journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 10 (2009), Issue 3, Article 65, 11 pp.

DRAGOMIR-AGARWAL TYPE INEQUALITIES FOR SEVERAL FAMILIES OF QUADRATURES

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Received 05 May, 2009; accepted 15 September, 2009
Communicated by S.S. Dragomir

ABSTRACT. Inequalities estimating the absolute value of the difference between the integral and the quadrature, i.e. the Dragomir-Agarwal-type inequalities, are given for the general 3, 4 and 5-point quadrature formulae, both classical and corrected. Beside values of the function in the chosen nodes, "corrected" quadrature formula includes values of the first derivative at the end points of the interval and has a higher accuracy than the adjoint classical quadrature formula.

Key words and phrases: Dragomir-Agarwal-type inequalities, k-convex functions, General 3-point, 4-point and 5-point quadrature formulae, Corrected quadrature formulae.

2000 *Mathematics Subject Classification.* 26D15, 65D30, 65D32.

1. INTRODUCTION

The well-known Hermite-Hadamard inequality states that if $f : [a,b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This pair of inequalities has been improved and extended in a number of ways. One of the directions estimated the difference between the middle and rightmost term in (1.1). For example, Dragomir and Agarwal presented the following result in [2]: suppose $f : I \subseteq \mathbb{R} \to \mathbb{R}$

The research of the authors was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grants 058-1170889-1050 (first author) and 117-1170889-0888 (second author).
125-09

is differentiable on I and $|f'|^q$ is convex on $[a,b]$ for some $q \geq 1$, where I is an open interval in $\mathbb{R}$ and $a,b \in I$ $(a<b)$. Then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}.$$

Generalizations to higher-order convexity for this type of inequality were given in [3]. Related results for Euler-midpoint, Euler-twopoint, Euler-Simpson, dual Euler-Simpson, Euler-Maclaurin, Euler-Simpson 3/8 and Euler-Boole formulae were given in [11]. Furthermore, related results for the general Euler 2-point formulae were given in [10], unifying the cases of Euler trapezoid, Euler midpoint and Euler-twopoint formulae.

The aim of this paper is to give related results for the general 3, 4 and 5-point quadrature formulae, as well as for the corrected general 3, 4 and 5-point quadrature formulae. In addition to values of the function at the chosen nodes, "corrected" quadrature formulae include values of the first derivative at the end points of the interval and also have higher accuracy than adjoint classical quadrature formulae. They are sometimes called "quadratures with end corrections".

Our first course of action was to obtain the quadrature formulae. This was done using the extended Euler formulae, in which Bernoulli polynomials play an important role. For the reader's convenience, let us recall some basic properties of Bernoulli polynomials. Bernoulli polynomials $B_k(t)$ are uniquely determined by

$$B_k'(x) = kB_{k-1}(x), \quad B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0,\ B_0(t) = 1.$$

For the kth Bernoulli polynomial we have $B_k(1-x) = (-1)^k B_k(x),\ x \in \mathbb{R},\ k \geq 1.$

The kth Bernoulli number B_k is defined by $B_k = B_k(0)$. For $k \geq 2$, we have $B_k(1) = B_k(0) = B_k$. Note that $B_{2k-1} = 0,\ k \geq 2$ and $B_1(1) = -B_1(0) = 1/2$.

$B_k^*(x)$ are periodic functions of period 1 defined by $B_k^*(x+1) = B_k^*(x),\ x \in \mathbb{R}$, and related to Bernoulli polynomials as $B_k^*(x) = B_k(x),\ 0 \leq x < 1$. For $k \geq 2$, $B_k^*(t)$ is a continuous function, while $B_1^*(x)$ is a discontinuous function with a jump of -1 at each integer. For further details on Bernoulli polynomials, see [1] and [9].

