left|a_k\right|\leq 1-\alpha \tag{1.3}$$

and to be in $K(\alpha)$ is that

$$\sum_{k=2}^{\infty}k(k-\alpha)\left|a_k\right|\leq 1-\alpha. \tag{1.4}$$

For functions of the form (1.2), Silverman [6] proved that the above sufficient conditions are also necessary.

Let $\phi(z)\in S$ be a fixed function of the form

$$\phi(z)=z+\sum_{k=2}^{\infty}c_kz^k,\quad (c_k\geq c_2>0,k\geq 2). \tag{1.5}$$

Very recently, Frasin [2] defined the class $H_{\phi}(c_k,\delta)$ consisting of functions $f(z)$, of the form (1.1) which satisfy the inequality

$$\sum_{k=2}^{\infty}c_k\left|a_k\right|\leq \delta, \tag{1.6}$$

where $\delta>0$.

He shows that for suitable choices of c_k and δ, $H_{\phi}(c_k,\delta)$ reduces to various known subclasses of S studied by various authors (for a detailed study, see [2] and the references therein).

In the present paper, we determine sharp lower bounds for $\operatorname{Re}\left\{\frac{f(z)*\psi(z)}{f_n(z)*\psi(z)}\right\}$ and $\operatorname{Re}\left\{\frac{f_n(z)*\psi(z)}{f(z)*\psi(z)}\right\}$, where

$$f_n(z)=z+\sum_{k=2}^{n}a_kz^k$$

is a sequence of partial sums of a function

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k$$

belonging to the class $H_{\phi}(c_k,\delta)$ and

$$\psi(z)=z+\sum_{k=2}^{\infty}\lambda_kz^k,\quad (\lambda_k\geq 0)$$

is analytic in open unit disc U and the operator "*" stands for the Hadamard product or convolution of two power series, which is defined for two functions $f,g\in A$, where $f(z)$ and $g(z)$ are of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k\quad\text{and}\quad g(z)=z+\sum_{k=2}^{\infty}b_kz^k$$

as

$$(f * g)(z) = f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

In this paper, we extend the results of Silverman [5], Frasin ([1], [2]) Rosy et al. [4] as well as Raina and Bansal [3] and we point out that some conditions on the results of Frasin ([2, Theorem 2.7], [1, Theorem 2]), Rosy et al. ([4, Theorem 4.2, 4.3]), Raina and Bansal ([3, Theorem 6.2]) are incorrect and we correct them. It is seen that this study not only gives a particular case of the results ([1] – [5]) but also gives rise to several new results.

2. Main Results

Theorem 2.1. *If* $f \in H_\phi(c_k, \delta)$ *and* $\psi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k$, $\lambda_k \geq 0$, *then*

$$\operatorname{Re}\left\{\frac{f(z) * \psi(z)}{f_n(z) * \psi(z)}\right\} \geq \frac{c_{n+1} - \lambda_{n+1}\delta}{c_{n+1}} \qquad (z \in U) \tag{2.1}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z) * \psi(z)}{f(z) * \psi(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1} + \lambda_{n+1}\delta} \qquad (z \in U), \tag{2.2}$$

where

$$c_k \geq \begin{cases} \lambda_k \delta & \text{if} \quad k = 2, 3, \ldots, n, \\ \frac{\lambda_k c_{n+1}}{\lambda_{n+1}} & \text{if} \quad k = n+1, n+2, \ldots \end{cases}$$

The results (2.1) *and* (2.2) *are sharp with the function given by*

$$f(z) = z + \frac{\delta}{c_{n+1}} z^{n+1}, \tag{2.3}$$

where $0 < \delta \leq \frac{c_{n+1}}{\lambda_{n+1}}$.

Proof. Define the function $\omega(z)$ by

$$\begin{aligned} \frac{1+\omega(z)}{1-\omega(z)} &= \frac{c_{n+1}}{(\lambda_{n+1})\delta}\left[\frac{f(z) * \psi(z)}{f_n(z) * \psi(z)} - \left(\frac{c_{n+1} - \delta\lambda_{n+1}}{c_{n+1}}\right)\right] \\ &= \frac{1 + \sum_{k=2}^{n} \lambda_k a_k z^{k-1} + \frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k a_k z^{k-1}}{1 + \sum_{k=2}^{n} \lambda_k a_k z^{k-1}}. \end{aligned} \tag{2.4}$$

