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NEW INEQUALITIES INVOLVING THE ZETA FUNCTION

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ABSTRACT. Inequalities involving the Euler zeta function are proved. Applications of the inequalities in estimating the zeta function at odd integer values in terms of the known zeta function at even integer values are discussed.

Key words and phrases: Euler's zeta function, Zeta function inequality, Approximation of the zeta function at odd integral values, Bounds.

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1. INTRODUCTION

The Zeta function

(1.1) $$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}, \qquad x > 1$$

was originally introduced in 1737 by the Swiss mathematician Leonhard Euler (1707-1783) for real x who proved the identity

(1.2) $$\zeta(x) := \prod_p \left(1 - \frac{1}{p^x}\right)^{-1}, \qquad x > 1,$$

where p runs through all primes. It was Riemann who allowed x to be a complex variable and showed that even though both sides of (1.1) and (1.2) diverge for Re$(x) \leq 1$, the function has a continuation to the whole complex plane with a simple pole at $x = 1$ with residue 1. The function plays a very significant role in the theory of the distribution of primes. One of the most striking properties of the zeta function, discovered by Riemann himself, is the functional equation

(1.3) $$\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x)$$

that can be written in symmetric form to give

(1.4) $$\pi^{-\frac{x}{2}} \Gamma\left(\frac{x}{2}\right) \zeta(x) = \pi^{-\left(\frac{1-x}{2}\right)} \Gamma\left(\frac{1-x}{2}\right) \zeta(1-x).$$

The function $\Gamma(x)$ is the Gamma function

(1.5) $$\Gamma(x) := \int_0^1 (-\log t)^{x-1} dt, \qquad x > 0,$$

introduced by Euler in 1730. The gamma function has the integral representation

(1.6) $$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt, \qquad x > 0,$$

or equivalently

(1.7) $$\Gamma(x) := 2\int_0^\infty e^{-t^2} t^{2x-1} dt, \qquad x > 0,$$

and satisfies the important relations

(1.8) $$\Gamma(x+1) = x\Gamma(x),$$

(1.9) $$\Gamma(x)\Gamma(1-x) = \pi \csc x, \qquad x \text{ non integer}.$$

In addition to the relation (1.3) between the zeta and the gamma function, these functions are also connected via the integrals [3]

(1.10) $$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1} dt}{e^t - 1}, \qquad x > 1,$$

and

(1.11) $$\zeta(x) = \frac{1}{C(x)} \int_0^\infty \frac{t^{x-1} dt}{e^t + 1}, \qquad x > 0,$$

where

(1.12) $$C(x) := \Gamma(x)\left(1 - 2^{1-x}\right).$$

The zeta function is monotonically decreasing in the interval $(0, 1)$. It has a simple pole at $x = 1$ with residue 1. It is monotonically decreasing once again in the interval $(1, \infty)$. Thus one has the inequality

$$\zeta(x+1) \le \zeta(x), \qquad x > 0. \tag{1.13}$$

It is the intention of the current paper to obtain better upper bounds than (1.13) and also procure lower bounds. These bounds are procured from utilising a functional equation involving the zeta function evaluated at a distance of one apart. This enables the approximation of the zeta function at odd integer arguments in terms of the explicitly known zeta values at the even integers with *a priori* bounds on the error.

2. MAIN RESULTS

The following identity will prove crucial in obtaining bounds for the Zeta function.

Lemma 2.1. *The following identity involving the Zeta function holds. Namely,*

$$\int_0^\infty \frac{t^x}{(e^t+1)^2} dt = C(x+1)\zeta(x+1) - xC(x)\zeta(x), \qquad x > 0, \tag{2.1}$$

where $C(x)$ is as given by (1.12).

Proof. Consider the auxiliary function

$$f(t) := \frac{1}{e^t+1}, \qquad t > 0 \tag{2.2}$$

that has the derivative given by

$$f'(t) = -f(t) + \frac{1}{(e^t+1)^2}. \tag{2.3}$$

Taking the Mellin transform of both sides in the real variable α in (2.2)–(2.3) and using (1.11), we find

$$M[f;\alpha] = C(\alpha)\zeta(\alpha), \text{ and} \tag{2.4}$$

$$M[f';\alpha] = -C(\alpha)\zeta(\alpha) + M\left[\frac{1}{(e^t+1)^2}, \alpha\right]. \tag{2.5}$$

However, $M[f;\alpha]$ and $M[f';\alpha]$ are related via

$$M[f';\alpha] = -(\alpha-1)M[f;\alpha-1], \tag{2.6}$$

provided $t^{\alpha-1}f(t)$ vanishes at zero and infinity. Hence, from (2.4)–(2.6), we find

$$M\left[\frac{1}{(e^t+1)^2};\alpha\right] = C(\alpha)\zeta(\alpha) - (\alpha-1)C(\alpha-1)\zeta(\alpha-1). \tag{2.7}$$

Replacing α by $x+1$ in (2.7) readily produces the stated result (2.1). □

Theorem 2.2. *The Zeta function satisfies the bounds*

$$(1-b(x))\zeta(x) + \frac{b(x)}{8} \le \zeta(x+1) \le (1-b(x))\zeta(x) + \frac{b(x)}{2}, \qquad x > 0, \tag{2.8}$$

where

$$b(x) := \frac{1}{2^x-1}. \tag{2.9}$$

Proof. From identity (2.1) let

$$A(x) := \int_0^\infty \frac{t^x}{(e^t+1)^2}dt, \qquad x > 0. \tag{2.10}$$

Now,

$$\frac{1}{e^t+1} = \frac{e^{-t}}{1+e^{-t}} \tag{2.11}$$

and so

$$\frac{e^{-t}}{2} = \frac{e^{-t}}{\max_{t\in\mathbb{R}^+}(1+e^{-t})} \le \frac{1}{e^t+1} \le \frac{e^{-t}}{\min_{t\in\mathbb{R}^+}(1+e^{-t})} = e^{-t}. \tag{2.12}$$