2. PRELIMINARIES

General 3-point quadrature formulas were obtained in [5] and general corrected 3-point quadrature formulas in [6]; general closed 4-point quadrature formulas were considered in [7] and finally, general closed 5-point quadrature formulas were derived in [8]. Namely, if $f : [0,1] \to \mathbb{R}$ is such that $f^{(n-1)}$ is continuous and of bounded variation on $[0,1]$ for some $n \geq 1$, then we have

$$\int_0^1 f(t)dt - Q_\alpha(x) + T_{n-1}^\alpha(x) = \frac{1}{n!}\int_0^1 F_n^\alpha(x,t)df^{(n-1)}(t), \tag{2.1}$$

for $\alpha = Q3,\ CQ3$ and $x \in [0,1/2)$, for $\alpha = Q4,\ CQ4$ and $x \in (0,1/2]$, and for $\alpha = Q5,\ CQ5$ and $x \in (0,1/2)$, where

$$Q_{Q3}(x) := Q\left(x, \frac{1}{2}, 1-x\right) = \frac{f(x) + 24B_2(x)f\left(\frac{1}{2}\right) + f(1-x)}{6(1-2x)^2},$$

$$Q_{CQ3}(x) := Q_C\left(x, \frac{1}{2}, 1-x\right) = \frac{7f(x) - 480B_4(x)f\left(\frac{1}{2}\right) + 7f(1-x)}{30(1-2x)^2(1+4x-4x^2)},$$

$$Q_{Q4}(x) := Q(0,x,1-x,1) = \frac{-6B_2(x)f(0)+f(x)+f(1-x)-6B_2(x)f(1)}{12x(1-x)},$$

$$Q_{CQ4}(x) := Q_C(0,\ x,\ 1-x,\ 1) = \frac{30B_4(x)f(0)+f(x)+f(1-x)+30B_4(x)f(1)}{60x^2(1-x)^2},$$

$$\begin{aligned} Q_{Q5}(x) &:= Q\left(0,\ x,\ \frac{1}{2},\ 1-x,\ 1\right) \\ &= \frac{1}{60x(1-x)(1-2x)^2}\Big[f(x)+f(1-x) \\ &\qquad -(10x^2-10x+1)(1-2x)^2(f(0)+f(1)) \\ &\qquad +32x(1-x)(5x^2-5x+1)f\left(\frac{1}{2}\right)\Big], \end{aligned}$$

$$\begin{aligned} Q_{CQ5}(x) &:= Q_C\left(0,\ x,\ \frac{1}{2},\ 1-x,\ 1\right) \\ &= \frac{1}{420x^2(1-x)^2(1-2x)^2}\,[f(x)+f(1-x) \\ &\qquad +(98x^4-196x^3+102x^2-4x-1)(1-2x)^2(f(0)+f(1)) \\ &\qquad +64x^2(1-x)^2(14x^2-14x+3)f\left(1/2\right)] \end{aligned}$$

and

$$T^{\alpha}_{n-1}(x) = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \frac{1}{(2k)!}\, G^{\alpha}_{2k}(x,0)\, [f^{(2k-1)}(1)-f^{(2k-1)}(0)],$$

$$F^{\alpha}_n(x,t) = G^{\alpha}_n(x,t) - G^{\alpha}_n(x,0), \tag{2.2}$$

and finally,

$$G^{Q3}_n(x,t) = \frac{B^*_n(x-t)+24B_2(x)\cdot B^*_n\left(\frac{1}{2}-t\right)+B^*_n(1-x-t)}{6(1-2x)^2}, \tag{2.3}$$

$$G^{CQ3}_n(x,t) = \frac{7B^*_n(x-t)-480B_4(x)\cdot B^*_n\left(\frac{1}{2}-t\right)+7B^*_n(1-x-t)}{30(1-2x)^2(1+4x-4x^2)}, \tag{2.4}$$

$$G^{Q4}_n(x,t) = \frac{B^*_n(x-t)-12B_2(x)\cdot B^*_n(1-t)+B^*_n(1-x-t)}{12x(1-x)}, \tag{2.5}$$

$$G^{CQ4}_n(x,t) = \frac{60B_4(x)\cdot B^*_n(1-t)+B^*_n(x-t)+B^*_n(1-x-t)}{60x^2(1-x)^2}, \tag{2.6}$$

$$\begin{aligned} G^{Q5}_n(x,t) &= \frac{10x^2-10x+1}{30x(x-1)}\, B^*_n(1-t) \\ &\quad + \frac{B^*_n(x-t)+B^*_n(1-x-t)}{60x(1-x)(1-2x)^2} + \frac{8(5x^2-5x+1)}{15(1-2x)^2}\, B^*_n\left(\frac{1}{2}-t\right), \end{aligned} \tag{2.7}$$

$$\begin{aligned} G^{CQ5}_n(x,t) &= \frac{98x^4-196x^3+102x^2-4x-1}{210x^2(1-x)^2}\, B^*_n(1-t) \\ &\quad + \frac{B^*_n(x-t)+B^*_n(1-x-t)}{420x^2(1-x)^2(1-2x)^2} + \frac{16(14x^2-14x+3)}{105(1-2x)^2}\, B^*_n\left(\frac{1}{2}-t\right). \end{aligned} \tag{2.8}$$

The following lemma was the key result for obtaining the results in [5], [6], [7] and [8], and we shall need it here as well.