It suffices to show that $|\omega(z)| \leq 1$. Now, from (2.4) we can write

$$\omega(z) = \frac{\frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k a_k z^{k-1}}{2 + 2\sum_{k=2}^{n} \lambda_k a_k z^{k-1} + \frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k a_k z^{k-1}}.$$

Hence we obtain

$$|\omega(z)| \leq \frac{\frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k |a_k|}{2 - 2\sum_{k=2}^{n} \lambda_k |a_k| - \frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k |a_k|}.$$

Now $|\omega(z)| \leq 1$ if

$$2\frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k |a_k| \leq 2 - 2\sum_{k=2}^{n} \lambda_k |a_k|$$

or, equivalently,

$$\sum_{k=2}^{n} \lambda_k |a_k| + \frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k |a_k| \leq 1. \tag{2.5}$$

It suffices to show that the L.H.S. of (2.5) is bounded above by $\sum_{k=2}^{\infty}\frac{c_k}{\delta}\left|a_k\right|$, which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{c_k-\delta\lambda_k}{\delta}\right)\left|a_k\right|+\sum_{k=n+1}^{\infty}\left(\frac{\lambda_{n+1}c_k-c_{n+1}\lambda_k}{\lambda_{n+1}\delta}\right)\left|a_k\right|\geq 0. \tag{2.6}$$

To see that the function given by (2.3) gives a sharp result we observe that for $z=re^{i\pi/n}$

$$\begin{aligned}\frac{f\left(z\right)*\psi\left(z\right)}{f_n\left(z\right)*\psi\left(z\right)}&=1+\frac{\delta}{c_{n+1}}\lambda_{n+1}z^n\to 1-\frac{\delta}{c_{n+1}}\lambda_{n+1}\\&=\frac{c_{n+1}-\delta\lambda_{n+1}}{c_{n+1}}\end{aligned}$$

when $r\to 1^-$.

To prove the second part of this theorem, we write

$$\begin{aligned}\frac{1+\omega\left(z\right)}{1-\omega\left(z\right)}&=\frac{c_{n+1}+\lambda_{n+1}\delta}{\lambda_{n+1}\delta}\left[\frac{f_n\left(z\right)*\psi\left(z\right)}{f\left(z\right)*\psi\left(z\right)}-\left(\frac{c_{n+1}}{c_{n+1}+\lambda_{n+1}\delta}\right)\right]\\&=\frac{1+\sum_{k=2}^{n}\lambda_k a_k z^{k-1}-\frac{c_{n+1}}{\lambda_{n+1}\delta}\sum_{k=n+1}^{\infty}\lambda_k a_k z^{k-1}}{1+\sum_{k=2}^{\infty}\lambda_k a_k z^{k-1}},\end{aligned}$$

where

$$\left|\omega\left(z\right)\right|\leq\frac{\left(\frac{c_{n+1}+\lambda_{n+1}\delta}{\lambda_{n+1}\delta}\right)\sum_{k=n+1}^{\infty}\lambda_k\left|a_k\right|}{2-2\sum_{k=2}^{n}\lambda_k\left|a_k\right|-\frac{c_{n+1}-\lambda_{n+1}\delta}{\lambda_{n+1}\delta}\sum_{k=n+1}^{\infty}\lambda_k\left|a_k\right|}\leq 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{n}\lambda_k\left|a_k\right|+\frac{c_{n+1}}{\left(\lambda_{n+1}\right)\delta}\sum_{k=n+1}^{\infty}\lambda_k\left|a_k\right|\leq 1.$$

Making use of (1.6), we get (2.6). Finally, equality holds in (2.2) for the function $f\left(z\right)$ given by (2.3). □

Taking $\psi\left(z\right)=\frac{z}{1-z}$ in Theorem 2.1, we obtain the following result given by Frasin in [2].

Corollary 2.2. *If $f\in H_\phi\left(c_k,\delta\right)$, then*

$$\operatorname{Re}\left\{\frac{f\left(z\right)}{f_n\left(z\right)}\right\}\geq\frac{c_{n+1}-\delta}{c_{n+1}}\qquad\left(z\in U\right) \tag{2.7}$$

and

$$\operatorname{Re}\left\{\frac{f_n\left(z\right)}{f\left(z\right)}\right\}\geq\frac{c_{n+1}}{c_{n+1}+\delta}\qquad\left(z\in U\right), \tag{2.8}$$

where

$$c_k\geq\begin{cases}\delta & \textit{if}\quad k=2,3,\ldots,n,\\ c_{n+1} & \textit{if}\quad k=n+1,n+2,\ldots\end{cases}$$

The results (2.7) *and* (2.8) *are sharp with the function given by* (2.3).

