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ON SOME NEW NONLINEAR RETARDED INTEGRAL INEQUALITIES

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ABSTRACT. In the present paper we establish new nonlinear retarded integral inequalities which can be used as tools in certain applications. Some applications are also given to illustrate the usefulness of our results.

Key words and phrases: Retarded integral inequalities, Explicit bounds, Two independent variable generalizations, Partial derivatives, Estimate on the solution.

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1. INTRODUCTION

In [3] Lipovan obtained a useful upper bound on the following inequality:

$$u^2(t) \le c^2 + \int_0^{\alpha(t)} \left[f(s)\, u^2(s) + g(s)\, u(s) \right] ds, \tag{1.1}$$

and its variants, under some suitable conditions on the functions involved in (1.1). In fact, the results given in [3] are the retarded versions of the inequalities established by Pachpatte in [4] (see also [5]). However, the bounds provided on such inequalities in [3] (see also [1, p. 142]) are not directly applicable in the study of certain retarded differential and integral equations. It is desirable to find new inequalities of the above type, which will prove their importance in achieving a diversity of desired goals. The main purpose of this paper is to establish explicit bounds on the general versions of (1.1) which can be used more effectively in the study of certain classes of retarded differential and integral equations. The two independent variable generalizations of the main results and some applications are also given.

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2. STATEMENT OF RESULTS

In what follows, $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}_+ = [0, \infty)$, $I = [t_0, T)$, $I_1 = [x_0, X)$, $I_2 = [y_0, Y)$ are the given subsets of $\mathbb{R}$; $\Delta = I_1 \times I_2$ and $'$ denotes the derivative. The first order partial derivatives of a function $z(x, y)$ for $x, y \in \mathbb{R}$ with respect to x and y are denoted by $D_1 z(x, y)$ and $D_2 z(x, y)$ respectively. Let $C(M, N)$ denote the class of continuous functions from the set M to the set N.

Our main results are given in the following theorem.

Theorem 2.1. *Let $u, a_i, b_i \in C(I, \mathbb{R}_+)$ and $\alpha_i \in C^1(I, I)$ be nondecreasing with $\alpha_i(t) \le t$ on I for $i = 1, \ldots, n$. Let $p > 1$ and $c \ge 0$ be constants.*

(c_1) *If*

$$u^p(t) \le c + p \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [a_i(s) u^p(s) + b_i(s) u(s)] \, ds, \tag{2.1}$$

for $t \in I$, then

$$u(t) \le \left\{ A(t) \exp\left((p-1) \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} a_i(\sigma) \, d\sigma \right) \right\}^{\frac{1}{p-1}}, \tag{2.2}$$

for $t \in I$, where

$$A(t) = \{c\}^{\frac{p-1}{p}} + (p-1) \sum_{i=1}^{n} \int_{\alpha_i(t)}^{\alpha_i(t)} b_i(\sigma) \, d\sigma, \tag{2.3}$$

for $t \in I$.

(c_2) *Let $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $w(u) > 0$ on $(0, \infty)$. If for $t \in I$,*

$$u^p(t) \le c + p \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [a_i(s) u(s) w(u(s)) + b_i(s) u(s)] \, ds, \tag{2.4}$$

then for $t_0 \le t \le t_1$,

$$u(t) \le \left\{ G^{-1} \left[G(A(t)) + (p-1) \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} a_i(\sigma) \, d\sigma \right] \right\}^{\frac{1}{p-1}}, \tag{2.5}$$

where $A(t)$ is defined by (2.3), G^{-1} is the inverse function of

$$G(r) = \int_{r_0}^{r} \frac{ds}{w\left(s^{\frac{1}{p-1}}\right)}, \quad r > 0, \tag{2.6}$$

$r_0 > 0$ is arbitrary and $t_1 \in I$ is chosen so that

$$G(A(t)) + (p-1) \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} a_i(\sigma) \, d\sigma \in Dom\left(G^{-1}\right),$$

for all t lying in the interval $t_0 \le t \le t_1$.

Remark 2.2. If we take $p = 2$, $n = 1$, $\alpha_1 = \alpha$, $a_1 = f$, $b_1 = g$ in Theorem 2.1, then we recapture the inequalities given in [3] (see Corollary 2 and Theorem 1).