Lemma 2.1. *For* $x \in \{0\} \cup [1/6,\ 1/2)$ *and* $n \geq 2$*,* $G_{2n-1}^{Q3}(x,t)$ *has no zeros in variable* t *on* $(0,\ 1/2)$*. The sign of the function is determined by:*

$$(-1)^{n+1} G_{2n-1}^{Q3}(x,t) > 0 \quad \text{for } \ x \in [1/6, 1/2) \quad \textit{and} \quad (-1)^n G_{2n-1}^{Q3}(0,t) > 0.$$

For $x \in \left[0,\ 1/2 - \sqrt{15}/10\right] \cup [1/6,\ 1/2)$ *and* $n \geq 3$*,* $G_{2n-1}^{CQ3}(x,t)$ *has no zeros in variable* t *on* $(0,\ 1/2)$*. The sign of the function is determined by:*

$$\begin{aligned}
&(-1)^n G_{2n-1}^{CQ3}(x,t) > 0 \quad \text{for } \ x \in [0,\ 1/2 - \sqrt{15}/10],\\
&(-1)^{n+1} G_{2n-1}^{CQ3}(x,t) > 0 \quad \text{for } \ x \in [1/6,\ 1/2).
\end{aligned}$$

For $x \in (0,\ 1/2 - \sqrt{3}/6] \cup [1/3,\ 1/2]$ *and* $n \geq 2$*,* $G_{2n-1}^{Q4}(x,t)$ *has no zeros in variable* t *on* $(0,\ 1/2)$*. The sign of the function is determined by:*

$$\begin{aligned}
&(-1)^{n+1} G_{2n-1}^{Q4}(x,t) > 0 \quad \text{for } \ x \in (0,\ 1/2 - \sqrt{3}/6],\\
&(-1)^n G_{2n-1}^{Q4}(x,t) > 0 \quad \text{for } \ x \in [1/3,\ 1/2].
\end{aligned}$$

For $x \in (0,\ 1/2 - \sqrt{5}/10] \cup [1/3,\ 1/2]$ *and* $n \geq 3$*,* $G_{2n-1}^{CQ4}(x,t)$ *has no zeros in variable* t *on* $(0,\ 1/2)$*. The sign of the function is determined by:*

$$\begin{aligned}
&(-1)^{n+1} G_{2n-1}^{CQ4}(x,t) > 0 \quad \text{for } \ x \in (0,\ 1/2 - \sqrt{5}/10],\\
&(-1)^n G_{2n-1}^{CQ4}(x,t) > 0 \quad \text{for } \ x \in [1/3,\ 1/2].
\end{aligned}$$

For $x \in (0,\ 1/2 - \sqrt{15}/10] \cup [1/5,\ 1/2)$ *and* $n \geq 3$*,* $G_{2n-1}^{Q5}(x,t)$ *has no zeros in variable* t *on* $(0,\ 1/2)$*. The sign of the function is determined by:*

$$\begin{aligned}
&(-1)^n G_{2n-1}^{Q5}(x,t) > 0 \quad \textit{for } \ x \in \left(0,\ 1/2 - \sqrt{15}/10\right],\\
&(-1)^{n+1} G_{2n-1}^{Q5}(x,t) > 0 \quad \textit{for } \ x \in [1/5,\ 1/2)\,.
\end{aligned}$$