If we put $\psi\left(z\right)=\frac{z}{\left(1-z\right)^2}$ in Theorem 2.1, we obtain:

Corollary 2.3. *If* $f \in H_\phi(c_k, \delta)$, *then*

$$\operatorname{Re}\frac{f'(z)}{f_n'(z)} \geq \frac{c_{n+1} - (n+1)\delta}{c_{n+1}} \qquad (z \in U) \tag{2.9}$$

and

$$\operatorname{Re}\frac{f_n'(z)}{f'(z)} \geq \frac{c_{n+1}}{c_{n+1} + (n+1)\delta} \qquad (z \in U), \tag{2.10}$$

where

$$c_k \geq \begin{cases} k\delta & \text{if} \quad k = 2, 3, \dots, n, \\ \frac{kc_{n+1}}{n+1} & \text{if} \quad k = n+1, n+2, \dots \end{cases} \tag{2.11}$$

The results (2.9) *and* (2.10) *are sharp with the function given by* (2.3).

Remark 1. Frasin has shown in Theorem 2.7 of [2] that for $f \in H_\phi(c_k, \delta)$, inequalities (2.9) and (2.10) hold with the condition

$$c_k \geq \begin{cases} k\delta & \text{if} \quad k = 2, 3, \dots, n, \\ k\delta\left(1 + \frac{c_{n+1}}{n+1}\right) & \text{if} \quad k = n+1, n+2, \dots \end{cases} \tag{2.12}$$

However, it can be easily seen that the condition (2.12) for $k = n+1$ gives

$$c_{n+1} \geq (n+1)\delta\left(1 + \frac{c_{n+1}}{(n+1)\delta}\right)$$

or, equivalently $\delta \leq 0$, which contradicts the initial assumption $\delta > 0$. So Theorem 2.7 of [2] does not seem suitable with the condition (2.12), but our condition (2.11) remedies this problem.

Taking $\psi(z) = \frac{z}{1-z}$, $c_k = \frac{[(1+\beta)k-(\alpha+\beta)]}{1-\alpha}\binom{k+\lambda-1}{k}$, where $\lambda \geq 0$, $\beta \geq 0$, $-1 \leq \alpha < 1$ and $\delta = 1$ in Theorem 2.1, we obtain the following result given by Rosy et al. in [4].

Corollary 2.4. *If* f *is of the form* (1.1) *and satisfies the condition* $\sum_{k=2}^{\infty} c_k |a_k| \leq 1$, *where* $c_k = \frac{[(1+\beta)k-(\alpha+\beta)]}{1-\alpha}\binom{k+\lambda-1}{k}$, $\lambda \geq 0$, $\beta \geq 0$, $-1 \leq \alpha < 1$, *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{c_{n+1} - 1}{c_{n+1}} \qquad (z \in U) \tag{2.13}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1} + 1} \qquad (z \in U). \tag{2.14}$$

The results (2.13) *and* (2.14) *are sharp with the function given by*

$$f(z) = z + \frac{1}{c_{n+1}} z^{n+1}. \tag{2.15}$$

Taking

$$\psi(z) = \frac{z}{(1-z)^2}, \qquad c_k = \frac{[(1+\beta)k - (\alpha+\beta)]}{1-\alpha}\binom{k+\lambda-1}{k},$$

where $\lambda \geq 0$, $\beta \geq 0$, $-1 \leq \alpha < 1$ and $\delta = 1$ in Theorem 2.1, we obtain

Corollary 2.5. *If f is of the form* (1.1) *and satisfies the condition*

$$\sum_{k=2}^{\infty} c_k |a_k| \leq 1,$$

where

$$c_k = \frac{[(1+\beta)k - (\alpha+\beta)]}{1-\alpha} \binom{k+\lambda-1}{k}, \qquad (\lambda \geq 0,\ \beta \geq 0,\ -1 \leq \alpha < 1),$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{c_{n+1}-(n+1)}{c_{n+1}} \qquad (z \in U) \tag{2.16}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1}+(n+1)} \qquad (z \in U), \tag{2.17}$$

where

$$c_k \geq \begin{cases} k & \text{if} \quad k = 2, 3, \ldots, n, \\ \frac{kc_{n+1}}{n+1} & \text{if} \quad k = n+1, n+2, \ldots \end{cases} \tag{2.18}$$

The results (2.16) *and* (2.17) *are sharp with the function given by* (2.15).