The following theorem deals with the two independent variable versions of the inequalities established in Theorem 2.1 which can be used in certain applications.

Theorem 2.3. *Let $u, a_i, b_i \in C(\Delta, \mathbb{R}_+)$ and $\alpha_i \in C^1(I_1, I_1)$, $\beta_i \in C^1(I_2, I_2)$ be nondecreasing with $\alpha_i(x) \le x$ on I_1, $\beta_i(y) \le y$ on I_2 for $i = 1, \ldots, n$. Let $p > 1$ and $c \ge 0$ be constants.*

(d_1) *If*

$$u^p(x,y) \le c + p \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} [a_i(s,t)\, u^p(s,t) + b_i(s,t)\, u(s,t)]\, dt ds, \tag{2.7}$$

for $(x,y) \in \Delta$, then

$$u(x,y) \le \left\{ B(x,y) \exp\left((p-1) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} a_i(\sigma,\tau)\, d\tau d\sigma \right) \right\}^{\frac{1}{p-1}}, \tag{2.8}$$

for $(x,y) \in \Delta$, where

$$B(x,y) = \{c\}^{\frac{p-1}{p}} + (p-1) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\sigma,\tau)\, d\tau d\sigma, \tag{2.9}$$

for $(x,y) \in \Delta$.

(d_2) *Let w be as in Theorem 2.1, part (c_2). If for $(x,y) \in \Delta$,*

$$u^p(x,y) \le c + p \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} [a_i(s,t)\, u(s,t)\, w(u(s,t)) + b_i(s,t)\, u(s,t)]\, dt ds, \tag{2.10}$$

then, for $x_0 \le x \le x_1, y_0 \le y \le y_1$,

$$u(x,y) \le \left\{ G^{-1} \left[G(B(x,y)) + (p-1) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} a_i(\sigma,\tau)\, d\tau d\sigma \right] \right\}^{\frac{1}{p-1}}, \tag{2.11}$$

where $B(x,y)$ is defined by (2.9), G, G^{-1} are as in Theorem 2.1, part (c_2) and $x_1 \in I_1$, $y_1 \in I_2$ are chosen so that

$$G(B(x,y)) + (p-1) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} a_i(\sigma,\tau)\, d\tau d\sigma \in Dom\left(G^{-1}\right),$$

for all x, y lying in the interval $x_0 \le x \le x_1, y_0 \le y \le y_1$.

Remark 2.4. We note that the inequalities established in Theorem 2.3 can be extended very easily for functions involving more than two independent variables (see [5]). If we take $p = 2$, $n = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $a_1 = f$, $b_1 = g$ in Theorem 2.3, then we get the two independent variable generalizations of the inequalities given in [3] (see Corollary 2 and Theorem 1). For a slight variant of the inequality in Theorem 2.3 given in [3] and its two independent variable version, see [6].

3. PROOFS OF THEOREMS 2.1 AND 2.3

We give the details of the proofs for (c_1) and (d_2) only; the proofs of (c_2) and (d_1) can be completed by following the proofs of the above mentioned inequalities.

From the hypotheses we observe that $\alpha_i'(t) \ge 0$ for $t \in I$, $\alpha_i'(x) \ge 0$ for $x \in I_1$, $\beta_i(y) \ge 0$ for $y \in I_2$.

(c_1) Let $c > 0$ and define a function $z(t)$ by the right hand side of (2.1). Then $z(t) > 0$, $z(t_0) = c$, $z(t)$ is nondecreasing for $t \in I$, $u(t) \le \{z(t)\}^{\frac{1}{p}}$ and

$$\begin{aligned} z'(t) &= p\sum_{i=1}^{n}\left[a_i(\alpha_i(t))\,u^p(\alpha_i(t)) + b_i(\alpha_i(t))\,u(\alpha_i(t))\right]\alpha_i'(t) \\ &\le p\sum_{i=1}^{n}\left[a_i(\alpha_i(t))\,z(\alpha_i(t)) + b_i(\alpha_i(t))\{z(\alpha_i(t))\}^{\frac{1}{p}}\right]\alpha_i'(t) \\ &= p\sum_{i=1}^{n}\left[a_i(\alpha_i(t))\{z(\alpha_i(t))\}^{1-\frac{1}{p}} + b_i(\alpha_i(t))\right]\{z(\alpha_i(t))\}^{\frac{1}{p}}\alpha_i'(t) \\ &\le p\sum_{i=1}^{n}\left[a_i(\alpha_i(t))\{z(\alpha_i(t))\}^{\frac{p-1}{p}} + b_i(\alpha_i(t))\right]\{z(t)\}^{\frac{1}{p}}\alpha_i'(t) \end{aligned}$$

i.e.