Thus,

$$\frac{e^{-2t}}{4} \le \frac{1}{(e^t+1)^2} \le e^{-2t}$$

producing from (2.10)

$$\frac{\Gamma(x+1)}{2^{x+3}} \le A(x) \le \frac{\Gamma(x+1)}{2^{x+1}}, \tag{2.13}$$

where we have used the fact that

$$\int_0^\infty e^{-st}t^x dt = \frac{\Gamma(x+1)}{s^{x+1}}. \tag{2.14}$$

The result (2.8) is procured on using the identity (2.1), the definition (2.10) and the bounds (2.13) on noting from (2.13) and (2.9), that

$$\frac{xC(x)}{C(x+1)} = 1 - b(x) \tag{2.15}$$

and

$$\frac{\Gamma(x+1)}{2^{x+\gamma}C(x+1)} = \frac{b(x)}{2^\gamma}. \tag{2.16}$$

The theorem is thus proved. □

Remark 2.3. The lower bound $(1-b(x))\zeta(x)$ is obtained for $\zeta(x+1)$ if we use the result, from (2.10), that $0 \le A(x)$ rather than the sharper bound as given in (2.13). The lower bound for $\zeta(x+1)$ as given in (2.8) is better by the amount $\frac{b(x)}{8} > 0$.

Further, the bound (1.13), is obviously inferior to the upper bound in (2.8) since

$$(1-b(x))\zeta(x) + \frac{b(x)}{2} = \zeta(x) + b(x)\left[\frac{1}{2} - \zeta(x)\right]$$

and for $x > 1$, $\zeta(x) > 1$ giving $b(x)\left[\frac{1}{2} - \zeta(x)\right] < 0$.

Corollary 2.4. *The bound*

$$\left|\zeta(x+1) - (1-b(x))\zeta(x) - \frac{5}{16}b(x)\right| \le \frac{3}{16}b(x) \tag{2.17}$$

holds, where $b(x)$ is as given by (2.9).

Proof. Let

$$L(x) = (1 - b(x))\zeta(x) + \frac{b(x)}{8}, \;\; U(x) = (1 - b(x))\zeta(x) + \frac{b(x)}{2} \tag{2.18}$$

then from (2.8) we have

$$L(x) \le \zeta(x+1) \le U(x).$$

Hence

$$-\frac{U(x) - L(x)}{2} \le \zeta(x+1) - \frac{U(x) + L(x)}{2} \le \frac{U(x) - L(x)}{2}$$

which may be expressed as the stated result (2.17) on noting the obvious correspondences and simplification. □

Remark 2.5. The form (2.17) is a useful one since we may write

$$\zeta(x+1) = (1 - b(x))\zeta(x) + \frac{5}{16} b(x) + E(x), \tag{2.19}$$

where

$$|E(x)| < \varepsilon$$

for

$$x > x^* := \ln 2 \cdot \ln\left(1 + \frac{3}{16\varepsilon}\right).$$

That is, we may approximate $\zeta(x+1)$ by $(1 - b(x))\zeta(x) + \frac{5}{16} b(x)$ within an accuracy of ε for $x > x^*$.

We note that both the result of Theorem 2.2 and Corollary 2.4 as expressed in (2.8) and (2.17) respectively rely on approximating $\zeta(x+1)$ in terms of $\zeta(x)$. The following result involves approximating $\zeta(x+1)$ in terms of $\zeta(x+2)$, the subsequent zeta values within a distance of one rather than the former zeta values.

Theorem 2.6. *The zeta function satisfies the bounds*

$$L_2(x) \le \zeta(x+1) \le U_2(x), \tag{2.20}$$

where

$$L_2(x) = \frac{\zeta(x+2) - \frac{b(x+1)}{2}}{1 - b(x+1)} \;\; \textit{and} \;\; U_2(x) = \frac{\zeta(x+2) - \frac{b(x+1)}{8}}{1 - b(x+1)}. \tag{2.21}$$

Proof. From (2.8) we have

$$0 \le \frac{b(x)}{8} \le \zeta(x+1) - (1 - b(x))\zeta(x) \le \frac{b(x)}{2}$$

and so

$$-\frac{b(x)}{2} \le (1 - b(x))\zeta(x) - \zeta(x+1) \le -\frac{b(x)}{8}$$

to produce

$$\zeta(x+1) - \frac{b(x)}{2} \le (1 - b(x))\zeta(x) \le \zeta(x+1) - \frac{b(x)}{8}.$$

A rearrangement and change of x to $x+1$ produces the stated result (2.20) – (2.21). □

The following corollary is valid in which $\zeta(x+1)$ may be approximated in terms of $\zeta(x+2)$ and an explicit bound is provided for the error.

Corollary 2.7. *The bound*

$$\left|\zeta(x+1)-\frac{\zeta(x+2)-\frac{5}{16}b(x+1)}{1-b(x+1)}\right|\le\frac{3}{16}\cdot\frac{b(x+1)}{1-b(x+1)} \tag{2.22}$$

holds, where $b(x)$ is as defined by (2.9).

Proof. The proof is straight forward and follows that of Corollary 2.4 with $L(x)$ and $U(x)$ replaced by $L_2(x)$ and $U_2(x)$ as defined by (2.21). □

Corollary 2.8. *The zeta function satisfies the bounds*

$$\max\{L(x),L_2(x)\}\le\zeta(x+1)\le\min\{U(x),U_2(x)\}, \tag{2.23}$$

where $L(x), U(x)$ are given by (2.18) and $L_2(x), U_2(x)$ by (2.21).

Remark 2.9. Some experimentation using the Maple computer algebra package indicates that the lower bound $L_2(x)$ is better than the lower bound $L(x)$ for $x > x_* = 0.542925\ldots$ and vice versa for $x < x_*$. In a similar manner the upper bound $U_2(x)$ is better than $U(x)$ for $x < x^* = 2.96415283\ldots$ and vice versa for $x > x^*$. The results of this section will be utilised in the next section to obtain bounds for odd integer values of the zeta function, namely, $\zeta(2n+1), n\in\mathbb{N}$.

Remark 2.10. The Figure 2.1 plots $\zeta(x+1)$, its approximation by an expression involving $\zeta(x)$ and the bound as given by (2.17). For $x=4$ the approximation of $\zeta(5)$ has a bound on the error of 0.0125. Figure 2.2 shows a plot of $\zeta(x+1)$ and its approximation by an expression involving $\zeta(x+2)$ (which are indistinguishable) and the bound as given by (2.22).