Remark 2. Rosy et al. has obtained inequalities (2.16) & (2.17) in Theorem 4.2 & 4.3 of [4] without any restriction on c_k. However, when we critically observe the proof of Theorem 4.2 we find that inequality (4.16) of [4, Theorem 4.2]

$$\sum_{k=2}^{n} (c_k - k)|a_k| + \sum_{k=n+1}^{\infty} \left(c_k - \frac{c_{n+1}k}{n+1}\right)|a_k| \geq 0$$

cannot hold if condition (2.18) does not occur. So Theorems 4.2 & 4.3 of [4] are not proper and proper results are mentioned in Corollary 2.5.

Taking $\psi(z) = \frac{z}{1-z}$, $c_k = \lambda_k - \alpha\mu_k$, $\delta = 1-\alpha$, where $0 \leq \alpha < 1$, $\lambda_k \geq 0$, $\mu_k \geq 0$, and $\lambda_k \geq \mu_k$ $(k \geq 2)$ in Theorem 2.1, we obtain the following result given by Frasin in [1].

Corollary 2.6. *If f is of the form* (1.1) *and satisfies the condition*

$$\sum_{k=2}^{\infty} (\lambda_k - \alpha\mu_k)|a_k| \leq 1-\alpha,$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1} - 1 + \alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} \qquad (z \in U) \tag{2.19}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{\lambda_{n+1} - \alpha\mu_{n+1} + 1 - \alpha} \qquad (z \in U), \tag{2.20}$$

where

$$\lambda_k - \alpha\mu_k \geq \begin{cases} 1-\alpha & \text{if} \quad k = 2, 3, \ldots, n, \\ \lambda_{n+1} - \alpha\mu_{n+1} & \text{if} \quad k = n+1, n+2, \ldots \end{cases}$$

The results (2.19) *and* (2.20) *are sharp with the function given by*

$$f(z) = z + \frac{1-\alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} z^{n+1}. \tag{2.21}$$

Taking $\psi(z) = \frac{z}{(1-z)^2}$, $c_k = \lambda_k - \alpha\mu_k$, $\delta = 1-\alpha$ where $0 \le \alpha < 1$, $\lambda_k \ge 0$, $\mu_k \ge 0$, and $\lambda_k \ge \mu_k$ $(k \ge 2)$ in Theorem 2.1, we obtain:

Corollary 2.7. *If f is of the form* (1.1) *and satisfies the condition*

$$\sum_{k=2}^{\infty} (\lambda_k - \alpha\mu_k)|a_k| \le 1-\alpha,$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \ge \frac{\lambda_{n+1} - \alpha\mu_{n+1} - (n+1)(1-\alpha)}{\lambda_{n+1} - \alpha\mu_{n+1}} \qquad (z \in U) \tag{2.22}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \ge \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{\lambda_{n+1} - \alpha\mu_{n+1} + (n+1)(1-\alpha)} \qquad (z \in U), \tag{2.23}$$

where

$$\lambda_k - \alpha\mu_k \ge \begin{cases} k(1-\alpha) & \text{if} \quad k = 2, 3, \dots, n, \\ \frac{k(\lambda_{n+1} - \alpha\mu_{n+1})}{n+1} & \text{if} \quad k = n+1, n+2, \dots \end{cases} \tag{2.24}$$

The results (2.22) *and* (2.23) *are sharp with the function given by* (2.21).

Remark 3. Frasin has obtained inequalities (2.22) & (2.23) in Theorem 2 of [1] under the condition

$$\lambda_{k+1} - \alpha\mu_{k+1} \ge \begin{cases} k(1-\alpha) & \text{if} \quad k = 2, 3, \dots, n, \\ k(1-\alpha) + \frac{k(\lambda_{n+1} - \alpha\mu_{n+1})}{n+1} & \text{if} \quad k = n+1, n+2, \dots \end{cases} \tag{2.25}$$

However, when we critically observe the proof of Theorem 2 of [1], we find that the last inequality of this theorem

$$\sum_{k=2}^{n}\left(\frac{\lambda_k - \alpha\mu_k}{1-\alpha} - k\right)|a_k| + \sum_{k=n+1}^{\infty}\left(\frac{\lambda_k - \alpha\mu_k}{1-\alpha} - \left(1 + \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{(n+1)(1-\alpha)}\right)k\right)|a_k| \ge 0 \tag{2.26}$$

cannot hold for the function given by (2.21) for supporting the sharpness of the results (2.22) & (2.23). So condition 2.25 of Theorem 2 in [1] is incorrect and the corrected results are mentioned in Corollary 2.7.