$$\frac{z'(t)}{\{z(t)\}^{\frac{1}{p}}} \le p\sum_{i=1}^{n}\left[a_i(\alpha_i(t))\{z(\alpha_i(t))\}^{\frac{p-1}{p}} + b_i(\alpha_i(t))\right]\alpha_i'(t). \tag{3.1}$$

By taking $t = s$ in (3.1) and integrating it with respect to s from t_0 to t we get

$$\begin{aligned} \{z(t)\}^{\frac{p-1}{p}} &\le \{c\}^{\frac{p-1}{p}} \\ &\quad + (p-1)\int_{t_0}^{t}\sum_{i=1}^{n}\left[a_i(\alpha_i(s))\{z(\alpha_i(s))\}^{\frac{p-1}{p}} + b_i(\alpha_i(s))\right]\alpha_i'(s)\,ds. \end{aligned} \tag{3.2}$$

Making the change of variables on the right hand side in (3.2)and rewriting we get

$$\{z(t)\}^{\frac{p-1}{p}} \le A(t) + (p-1)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} a_i(\sigma)\{z(\sigma)\}^{\frac{p-1}{p}}\,d\sigma.$$

Clearly $A(t)$ is a continuous, positive and nondecreasing function for $t \in I$. Now by following the idea used in the proof of Theorem 1 in [3] (see also [6]) we get

$$\{z(t)\}^{\frac{p-1}{p}} \le A(t)\exp\left((p-1)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} a_i(\sigma)\,d\sigma\right). \tag{3.3}$$

Using (3.3) in $u(t) \le \{z(t)\}^{\frac{1}{p}}$ we get the desired inequality in (2.2).

If $c \ge 0$ we carry out the above procedure with $c + \varepsilon$ instead of c, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass the limit $\varepsilon \to 0$ to obtain (2.2).

(d_2) Let $c > 0$ and define a function $z(x, y)$ by the right hand side of (2.10). Then $z(x, y) > 0$, $z(x_0, y) = z(x, y_0) = c$, $z(x, y)$ is nondecreasing in $(x, y) \in \Delta$, $u(x, y) \le \{z(x, y)\}^{\frac{1}{p}}$ and

$$\begin{aligned} &D_2D_1z(x,y) \\ &= p\sum_{i=1}^{n}\left[a_i(\alpha_i(x),\beta_i(y))\,u(\alpha_i(x),\beta_i(y))\,w(u(\alpha_i(x),\beta_i(y)))\right. \\ &\qquad \left.+ b_i(\alpha_i(x),\beta_i(y))\,u(\alpha_i(x),\beta_i(y))\right]\beta_i'(y)\,\alpha_i'(x) \end{aligned} \tag{3.4}$$

$$\begin{aligned}&\leq p\sum_{i=1}^{n}\Big[a_i\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)\left\{z\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)\right\}^{\frac{1}{p}} w\left(\left\{z\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)\right\}^{\frac{1}{p}}\right)\\&\qquad +b_i\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)\left\{z\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)\right\}^{\frac{1}{p}}\Big]\beta_i'\left(y\right)\alpha_i'\left(x\right)\\&\leq p\sum_{i=1}^{n}\left[a_i\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)w\left(\left\{z\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)\right\}^{\frac{1}{p}}\right)\right.\\&\qquad\left.+b_i\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)\right]\left\{z\left(x,y\right)\right\}^{\frac{1}{p}}\beta_i'\left(y\right)\alpha_i'\left(x\right).\end{aligned}$$

From (3.4) we observe that

$$\begin{aligned}\frac{D_2D_1z\left(x,y\right)}{\left\{z\left(x,y\right)\right\}^{\frac{1}{p}}}&\leq p\sum_{i=1}^{n}\left[a_i\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)w\left(\left\{z\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)\right\}^{\frac{1}{p}}\right)\right.\\&\qquad\left.+b_i\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)\right]\beta_i'\left(y\right)\alpha_i'\left(x\right)+\frac{D_1z\left(x,y\right)\left[D_2\left\{z\left(x,y\right)\right\}^{\frac{1}{p}}\right]}{\left[\left\{z\left(x,y\right)\right\}^{\frac{1}{p}}\right]^2},\end{aligned}$$

i.e.