3. Approximation of the Zeta functions at odd integers

In the series expansion

$$\frac{te^{xt}}{e^t-1}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}, \tag{3.1}$$

where $B_m(x)$ are the Bernoulli polynomials (after Jacob Bernoulli), $B_m(0)=B_m$ are the Bernoulli numbers. They occurred for the first time in the formula [1, p. 804]

$$\sum_{k=1}^{m}k^n=\frac{B_{n+1}(m+1)-B_{n+1}}{n+1},\qquad n,m=1,2,3,\ldots\,. \tag{3.2}$$

One of Euler's most celebrated theorems discovered in 1736 (Institutiones Calculi Differentialis, Opera (1), Vol. 10) is

$$\zeta(2n)=(-1)^{n-1}\frac{2^{2n-1}\pi^{2n}}{(2n)!}B_{2n};\qquad n=1,2,3,\ldots\,. \tag{3.3}$$

The result may also be obtained in a straight forward fashion from (1.11) and a change of variable on using the fact that

$$B_{2n}=(-1)^{n-1}\cdot 4n\int_0^{\infty}\frac{t^{2n-1}}{e^{2\pi t}-1}dt \tag{3.4}$$

from Whittaker and Watson [9, p. 126].

Despite several efforts to find a formula for $\zeta(2n+1)$, (for example [5, 7, 11]), there seems to be no elegant representation for the zeta function at the odd integer values. Several series representations for the value $\zeta(2n+1)$ have been proved by Srivastava, Tsumura, Zhang and Williams.

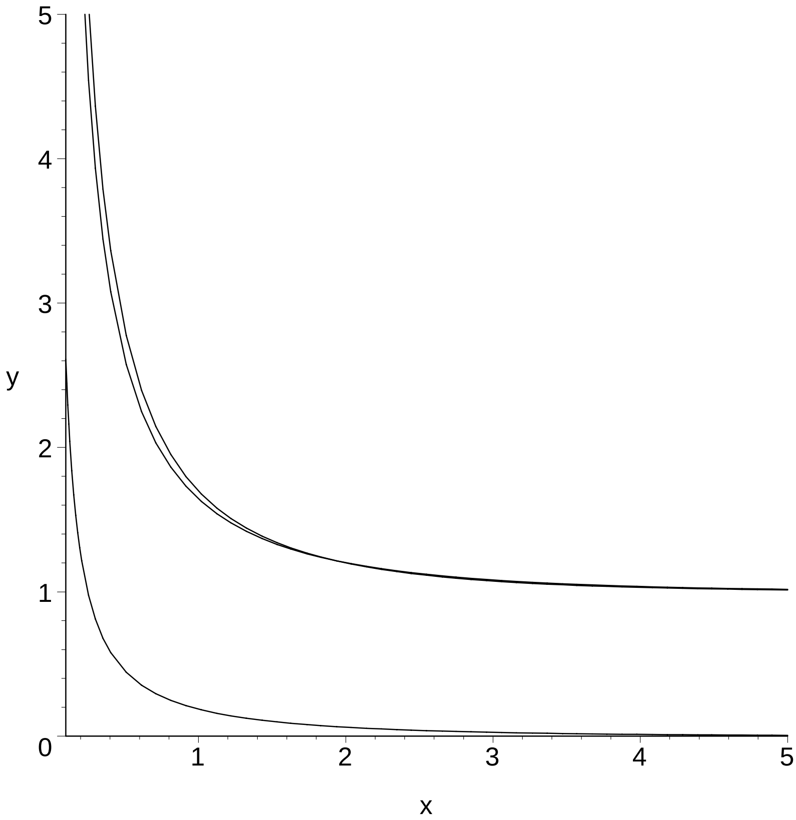

Figure 2.1: Plot of $\zeta(x+1)$, its approximation $(1-b(x))\zeta(x)+\frac{5}{16}b(x)$ and error bound $\frac{3}{16}b(x)$ where $b(x)$ is as given by (2.9). This represents the implementation of Corollary 2.4.

From a long list of these representations, [5, 7], we quote only a few

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\pi^{2n}\Bigg[&\frac{H_{2n+1}-\log\pi}{(2n+1)!}\\ &+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{\pi^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\frac{\zeta(2k)}{2^{2k}}\Bigg],\end{aligned} \tag{3.5}$$

$$\begin{aligned}\zeta(2n+1) = (-1)^{n}\frac{(2\pi)^{2n}}{n(2^{2n+1}-1)}\Bigg[&\sum_{k=1}^{n-1}\frac{(-1)^{k-1}k}{(2n-2k)!}\frac{\zeta(2k)}{\pi^{2k}}\\ &+\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k)!}\frac{\zeta(2k)}{2^{2k}}\Bigg], \quad \text{and}\end{aligned} \tag{3.6}$$

$$\begin{aligned}\zeta(2n+1) = (-1)^{n}\frac{(2\pi)^{2n}}{(2n-1)2^{2n}+1}\Bigg[&\sum_{k=1}^{n-1}\frac{(-1)^{k-1}k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{\pi^{2k}}\\ &+\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k+1)!}\frac{\zeta(2k)}{2^{2k}}\Bigg], \qquad n=1,2,3,\dots\,.\end{aligned} \tag{3.7}$$