For $x \in (0,\ 1/2 - \sqrt{21}/14] \cup [3/7 - \sqrt{2}/7,\ 1/2)$ *and* $n \geq 4$*,* $G_{2n-1}^{CQ5}(x,t)$ *has no zeros in variable* t *on* $(0,\ 1/2)$*. The sign of the function is determined by:*

$$\begin{aligned}
&(-1)^n G_{2n-1}^{CQ5}(x,t) > 0 \quad \textit{for } \ x \in \left(0,\ 1/2 - \sqrt{21}/14\right],\\
&(-1)^{n+1} G_{2n-1}^{CQ5}(x,t) > 0 \quad \textit{for } \ x \in \left[3/7 - \sqrt{2}/7,\ 1/2\right),
\end{aligned}$$

where G_{2n-1}^{Q3} *is as in (2.3),* G_{2n-1}^{CQ3} *as in (2.4),* G_{2n-1}^{Q4} *as in (2.5),* G_{2n-1}^{CQ4} *as in (2.6),* G_{2n-1}^{Q5} *as in (2.7) and* G_{2n-1}^{CQ5} *as in (2.8).*

Applying properties of Bernoulli polynomials, it easily follows that functions G_n^α for $\alpha = Q3,\ CQ3,\ Q4,\ CQ4,\ Q5,\ CQ5$ and $n \geq 1$, have the following properties:

$$G_n^\alpha(x, 1-t) = (-1)^n G_n^\alpha(x,t), \quad t \in [0,1], \tag{2.9}$$

$$\frac{\partial^j G_n^\alpha(x,t)}{\partial t^j} = (-1)^j \frac{n!}{(n-j)!} G_{n-j}^\alpha(x,t), \ \ j = 1, 2, \dots, n, \tag{2.10}$$

also, that $G_{2n-1}^\alpha(x,0) = 0$ for $n \geq 1$, and so $F_{2n-1}^\alpha(x,t) = G_{2n-1}^\alpha(x,t)$.

These properties and Lemma 2.1 yield that functions F_{2n}^α, defined by (2.2), are monotonous on $(0, 1/2)$ and $(1/2, 1)$, have constant sign on $(0,1)$, so the functions $|F_{2n}^\alpha(t)|$ attain their maximal value at $t = 1/2$. Finally, using (2.9) and (2.10), it is not hard to establish that under the assumptions of Lemma 2.1 we have:

$$\int_0^1 |F_{2n}^\alpha(x,t)|dt = 2\int_0^1 t|F_{2n}^\alpha(x,t)|dt = |G_{2n}^\alpha(x,0)|, \tag{2.11}$$

$$\int_0^1 |G_{2n-1}^\alpha(x,t)|dt = 2\int_0^1 t|G_{2n-1}^\alpha(x,t)|dt = \frac{1}{n}\left|F_{2n}^\alpha(x,1/2)\right|. \tag{2.12}$$

Now that we have stated all the previously obtained results which form a basis for the results of this paper, we proceed to the main result.

3. MAIN RESULT

To shorten notation, we denote the left-hand side of (2.1) by $\int_0^1 f(t)dt - \Delta_n^\alpha(x)$, i.e.

$$\Delta_n^\alpha(x) := Q_\alpha(x) - T_{n-1}^\alpha(x)$$

for $\alpha = Q3, CQ3, Q4, CQ4, Q5, CQ5$ and $n \geq 1$.

Theorem 3.1. *Let $f : [0,1] \to \mathbb{R}$ be n-times differentiable. If $|f^{(n)}|^p$ is convex for some $p \geq 1$ and*

- *$n \geq 3$ and*
  (1) $\alpha = Q3$ *and* $x \in \{0\} \cup [1/6,\ 1/2)$,
  (2) $\alpha = Q4$ *and* $x \in (0,\ 1/2 - \sqrt{3}/6] \cup [1/3,\ 1/2]$,
- *$n \geq 5$ and*
  (1) $\alpha = CQ3$ *and* $x \in [0,\ 1/2 - \sqrt{15}/10] \cup [1/6,\ 1/2)$,
  (2) $\alpha = CQ4$ *and* $x \in (0,\ 1/2 - \sqrt{5}/10] \cup [1/3,\ 1/2]$,
  (3) $\alpha = Q5$ *and* $x \in (0,\ 1/2 - \sqrt{15}/10] \cup [1/5,\ 1/2)$,
- *$n \geq 7$ and*
  (1) $\alpha = CQ5$ *and* $x \in (0,\ 1/2 - \sqrt{21}/14] \cup [3/7 - \sqrt{2}/7,\ 1/2)$,