$$\begin{aligned}D_2\left(\frac{D_1z\left(x,y\right)}{\left\{z\left(x,y\right)\right\}^{\frac{1}{p}}}\right)&\leq p\sum_{i=1}^{n}\left[a_i\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)w\left(\left\{z\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)\right\}^{\frac{1}{p}}\right)\right.\\&\qquad\left.+b_i\left(\alpha_i\left(x\right),\beta_i\left(y\right)\right)\right]\beta_i'\left(y\right)\alpha_i'\left(x\right),\end{aligned}\tag{3.5}$$

for $(x,y)\in\Delta$. By keeping x fixed in (3.5), we set $y=t$ and then, by integrating with respect to t from y_0 to y and using the fact that $D_1z\left(x,y_0\right)=0$, we have

$$\begin{aligned}\frac{D_1z\left(x,y\right)}{\left\{z\left(x,y\right)\right\}^{\frac{1}{p}}}&\leq p\int_{y_0}^{y}\sum_{i=1}^{n}\left[a_i\left(\alpha_i\left(x\right),\beta_i\left(t\right)\right)w\left(\left\{z\left(\alpha_i\left(x\right),\beta_i\left(t\right)\right)\right\}^{\frac{1}{p}}\right)\right.\\&\qquad\left.+b_i\left(\alpha_i\left(x\right),\beta_i\left(t\right)\right)\right]\beta_i'\left(t\right)\alpha_i'\left(x\right)dt.\end{aligned}\tag{3.6}$$

Now by keeping y fixed in (3.6) and setting $x=s$ and integrating with respect to s from x_0 to x we have

$$\begin{aligned}\left\{z\left(x,y\right)\right\}^{\frac{p-1}{p}}&\leq\left\{c\right\}^{\frac{p-1}{p}}+(p-1)\\&\quad\times\int_{x_0}^{x}\int_{y_0}^{y}\sum_{i=1}^{n}\left[a_i\left(\alpha_i\left(s\right),\beta_i\left(t\right)\right)w\left(\left\{z\left(\alpha_i\left(s\right),\beta_i\left(t\right)\right)\right\}^{\frac{1}{p}}\right)\right.\\&\qquad\left.+b_i\left(\alpha_i\left(s\right),\beta_i\left(t\right)\right)\right]\beta_i'\left(t\right)\alpha_i'\left(s\right)dtds.\end{aligned}\tag{3.7}$$

By making the change of variables on the right hand side of (3.7) and rewriting we have

$$\left\{z\left(x,y\right)\right\}^{\frac{p-1}{p}}\leq B\left(x,y\right)+(p-1)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}a_i\left(\sigma,\tau\right)w\left(\left\{z\left(\sigma,\tau\right)\right\}^{\frac{1}{p}}\right)d\tau d\sigma.\tag{3.8}$$

Now fix $\lambda\in I_1,\mu\in I_2$ such that $x_0\leq x\leq\lambda\leq x_1$, $y_0\leq y\leq\mu\leq y_1$. Then from (3.8) we observe that

$$\left\{z\left(x,y\right)\right\}^{\frac{p-1}{p}}\leq B\left(\lambda,\mu\right)+(p-1)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}a_i\left(\sigma,\tau\right)w\left(\left\{z\left(\sigma,\tau\right)\right\}^{\frac{1}{p}}\right)d\tau d\sigma,\tag{3.9}$$

for $x_0 \leq x \leq \lambda, y_0 \leq y \leq \mu$. Define a function $v(x, y)$ by the right hand side of (3.9). Then $v(x, y) > 0$, $v(x_0, y) = v(x, y_0) = B(\lambda, \mu)$, $v(x, y)$ is nondecreasing for $x_0 \leq x \leq \lambda, y_0 \leq y \leq \mu$, $\{z(x, y)\}^{\frac{p-1}{p}} \leq v(x, y)$ and