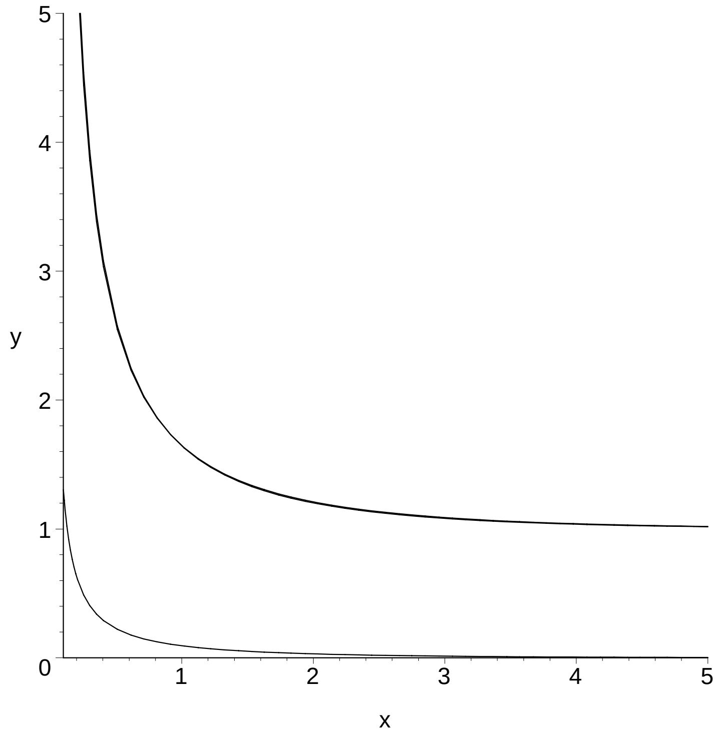

Figure 2.2: Plot of $\zeta(x+1)$, *its approximation* $\frac{\zeta(x+2)-\frac{5}{16}b(x+1)}{1-b(x+1)}$ *and its bound* $\frac{3}{16}\cdot\frac{b(x+1)}{1-b(x+1)}$ *where* $b(x)$ *is as given by (2.9). This represents the implementation of Corollary 2.7.*

There is also an integral representation for $\zeta(n+1)$ namely,

$$\zeta(2n+1) = (-1)^{n+1}\cdot\frac{(2\pi)^{2n+1}}{2\delta(n+1)!}\int_0^{\delta} B_{2n+1}(t)\cot(\pi t)\,dt, \tag{3.8}$$

where $\delta = 1$ or $\frac{1}{2}$ ([1, p. 807]). Recently, Cvijović and Klinkowski [2] have given the integral representations

$$\zeta(2n+1) = (-1)^{n+1}\cdot\frac{(2\pi)^{2n+1}}{2\delta(1-2^{-2n})(2n+1)!}\int_0^{\delta} B_{2n+1}(t)\tan(\pi t)\,dt, \tag{3.9}$$

and

$$\zeta(2n+1) = (-1)^{n}\cdot\frac{\pi^{2n+1}}{4\delta(1-2^{-(2n+1)})(2n)!}\int_0^{\delta} E_{2n}(t)\csc(\pi t)\,dt. \tag{3.10}$$

Both the series representations (3.5) – (3.7) and the integral representations (3.8) – (3.9) are however both somewhat difficult in terms of computational aspects and time considerations.

In the current section we explore how the results of Section 2 may be exploited to obtain bounds on $\zeta(2n+1)$ in terms of $\zeta(2n)$, which is explicitly given by (3.3).

Taking $x = 2n$ in the results of the previous section, we may obtain from (2.18) and (2.21), using (2.9), that

$$\begin{cases} L(2n) = \left(\frac{4^n-2}{4^n-1}\right)\zeta(2n) + \frac{1}{8(4^n-1)}; \\ U(2n) = \left(\frac{4^n-2}{4^n-1}\right)\zeta(2n) + \frac{1}{2(4^n-1)}; \\ L_2(2n) = \frac{2(2\cdot 4^n-1)\zeta(2n+2)-1}{4(4^n-1)}; \\ U_2(2n) = \frac{8(2\cdot 4^n-1)\zeta(2n+2)-1}{18(4^n-1)}. \end{cases} \tag{3.11}$$

Table 1 provides lower and upper bounds for $\zeta(2n+1)$ for $n = 1, \dots, 5$, utilising Theorems 2.2 and 2.6 for $x = 2n$ and so explicitly using (3.11). We notice that $L_2(n)$ is better than $L_1(2n)$ and $U(2n)$ is better than $U_2(2n)$ only for $n = 1$ (see also Remark 2.10). Tables 2 and 3 give the use of Corollaries 2.4 and 2.7 for $x = 2n$. Thus, the table provides $\zeta(2n+1)$, its approximation and the bound on the error.

n	$\mathbf{L}(2n)$	$\mathbf{L}_2(2n)$	$\zeta(2n+1)$	$\mathbf{U}(2n)$	$\mathbf{U}_2(2n)$
1	1.138289378	1.179377107	1.202056903	1.263289378	1.241877107
2	1.018501685	1.034587831	1.036927755	1.043501685	1.047087831
3	1.003178887	1.008077971	1.008349277	1.009131268	1.011054162
4	1.000629995	1.001976919	1.002008393	1.002100583	1.002712213
5	1.000138278	1.000490588	1.000494189	1.000504847	1.000673872

Table 1. Table of $L(2n)$, $L_2(2n)$, $\zeta(2n+1)$, $U(2n)$ and $U_2(2n)$ as given by (2.18) and (2.21) for $n = 1, \dots, 5$.

n	$\zeta(2n+1)$	$\frac{U(2n)+L(2n)}{2}$	$\frac{U(2n)-L(2n)}{2}$
1	1.202056903	1.200789378	0.06250000000
2	1.036927755	1.031001685	0.01250000000
3	1.008349277	1.006155077	0.002976190476
4	1.002008393	1.001365289	0.0007352941176
5	1.000494189	1.000321562	0.0001832844575

Table 2. Table of $\zeta(2n+1)$, its approximation $\frac{U(2n)+L(2n)}{2}$ and its bound $\frac{U(2n)-L(2n)}{2}$ for $n = 1, \dots, 5$ where $U(2n)$ and $L(2n)$ are given by (3.11).

n	$\zeta(2n+1)$	$\frac{U_2(2n)+L_2(2n)}{2}$	$\frac{U_2(2n)-L_2(2n)}{2}$
1	1.202056903	1.210627107	0.03125000000
2	1.036927755	1.040837831	0.00625000000
3	1.008349277	1.009566066	0.001488095238
4	1.002008393	1.002344566	0.0003676470588
5	1.000494189	1.000582230	0.00009164222874

Table 3. Table of $\zeta(2n+1)$, its approximation $\frac{U_2(2n)+L_2(2n)}{2}$ and its bound $\frac{U_2(2n)-L_2(2n)}{2}$ for $n = 1, \dots, 5$ where $U_2(2n)$ and $L_2(2n)$ are given by (3.11).