then we have

$$\left|\int_0^1 f(t)dt - \Delta_n^\alpha(x)\right| \leq C_\alpha(n,x) \cdot \left(\frac{|f^{(n)}(0)|^p + |f^{(n)}(1)|^p}{2}\right)^{\frac{1}{p}} \tag{3.1}$$

while if, under same conditions, $|f^{(n)}|$ is concave, then

$$\left|\int_0^1 f(t)dt - \Delta_n^\alpha(x)\right| \leq C_\alpha(n,x) \cdot \left|f^{(n)}\left(\frac{1}{2}\right)\right|, \tag{3.2}$$

where

$$C_\alpha(2k-1,\ x) = \frac{2}{(2k)!}\left|F_{2k}^\alpha\left(x, \frac{1}{2}\right)\right| \quad \textit{and} \quad C_\alpha(2k,\ x) = \frac{1}{(2k)!}\left|G_{2k}^\alpha(x,0)\right|$$

with functions F_{2k}^α defined as in (2.2) and G_{2k}^α as in (2.3), (2.4), (2.5), (2.6), (2.7) and (2.8).

Proof. First, recall that $F^{\alpha}_{2k-1}(x,t) = G^{\alpha}_{2k-1}(x,t)$. Now, starting from (2.1), we apply Hölder's and then Jensen's inequality for the convex function $|f^{(n)}|^p$, to obtain

$$\begin{aligned}
&n! \left| \int_0^1 f(t)dt - \Delta_n^{\alpha}(x) \right| \\
&\leq \int_0^1 |F_n^{\alpha}(x,t)| \cdot |f^{(n)}(t)|dt \\
&\leq \left(\int_0^1 |F_n^{\alpha}(x,t)|dt \right)^{1-\frac{1}{p}} \left(\int_0^1 |f^{(n)}((1-t)\cdot 0 + t \cdot 1)|^p \cdot |F_n^{\alpha}(x,t)|dt \right)^{\frac{1}{p}} \\
&\leq \left(\int_0^1 |F_n^{\alpha}(x,t)|dt \right)^{1-\frac{1}{p}} \left(|f^{(n)}(0)|^p \int_0^1 (1-t)|F_n^{\alpha}(x,t)|dt + |f^{(n)}(1)|^p \int_0^1 t|F_n^{\alpha}(x,t)|dt \right)^{\frac{1}{p}}.
\end{aligned}$$

Inequality (3.1) now follows from (2.11) and (2.12).

To prove (3.2), apply Jensen's integral inequality to (2.1) to obtain

$$\begin{aligned}
n! \left| \int_0^1 f(t)dt - \Delta_n^{\alpha}(x) \right| &\leq \int_0^1 |F_n^{\alpha}(x,t)| \cdot |f^{(n)}((1-t)\cdot 0 + t\cdot 1)|dt \\
&\leq \int_0^1 |F_n^{\alpha}(x,t)|dt \cdot \left| f^{(n)} \left(\frac{\int_0^1 ((1-t)\cdot 0 + t \cdot 1)|F_n^{\alpha}(x,t)|dt}{\int_0^1 |F_n^{\alpha}(x,t)|dt} \right) \right|.
\end{aligned}$$

□

Theorem 3.1 provides numerous interesting special cases. Particular choices of node will procure the Dragomir-Agarwal-type estimates for many classical quadrature formulas, as well as the adjoint corrected ones.

3.1. **CASE** $\alpha = Q3$ **and** $n = 3, 4$**.** For $x = 0$, Theorem 3.1 gives Dragomir-Agarwal-type estimates for Simpson's formula; for $x = 1/4$ it provides the estimates for the dual Simpson formula and for $x = 1/6$ for Maclaurin's formula. These were already obtained in [11]; Simpson's formula was also considered in [4].