$$v(x, y) \leq B(\lambda, \mu) + (p-1) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta(y)} a_i(\sigma, \tau) w\left(\{v(\sigma, \tau)\}^{\frac{1}{p-1}}\right) d\tau d\sigma,$$

for $x_0 \leq x \leq \lambda, y_0 \leq y \leq \mu$. Now by following the proof of Theorem 2.2, part (B_1) given in [7] (see also [6]) we get

$$v(x, y) \leq G^{-1}\left[G(B(\lambda, \mu)) + (p-1) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta(y)} a_i(\sigma, \tau) d\tau d\sigma\right], \tag{3.10}$$

for $x_0 \leq x \leq \lambda \leq x_1, y_0 \leq y \leq \mu \leq y_1$. Since (λ, μ) is arbitrary, we get the desired inequality in (2.11) from (3.10) and the fact that

$$u(x, y) \leq \{z(x, y)\}^{\frac{1}{p}} \leq \left\{[v(x, y)]^{\frac{p}{p-1}}\right\}^{\frac{1}{p}} = \{v(x, y)\}^{\frac{1}{p-1}}.$$

The proof of the case when $c \geq 0$ can be completed as mentioned in the proof of Theorem 2.1, part (c_1). The domain $x_0 \leq x \leq x_1$, $y_0 \leq y \leq y_1$ is obvious.

4. APPLICATIONS

In this section, we present some model applications which demonstrate the importance of our results to the literature.

First consider the differential equation involving several retarded arguments

$$x^{p-1}(t) x'(t) = f(t, x(t - h_1(t)), \ldots, x(t - h_n(t))), \tag{4.1}$$

for $t \in I$, with the given initial condition

$$x(t_0) = x_0 \tag{4.2}$$

where $p > 1$ and x_0 are constants, $f \in C(I \times \mathbb{R}^n, \mathbb{R})$ and for $i = 1, \ldots, n$, let $h_i \in C(I, \mathbb{R}_+)$ be nonincreasing and such that $t - h_i(t) \geq 0$, $t - h_i(t) \in C^1(I, I)$, $h_i'(t) < 1, h_i(t_0) = 0$. For the theory and applications of differential equations with deviating arguments, see [2].

The following theorem deals with the estimate on the solution of the problem (4.1) – (4.2).

Theorem 4.1. *Suppose that*

$$|f(t, u_1, \ldots, u_n)| \leq \sum_{i=1}^{n} b_i(t) |u_i|, \tag{4.3}$$

where $b_i(t)$ are as in Theorem 2.1. Let

$$Q_i = \max_{t \in I} \frac{1}{1 - h_i'(t)}, \qquad i = 1, \ldots, n. \tag{4.4}$$

If $x(t)$ is any solution of the problem (4.1) – (4.2), then

$$|x(t)| \leq \left\{|x_0|^{p-1} + (p-1) \sum_{i=1}^{n} \int_{t_0}^{t - h_i(t)} \bar{b}_i(\sigma) d\sigma\right\}^{\frac{1}{p-1}}, \tag{4.5}$$

for $t \in I$, where $\bar{b}_i(\sigma) = Q_i b_i(\sigma + h_i(s))$, $\sigma, s \in I$.

Proof. The solution $x(t)$ of the problem (4.1) – (4.2) can be written as

$$x^p(t) = x_0^p + p\int_{t_0}^{t} f(s, x(s-h_1(s)), \dots, x(s-h_n(s)))\,ds. \tag{4.6}$$

From (4.6), (4.3), (4.4) and making the change of variables we have

$$\begin{aligned} |x(t)|^p &\le |x_0|^p + p\int_{t_0}^{t}\sum_{i=1}^{n} b_i(s)\,|x(s-h_i(s))|\,ds \\ &\le |x_0|^p + p\sum_{i=1}^{n}\int_{t_0}^{t-h_i(t)} \bar{b}_i(\sigma)\,|x(\sigma)|\,d\sigma, \end{aligned} \tag{4.7}$$

for $t \in I$. Now a suitable application of the inequality in Theorem 2.1, part (c_1) (when $a_i = 0$) to (4.7) yields the required estimate in (4.5). □