Taking

$$\psi(z) = \frac{z}{1-z}, \qquad c_k = \frac{\{(1+\beta)k - (\alpha+\beta)\}\mu_k}{1-\alpha}$$

and $\delta = 1$, where $-1 \le \alpha < 1$, $\beta \ge 0$, $\mu_k \ge 0$ $(\forall k \in N \setminus \{1\})$ in Theorem 2.1, we obtain the following result given by Raina and Bansal in [3].

Corollary 2.8. *If f is of the form* (1.2) *and satisfies the condition* $\sum_{k=2}^{\infty} c_k |a_k| \le 1$, *where*

$$c_k = \frac{\{(1+\beta)k - (\alpha+\beta)\}\mu_k}{1-\alpha},$$

and $\langle\mu_k\rangle_{k=2}^{\infty}$ is a nondecreasing sequence such that

$$\mu_2 \ge \frac{1-\alpha}{2+\beta-\alpha}\left(0 < \frac{1-\alpha}{2+\beta-\alpha} < 1, \quad -1 \le \alpha < 1, \beta \ge 0\right),$$

then

$$\text{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{c_{n+1}-1}{c_{n+1}} \qquad (z \in U) \tag{2.27}$$

and

$$\text{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1}+1} \qquad (z \in U). \tag{2.28}$$

The results (2.27) *and* (2.28) *are sharp with the function given by*

$$f(z) = z - \frac{1}{c_{n+1}} z^{n+1}. \tag{2.29}$$

Taking $\psi(z) = \frac{z}{(1-z)^2}$, $c_k = \frac{\{(1+\beta)k-(\alpha+\beta)\}\mu_k}{1-\alpha}$ and $\delta = 1$, where $-1 \leq \alpha < 1$, $\beta \geq 0$, $\mu_k \geq 0$ $(\forall k \in N\setminus\{1\})$ in Theorem 2.1, we obtain the following result given by Raina and Bansal in [3].

Corollary 2.9. *If f is of the form* (1.2) *and satisfies the condition*

$$\sum_{k=2}^{\infty} c_k |a_k| \leq 1,$$

where

$$c_k = \frac{\{(1+\beta)k-(\alpha+\beta)\}\mu_k}{1-\alpha},$$

and $\langle\mu_k\rangle_{k=2}^{\infty}$ is a nondecreasing sequence such that

$$\mu_2 \geq \frac{2(1-\alpha)}{2+\beta-\alpha} \qquad \left(0 < \frac{1-\alpha}{2+\beta-\alpha} < 1, \quad -1 \leq \alpha < 1, \beta \geq 0\right).$$

Then

$$\text{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{c_{n+1}-(n+1)}{c_{n+1}} \qquad (z \in U) \tag{2.30}$$

and

$$\text{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1}+(n+1)} \qquad (z \in U), \tag{2.31}$$

where

$$c_k \geq \begin{cases} k & \textit{if} \quad k = 2, 3, \ldots, n, \\ \frac{kc_{n+1}}{n+1} & \textit{if} \quad k = n+1, n+2, \ldots \end{cases} \tag{2.32}$$

The results (2.30) *and* (2.31) *are sharp with the function given by* (2.29).

Remark 4. Raina and Bansal [3] have obtained inequalities (2.30) & (2.31) in Theorem 6.2 of [3] without any restriction on c_k . However, we easily see that condition (2.32) is must.

Remark 5. Taking $\psi(z) = \frac{z}{1-z}$, $c_k = (k-\alpha)$, $c_k = k(k-\alpha)$, $\delta = 1-\alpha$, $0 \leq \alpha < 1$ in Theorem 2.1, we obtain Theorems 1-3 given by Silverman in [5].

Remark 6. Taking $\psi(z) = \frac{z}{(1-z)^2}$, $c_k = (k-\alpha)$, $c_k = k(k-\alpha)$, $\delta = 1-\alpha$, $0 \leq \alpha < 1$ in Theorem 2.1, we obtain Theorems 4-5 given by Silverman in [5].

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