4. IMPROVEMENTS ON THE BOUNDS

The results of Sections 2 and 3 rely heavily on the representation (2.11) which allows us to obtain both lower and upper bounds as demonstrated by (2.12).

In this section we examine whether a different representation of $\frac{1}{e^t+1}$, other than that given by (2.11), would provide better bounds. Consider

$$H_\lambda(t) = \frac{e^{-\lambda t}}{e^{(1-\lambda)t} + e^{-\lambda t}}, \quad 0 \le \lambda \le 1. \tag{4.1}$$

We note that

$$H_0(t) = \frac{1}{e^t+1} \text{ and } H_1(t) = \frac{e^{-t}}{1+e^{-t}}.$$

Now if we denote the denominator of $H_\lambda(t)$ in (4.1) by $h_\lambda(t)$, then

$$h_\lambda(t) = e^{(1-\lambda)t} + e^{-\lambda t}$$

has a number of interesting properties.

We have already investigated the situation for $\lambda = 1$ in Section 2 to show that

$$1 = \lim_{t\to\infty} h_1(t) \le h_1(t) \le h_1(0) = 2. \tag{4.2}$$

For $0 \le \lambda < 1$ the upper bound is infinite. The lower bound however either occurs at $h_\lambda(0)$ for $0 \le \lambda \le \frac{1}{2}$ or at $t = t^*$ where $h'_\lambda(t^*) = 0$ giving the lower bound as $h_\lambda(t^*)$ for $\frac{1}{2} \le \lambda < 1$. Here, a simple calculation shows that for $\frac{1}{2} \le \lambda < 1$ to give

$$t^* = \ln\left(\frac{\lambda}{1-\lambda}\right), \quad \text{positive,} \tag{4.3}$$

and so

$$h_\lambda(t^*) = \frac{1}{1-\lambda}\left(\frac{1-\lambda}{\lambda}\right)^\lambda. \tag{4.4}$$

Figure 4.1 shows a plot of $\frac{1}{h_\lambda(t)}$ for $\lambda = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1 in order from bottom to top. The lower bound $H_\lambda(t)$ is zero for $0 \le \lambda < 1$ and $\frac{e^{-\lambda t}}{2}$ for $\lambda = 1$.

The upper bounds for $H_\lambda(t)$ are given by

$$H_\lambda(t) \le \begin{cases} \frac{e^{-\lambda t}}{2}, & 0 \le \lambda \le \frac{1}{2}; \\ \frac{e^{-\lambda t}}{h_\lambda(t^*)}, & \frac{1}{2} \le \lambda < 1; \\ e^{-\lambda t}, & \lambda = 1. \end{cases} \tag{4.5}$$

Now, from (2.10) and using (4.5) and (2.14), we have

$$A(x) = \int_0^\infty \frac{t^x}{(e^t+1)^2}dt \le \begin{cases} \frac{\Gamma(x+1)}{4(2\lambda)^{x+1}}, & 0 < \lambda \le \frac{1}{2}; \\ \frac{\Gamma(x+1)}{h_\lambda^2(t^*)(2\lambda)^{x+1}}, & \frac{1}{2} < \lambda < 1; \\ \frac{\Gamma(x+1)}{2^{x+1}}, & \lambda = 1. \end{cases}$$

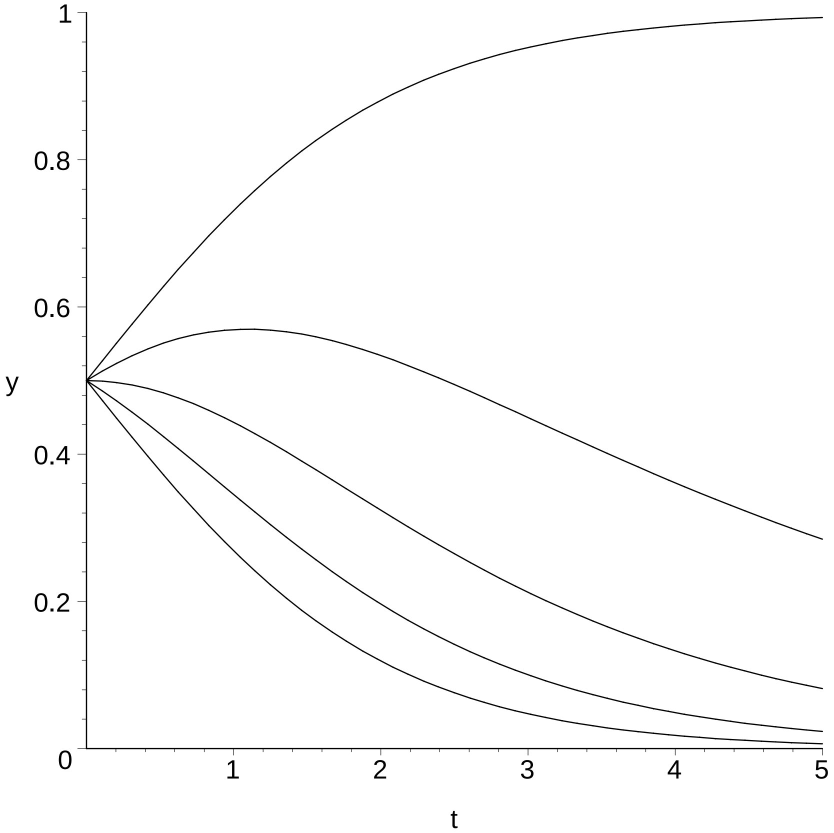

Figure 4.1: Plot of $\frac{1}{h_\lambda(t)}$ for $\lambda = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1 in order from bottom to top. The lower bound $H_\lambda(t)$ is zero for $0 \le \lambda < 1$ and $\frac{e^{-\lambda t}}{2}$ for $\lambda = 1$.