Next, we obtain an explicit bound on the solution of a retarded partial differential equation of the form

$$\begin{aligned} &D_2\left(z^{p-1}(x,y)\,D_1 z(x,y)\right) \\ &\qquad = F(x, y, z(x-h_1(x), y-g_1(y)), \dots, z(x-h_n(x), y-g_n(y))), \end{aligned} \tag{4.8}$$

for $(x,y) \in \Delta$, with the given initial boundary conditions

$$z(x, y_0) = e_1(x), \quad z(x_0, y) = e_2(y), \quad e_1(x_0) = e_2(y_0) = 0, \tag{4.9}$$

where $p > 1$ is a constant, $F \in C(\Delta \times \mathbb{R}^n, \mathbb{R})$, $e_1 \in C^1(I_1, \mathbb{R})$, $e_2 \in C^1(I_2, \mathbb{R})$, and $h_i \in C(I_1, \mathbb{R}_+)$, $g_i \in C(I_2, \mathbb{R}_+)$ are nonincreasing and such that $x - h_i(x) \ge 0$, $x - h_i(x) \in C^1(I_1, I_1)$, $y - g_i(y) \ge 0$, $y - g_i(y) \in C^1(I_2, I_2)$, $h_i'(t) < 1$, $g_i'(t) < 1$, $h_i(x_0) = g_i(y_0) = 0$ for $i = 1, \dots, n$; $x \in I_1, y \in I_2$. For the study of special versions of equation (4.8), we refer interested readers to [8].

Theorem 4.2. *Suppose that*

$$|F(x, y, u_1, \dots, u_n)| \le \sum_{i=1}^{n} b_i(x,y)\,|u_i|, \tag{4.10}$$

$$|e_1^p(x) + e_2^p(y)| \le c, \tag{4.11}$$

where $b_i(x,y)$ and c are as in Theorem 2.3. Let

$$M_i = \max_{x\in I_1}\frac{1}{1-h_i'(x)}, \quad N_i = \max_{y\in I_2}\frac{1}{1-g_i'(y)}, \quad i = 1, \dots, n. \tag{4.12}$$

If $z(x,y)$ is any solution of the problem (4.8) – (4.9), then

$$|z(x,y)| \le \left\{ \{c\}^{\frac{p-1}{p}} + (p-1)\sum_{i=1}^{n}\int_{\phi_i(x_0)}^{\phi_i(x)}\int_{\psi_i(y_0)}^{\psi_i(y)} \bar{b}_i(\sigma,\tau)\,d\tau\,d\sigma \right\}^{\frac{1}{p-1}}, \tag{4.13}$$

for $x \in I_1, y \in I_2$, where $\phi_i(x) = x - h_i(x)$, $x \in I_1$, $\psi_i(y) = y - \psi_i(y)$, $y \in I_2$, $\bar{b}_i(\sigma,\tau) = M_i N_i b_i(\sigma + h_i(s), \tau + g_i(t))$ for $\sigma, s \in I_1$; $\tau, t \in I_2$.

Proof. It is easy to see that the solution $z(x,y)$ of the problem (4.8) – (4.9) satisfies the equivalent integral equation

$$z^{p}(x,y)=e_{1}^{p}(x)+e_{2}^{p}(y)+p\int_{x_0}^{x}\int_{y_0}^{y}F(s,t,z(s-h_1(s),t-g_1(t)),\ldots,z(s-h_n(s),t-g_n(t)))dtds. \tag{4.14}$$

From (4.14), (4.10)-(4.12) and making the change of variables we have

$$\begin{aligned}|z(x,y)|^{p}&\le c+p\int_{x_0}^{x}\int_{y_0}^{y}\sum_{i=1}^{n}b_i(s,t)\,|z(s-h_i(s),t-g_i(t))|dtds\\&\le c+p\sum_{i=1}^{n}\int_{\phi_i(x_0)}^{\phi_i(x)}\int_{\psi_i(y_0)}^{\psi_i(y)}\bar{b}_i(\sigma,\tau)\,|z(\sigma,\tau)|d\tau d\sigma.\end{aligned} \tag{4.15}$$

Now a suitable application of the inequality given in Theorem 2.3, part (d_1) (when $a_i=0$) to (4.15) yields (4.13). □

Remark 4.3. From Theorem 4.1, it is easy to observe that the inequalities given in [3] cannot be used to obtain an estimate on the solution of the problem (4.1) – (4.2). Various other applications of the inequalities given here is left to another work.

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