For $x = 1/2 - \sqrt{3}/6$ ($\Leftrightarrow B_2(x) = 0$), the following estimates are obtained for the Gauss 2-point formula:

$$\Delta_n^{Q3}\left(\frac{3-\sqrt{3}}{6}\right) = \frac{1}{2} f\left(\frac{3-\sqrt{3}}{6}\right) + \frac{1}{2} f\left(\frac{3-\sqrt{3}}{6}\right)$$

and

$$\begin{aligned}
C_{Q3}\left(3,\ \frac{3-\sqrt{3}}{6}\right) &= \frac{9-4\sqrt{3}}{1728} \approx 1.2\cdot 10^{-3}, \\
C_{Q3}\left(4,\ \frac{3-\sqrt{3}}{6}\right) &= \frac{1}{4320} \approx 2.3 \cdot 10^{-4}.
\end{aligned}$$

3.2. **CASE** $\alpha = CQ3$ **and** $n = 5, 6$**.** For $x = 0$, the following estimates for the corrected Simpson's formula are produced:

$$\Delta_n^{CQ3}(0) = \frac{1}{30}\left[7f(0) + 16 f\left(\frac{1}{2}\right) + 7f(1)\right] - \frac{1}{60}[f'(1) - f'(0)]$$

and

$$C_{CQ3}(5,0) = \frac{1}{115200} \approx 8.68 \cdot 10^{-6},$$
$$C_{CQ3}(6,0) = \frac{1}{604800} \approx 1.65 \cdot 10^{-6}.$$

For $x = 1/6$, the following estimates for the corrected Maclaurin's formula are produced:

$$\Delta_n^{CQ3}\left(\frac{1}{6}\right) = \frac{1}{80}\left[27f\left(\frac{1}{6}\right) + 26f\left(\frac{1}{2}\right) + 27f\left(\frac{5}{6}\right)\right] + \frac{1}{240}[f'(1) - f'(0)]$$

and

$$C_{CQ3}\left(5,\ \frac{1}{6}\right) = \frac{1}{691200} \approx 1.45 \cdot 10^{-6},$$
$$C_{CQ3}\left(6,\ \frac{1}{6}\right) = \frac{31}{87091200} \approx 3.56 \cdot 10^{-7}.$$

For $x = 1/4$, the following estimates for the corrected dual Simpson's formula are produced:

$$\Delta_n^{CQ3}\left(\frac{1}{4}\right) = \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right] + \frac{1}{120}[f'(1) - f'(0)]$$

and

$$C_{CQ3}\left(5,\ \frac{1}{4}\right) = \frac{1}{115200} \approx 8.68 \cdot 10^{-6},$$
$$C_{CQ3}\left(6,\ \frac{1}{4}\right) = \frac{31}{19353600} \approx 1.6 \cdot 10^{-6}.$$

For $x = 1/2 - \sqrt{15}/10$ ($\Leftrightarrow\ G_2^{CQ3}(x,0) = 0$), the following estimates for the Gauss 3-point formula are produced:

$$\Delta_n^{CQ3}\left(\frac{5-\sqrt{15}}{10}\right) = \frac{1}{18}\left[5f\left(\frac{5-\sqrt{15}}{10}\right) + 8f\left(\frac{1}{2}\right) + 5f\left(\frac{5+\sqrt{15}}{10}\right)\right]$$

and

$$C_{CQ3}\left(5,\ \frac{5-\sqrt{15}}{10}\right) = \frac{25-6\sqrt{15}}{576000} \approx 3.06 \cdot 10^{-6},$$
$$C_{CQ3}\left(6,\ \frac{5-\sqrt{15}}{10}\right) = \frac{1}{2016000} \approx 4.96 \cdot 10^{-7}.$$

For $x = x_0 := 1/2 - \sqrt{225 - 30\sqrt{30}}\Big/30$ ($\Leftrightarrow\ B_4(x) = 0$) (the case when the weight next to $f(1/2)$ is annihilated), the following estimates for the corrected Gauss 2-point formula are produced:

$$\Delta_n^{CQ3}(x_0) = \frac{1}{2}f(x_0) + \frac{1}{2}f(1-x_0) - \frac{5-\sqrt{30}}{60}[f'(1) - f'(0)]$$

and

$$C_{CQ3}\left(5,\ \frac{15-\sqrt{225-30\sqrt{30}}}{30}\right) = \frac{46\sqrt{225-30\sqrt{30}}-120\sqrt{30-4\sqrt{30}}+150\sqrt{30}-825}{1728000} \approx 7.86\cdot 10^{-6},$$

$$C_{CQ3}\left(6,\ \frac{15-\sqrt{225-30\sqrt{30}}}{30}\right) = \frac{45-7\sqrt{30}}{4536000} \approx 1.47\cdot 10^{-6}.$$

3.3. **CASE** $\alpha = Q4$ **and** $n = 3, 4$**.** For $x = 1/3$, the estimates for the Simpson 3/8 formula from [11] are recaptured.