That is, from (2.16) we have $b(x) = \frac{\Gamma(x+1)}{2^x C(x+1)}$ and so

$$\frac{A(x)}{C(x+1)} \le \begin{cases} \frac{b(x)}{8\cdot\lambda^{x+1}}, & 0 < \lambda \le \frac{1}{2}; \\ \frac{b(x)}{2\cdot\lambda^{x-1}\left(\frac{1-\lambda}{\lambda}\right)^{2(\lambda-1)}}, & \frac{1}{2} < \lambda < 1; \\ \frac{b(x)}{2}, & \lambda = 1. \end{cases} \tag{4.6}$$

The following lemma provides the best upper bound for a given x. This involves deciding the λ that provides the sharpest bound for a given x.

Lemma 4.1. *For $A(x)$ as given by (2.10),*

$$\frac{A(x)}{C(x+1)} \le \frac{b(x)}{2\theta(\lambda^*, x)} \tag{4.7}$$

provides the sharpest bound where

$$\theta(\lambda, x) = \lambda^{x-1}\left(\frac{\lambda}{1-\lambda}\right)^{2(1-\lambda)} \tag{4.8}$$

and

$$\lambda^* = \frac{1}{z} \tag{4.9}$$

with z the solution of

$$z = 1 + e^{-\frac{x+1}{2}\cdot z}. \tag{4.10}$$

Proof. From (4.6), it may be noticed that a comparison of $4 \cdot \lambda^{x+1}$, $\theta(\lambda, x)$ and 1 needs to be made over its respective region of validity for λ.

It may be further noticed that the maximum of $4\lambda^{x+1}$ over $0 < \lambda \leq \frac{1}{2}$ is at $\lambda = \frac{1}{2}$, namely, $\left(\frac{1}{2}\right)^{x-1}$.

Differentiation of (4.8) with respect to λ gives

$$\frac{\theta'}{\theta} = \frac{x+1}{\lambda} + 2\ln\left(\frac{1-\lambda}{\lambda}\right)$$

the critical point, λ^*, which simplifies (4.9) – (4.10), may be shown to provide a maximum, as may be seen from an investigation of the second derivative, at $\lambda = \lambda^*$. □

Remark 4.2. Figure 4.2 demonstrates the upper bound for $2\frac{A(x)}{b(x)}$ from (4.6) for $0 < \lambda < 1$. Figure 5 is a plot of λ^* which satisfies (4.9) – (4.10), which is an increasing function from $0.676741\ldots$ to 1. Figure 6 shows a plot from top to bottom of $\theta(\lambda^*, x), 1$ and $2^{-(x-1)}$. We notice that $2^{-(x-1)} > 1$ only for $0 \leq x < 1$ while $\theta(\lambda^*, x) > 1$ for all x. That is, 1 is the best uniform lower bound so that $A(x) < \frac{b(x)}{2}$.

Theorem 4.3. *The Zeta function satisfies the bounds*

$$(1 - b(x))\zeta(x) + \frac{b(x)}{8} \leq \zeta(x+1) \leq (1 - b(x))\zeta(x) + \frac{b(x)}{2\theta(\lambda^*, x)} := U^*(x) \tag{4.11}$$

where $b(x)$ is as given by (2.9), $\theta(\lambda, x)$ by (4.8) and λ^* satisfies (4.9) – (4.10).

Proof. From Lemma 4.1 and identity (2.1) we have

$$\frac{A(x)}{C(x+1)} = \zeta(x+1) - \frac{xC(x)}{C(x+1)}\zeta(x) \leq \frac{b(x)}{2\theta(\lambda^*, x)}$$

and so

$$\zeta(x+1) \leq \frac{xC(x)}{C(x+1)}\zeta(x) + \frac{b(x)}{2\theta(\lambda^*, x)},$$

giving the right inequality in (4.11) on using (2.15).

The lower bound is that obtained previously in Theorem 2.2. It results from the $\lambda = 1$ case as discussed above in this section. □

Remark 4.4. The upper bound $U^*(x)$ in (4.11) is the best possible for a given x.

Theorem 4.5. *The Zeta function satisfies the bounds*

$$L_2^*(x) \leq \zeta(x+1) \leq U_2(x) \tag{4.12}$$

where

$$L_2^*(x) = \frac{\zeta(x+2) - \frac{b(x+1)}{2\theta(\lambda^*, x)}}{1 - b(x+1)} \tag{4.13}$$

with $U_2(x)$ defined by (2.20) and $\theta(\lambda^*, x)$ by (4.8) – (4.10).

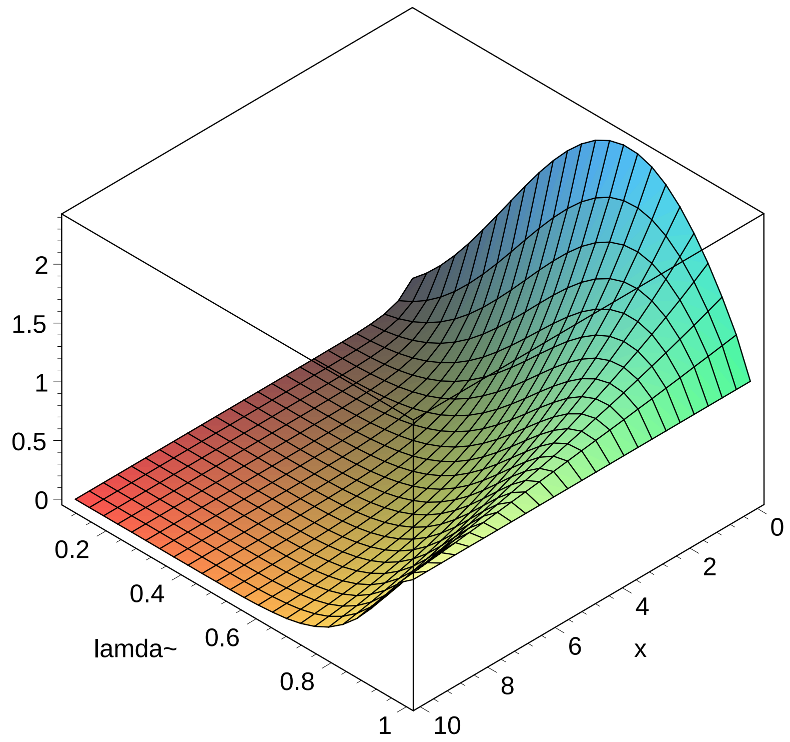

Figure 4.2: A plot of $4 \cdot \lambda^{x+1}$, $0 < \lambda \leq \frac{1}{2}$ *and* $\theta(\lambda, x) = \lambda^{x-1}\left(\frac{\lambda}{1-\lambda}\right)^{2(1-\lambda)}$, $\frac{1}{2} \leq \lambda < 1$.