3.4. **CASE** $\alpha = CQ4$ **and** $n = 5, 6$**.** For $x = 1/3$, the following estimates for the corrected Simpson 3/8 formula are produced:

$$\Delta_n^{CQ4}\left(\frac{1}{3}\right) = \frac{1}{80}\left[13f(0)+27f\left(\frac{1}{3}\right)+27f\left(\frac{2}{3}\right)+13f(1)\right]-\frac{1}{120}[f'(1)-f'(0)]$$

and

$$C_{CQ4}\left(5,\ \frac{1}{3}\right) = \frac{1}{691200} \approx 1.45\cdot 10^{-6},$$

$$C_{CQ4}\left(6,\ \frac{1}{3}\right) = \frac{1}{2721600} \approx 3.67\cdot 10^{-7}.$$

For $x = 1/2-\sqrt{5}/10$ $\left(\Leftrightarrow G_2^{CQ4}(x,0)=0\right)$, the following estimates for the Lobatto 4-point formula are produced:

$$\Delta_n^{CQ4}\left(\frac{1}{3}\right) = \frac{1}{12}\left[f(0)+5f\left(\frac{5-\sqrt{5}}{10}\right)+5f\left(\frac{5+\sqrt{5}}{10}\right)+f(1)\right]$$

and

$$C_{CQ4}\left(5,\ \frac{5-\sqrt{5}}{10}\right) = \frac{\sqrt{5}}{576000} \approx 3.88\cdot 10^{-6},$$

$$C_{CQ4}\left(6,\ \frac{5-\sqrt{5}}{10}\right) = \frac{1}{1512000} \approx 6.61\cdot 10^{-7}.$$

3.5. **CASE** $\alpha = Q5$ **and** $n = 5, 6$**.** For $x = 1/4$, the following estimates for Boole's formula from [11] are recaptured.

3.6. **CASE** $\alpha = CQ5$ **and** $n = 7, 8$**.** For $x = 1/2-\sqrt{21}/14$ $\left(\Leftrightarrow G_2^{CQ5}(x,0)=0\right)$, the following estimates for the Lobatto 5-point formula are produced:

$$\Delta_n^{CQ5}\left(\frac{1}{4}\right) = \frac{1}{180}\left[9f(0)+49f\left(\frac{7-\sqrt{21}}{14}\right)+64f\left(\frac{1}{2}\right)+49f\left(\frac{7+\sqrt{21}}{14}\right)+9f(1)\right]$$

and

$$C_{CQ5}\left(7,\ \frac{7-\sqrt{21}}{14}\right) = \frac{12\sqrt{21}-49}{1264435200} \ \approx\ 4.74\cdot 10^{-9},$$
$$C_{CQ5}\left(8,\ \frac{7-\sqrt{21}}{14}\right) = \frac{1}{1422489600} \ \approx\ 7.03\cdot 10^{-10}.$$

For $x = 1/4$, the following estimates for the corrected Boole's formula are produced:

$$\Delta_n^{CQ5}\left(\frac{1}{4}\right) = \frac{1}{1890}\left[217f(0) + 512f\left(\frac{1}{4}\right) + 432f\left(\frac{1}{2}\right) + 512f\left(\frac{3}{4}\right) + 217f(1)\right] - \frac{1}{252}[f'(1) - f'(0)]$$

and

$$C_{CQ5}\left(7,\ \frac{1}{4}\right) = \frac{17}{4877107200} \ \approx\ 3.49\cdot 10^{-9},$$
$$C_{CQ5}\left(8,\ \frac{1}{4}\right) = \frac{1}{1625702400} \ \approx\ 6.15\cdot 10^{-10}.$$