Proof. The proof is straightforward from Theorem 4.3 and utilising the methodology in proving Theorem 2.6 from Theorem 2.2. □

Results corresponding to Corollaries 2.4 and 2.7 are possible where $U(x)$ is replaced by $U^*(x)$ and $L_2(x)$ by $L_2^*(x)$.

Corollary 4.6. *The Zeta function satisfies the bounds*

$$\max\{L(x), L_2^*(x)\} \leq \zeta(x+1) \leq \min\{U^*(x), U_2(x)\}, \tag{4.14}$$

where $L(x)$ *is defined by (2.18),* $L_2^*(x)$ *by (4.13),* $U^*(x)$ *by (4.11) and* $U_2(x)$ *by (2.21).*

Remark 4.7. Experimentation using Maple indicates that the critical point is $x_* = 0.3346397\ldots$ with $L_2^*(x) > L(x)$ for $x > x_*$ and vice versa for $x < x_*$. Further, for $x > x^* = 2.755424387\ldots$ $U^*(x) < U_2(x)$ and vice versa for $x < x^*$. These critical points x_* and x^*, at which the bounds procured in terms of $\zeta(x)$ and $\zeta(x+2)$ cross, are to be compared to those in Remark 2.9.

Figure 4.5 gives a graphical representation of (4.14).

Table 4 gives lower and upper bounds for $\zeta(2n+1)$ for $n = 1, \ldots, 5$ utilizing Theorems 4.3 and 4.5 for $x = 2n$. We notice that $L_2^*(x)$ provides a stronger lower bound than $L_2(2n)$ from Table 4. As discussed in Remark 4.7, $L_2^*(x)$ is better than $L(x)$ in the region under the representation here. We also notice that $U^*(2n)$ here is better than $U(2n)$ of Table 1. Further, $U^*(2n)$ is better than $U_2(2n)$ for $n \geq 2$ in agreement with the comments of Remark 4.7.

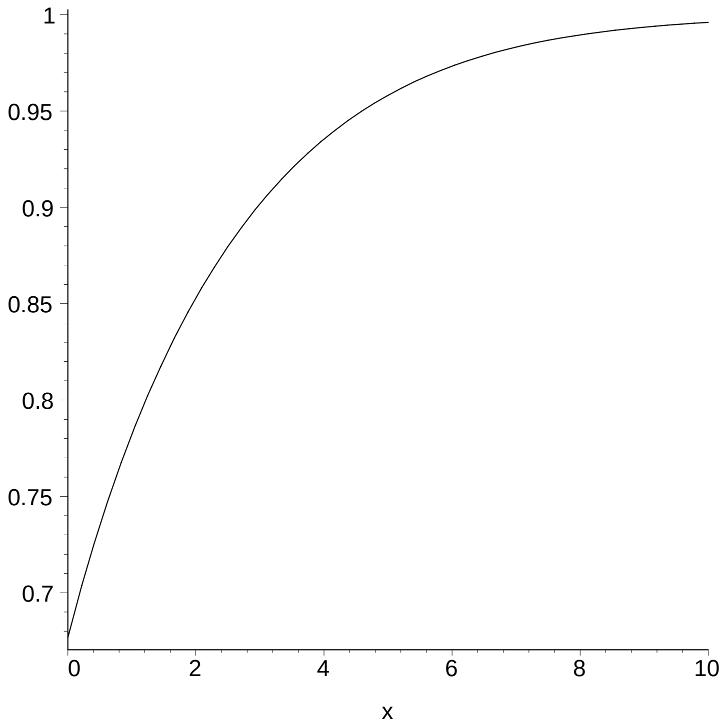

Figure 4.3: A plot of λ^*, satisfying (4.9) – (4.10) which produces $\max_{\lambda \in \left[\frac{1}{2},1\right)} \theta(\lambda, x) = \theta(\lambda^*, x)$.

n	$\mathbf{L}(2n)$	$\mathbf{L}_2^*(2n)$	$\zeta(2n+1)$	$\mathbf{U}^*(2n)$	$\mathbf{U}_2(2n)$
1	1.138289378	1.18644751	1.202056903	1.256754089	1.241877107
2	1.018501685	1.034776813	1.036927755	1.043123722	1.047087831
3	1.003178887	1.008088223	1.008349277	1.009110765	1.011054162
4	1.000629995	1.001977410	1.002008393	1.002099601	1.002712213
5	1.000138278	1.000490609	1.000494189	1.000504804	1.000673872

Table 4. Table of $L(2n)$, $L_2^*(2n)$, $\zeta(2n+1)$, $U^*(2n)$ and $U_2^*(2n)$ as given by (2.18), (4.13), (4.11) and (2.21) for $n = 1, \ldots, 5$.

n	$\zeta(2n+1)$	$\frac{U^*(2n)+L(2n)}{2}$	$\frac{U^*(2n)-L(2n)}{2}$
1	1.202056903	1.197521733	0.05923235565
2	1.036927755	1.030812704	0.01231101832
3	1.008349277	1.006144852	0.002965938819
4	1.002008393	1.001364798	0.0007348027621
5	1.000494189	1.000321541	0.0001832630327

Table 5. Table of $\zeta(2n+1)$, its approximation $\frac{U^*(2n)+L(2n)}{2}$ and its bound $\frac{U^*(2n)-L(2n)}{2}$ for $n = 1, \ldots, 5$ where $U^*(2n)$ and $L(2n)$ are given by (4.11) and (2.18).