Further, for $x = 1/2 - \sqrt{7}/14$ ($\Leftrightarrow\ 14x^2 - 14x + 3 = 0$), which is the case when the weight next to $f(1/2)$ is annihilated, the following estimates for the corrected Lobatto 4-point formula are produced:

$$\Delta_n^{CQ5}\left(\frac{7-\sqrt{7}}{14}\right) = \frac{1}{270}\left[37f(0) + 98f\left(\frac{7-\sqrt{7}}{14}\right) + 98f\left(\frac{7+\sqrt{7}}{14}\right) + 37f(1)\right] - \frac{1}{180}[f'(1) - f'(0)]$$

and

$$C_{CQ5}\left(7,\ \frac{7-\sqrt{7}}{14}\right) = \frac{343-16\sqrt{7}}{34139750400} \ \approx\ 8.81\cdot 10^{-9},$$
$$C_{CQ5}\left(8,\ \frac{7-\sqrt{7}}{14}\right) = \frac{1}{711244800} \ \approx\ 1.41\cdot 10^{-9}.$$

Finally, for

$$x = x_0 := \frac{1}{2} - \frac{\sqrt{45-2\sqrt{102}}}{14}$$
$$\left(\Leftrightarrow\ 98x^4 - 196x^3 + 102x^2 - 4x - 1 = 98\left(x^2 - x + \frac{1}{49} - \frac{\sqrt{102}}{98}\right)\left(x^2 - x + \frac{1}{49} + \frac{\sqrt{102}}{98}\right) = 0\right),$$

which is the case when the weight next to $f(0)$ and $f(1)$ is annihilated, the estimates for the corrected Gauss 3-point formula are produced:

$$\Delta_n^{CQ5}(x_0) = \frac{1977 + 16\sqrt{102}}{6930}[f(x_0) + f(1 - x_0)] + \frac{1488 - 16\sqrt{102}}{3465} f\left(\frac{1}{2}\right) - \frac{9 - \sqrt{102}}{420}[f'(1) - f'(0)]$$

and

$$C_{CQ5}(7,\ x_0) = \frac{24\sqrt{60933 - 6014\sqrt{102}} - 49(87 - 8\sqrt{102})}{3793305600} \approx 8.12 \cdot 10^{-9},$$
$$C_{CQ5}(8,\ x_0) = \frac{43 - 3\sqrt{102}}{9957427200} \approx 1.28 \cdot 10^{-9}.$$

Remark 1. An interesting fact to point out is that out of all the 3-point quadrature formulae, Maclaurin's formula gives the least estimate of error in Theorem 3.1; among the corrected 3-point quadrature formulae, the corrected Maclaurin's formula has the same property.

Further, among the closed 4-point quadrature formulas, the Simpson 3/8 formula gives the best estimate and the corrected Simpson 3/8 formula is the optimal corrected closed 4-point quadrature formula.

Finally, the node $x = 1/5$ produces the closed 5-point quadrature formula with the best error estimate, while the node $x = 3/7 - \sqrt{2}/7$ produces the corrected closed 5-point quadrature formula with the same property.

The proofs are similar to those in [5], [6], [7] and [8], respectively.

In view of the previous remark, let us consider the case $\alpha = Q5$, $n = 5, 6$ and $x = 1/5$. We have:

$$\Delta_n^{Q5}\left(\frac{1}{5}\right) = \frac{1}{432}\left[27f(0) + 125f\left(\frac{1}{5}\right) + 128f\left(\frac{1}{2}\right) + 125f\left(\frac{4}{5}\right) + 27f(1)\right]$$

and

$$C_{Q5}\left(5,\ \frac{1}{5}\right) = \frac{1}{1152000} \approx 8.68 \cdot 10^{-7},$$
$$C_{Q5}\left(6,\ \frac{1}{5}\right) = \frac{1}{5040000} \approx 1.98 \cdot 10^{-7}.$$

Further, for $\alpha = CQ5$, $n = 7, 8$ and $x = x_0 := 3/7 - \sqrt{2}/7$ we obtain:

$$\Delta_n^{CQ5}(x_0) = 0.10143\,[f(0) + f(1)] + 0.259261\,[f(x_0) + f(1 - x_0)] + 0.278617\,f\left(\frac{1}{2}\right) + 3.07832 \cdot 10^{-3}\,[f'(1) - f'(0)]$$

and

$$C_{CQ5}(7, x_0) = \frac{27 - 16\sqrt{2}}{3793305600} \approx 1.15 \cdot 10^{-9},$$
$$C_{CQ5}(8, x_0) = \frac{11 - 6\sqrt{2}}{9957427200} \approx 2.53 \cdot 10^{-10}.$$

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