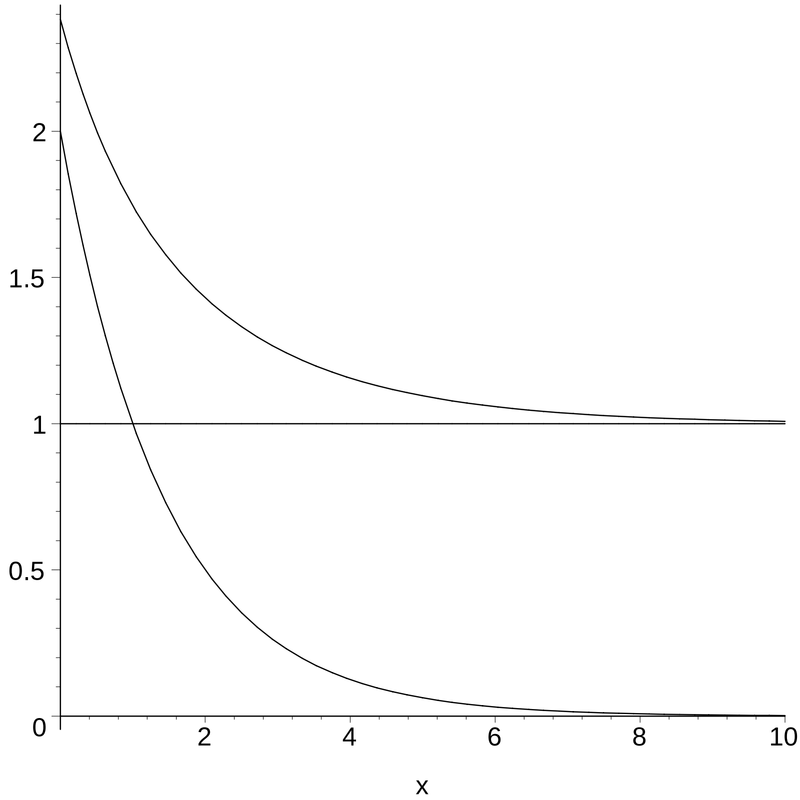

Figure 4.4: Diagram, from top to bottom, of $\theta(\lambda^*, x)$, 1 and $2^{-(x-1)}$ where λ^* satisfies (4.9) – (4.10).

n	$\zeta(2n+1)$	$\frac{U_2(2n)+L_2^*(2n)}{2}$	$\frac{U_2(2n)-L_2^*(2n)}{2}$
1	1.202056903	1.212260928	0.0296167783
2	1.036927755	1.040932323	0.006155509161
3	1.008349277	1.009571192	0.001482969409
4	1.002008393	1.002344812	0.0003674013812
5	1.000494189	1.000582241	0.00009163151632

Table 6. Table of $\zeta(2n+1)$, its approximation $\frac{U_2(2n)+L_2^*(2n)}{2}$ and its bound $\frac{U_2(2n)-L_2^*(2n)}{2}$ for $n = 1, \ldots, 5$ where $U_2(2n)$ and $L_2^*(2n)$ are given by (2.21) and (4.13).

Tables 5 and 6 provide approximations to $\zeta(2n+1)$ and a bound on the error from using $\zeta(2n)$ and $\zeta(2n+2)$ respectively, recalling that the zeta function is explicitly known at even integers (3.3). It may be noticed that the approximations of Table 5 seem to provide an under-estimate while those of Table 6 an over estimate for $\zeta(2n+1)$. Further, the results of Table 6 seem to be tighter than those of Table 5.

5. CONCLUDING REMARKS

An identity has been derived involving the zeta function values at a distance of one apart. Bounds are obtained for $\zeta(x+1)$ on approximations in terms of $\zeta(x)$ and $\zeta(x+2)$. For $x = 2n$, n a positive integer, the zeta values at even integers are explicitly known so that $\zeta(2n+1)$

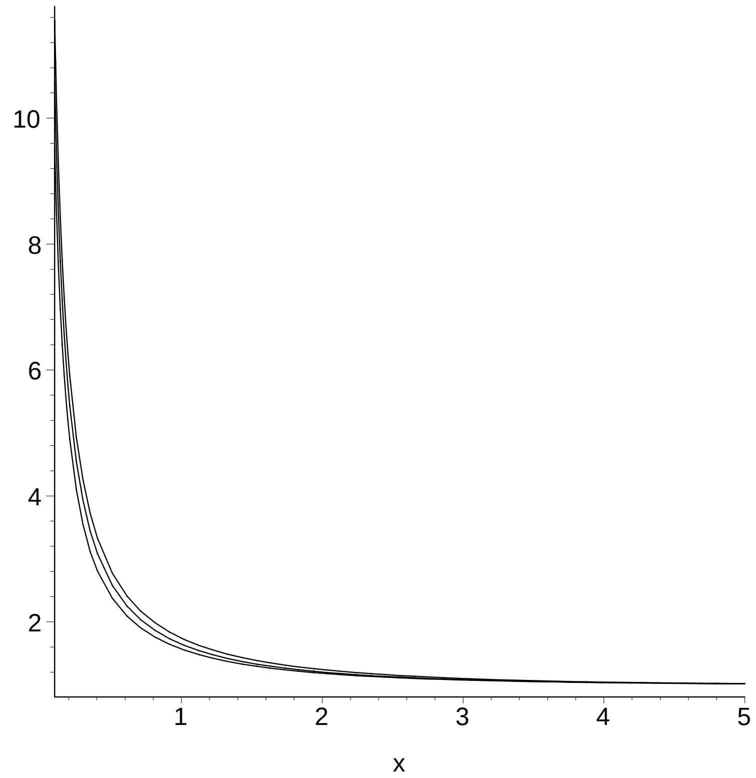

Figure 4.5: Plot of $m(x) = \max\{L(x), L_2^*(x)\}$, $\zeta(x+1)$ *and* $M(x) = \min\{U^*(x), U_2(x)\}$ *where* $L(x), L_2^*(x), U^*(x)$ *and* $U_2(x)$ *are defined by (2.18), (4.13), (4.11) and (2.21) respectively.*

has been accurately approximated or bounded in terms of explicitly known expressions. A priori bounds on the error have also been derived in the current development.

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