

JÓZSEF SÁNDOR

Department of Mathematics and Computer Science
Babeş-Bolyai University, Cluj-Napoca
Romania.

EMail: jsandor@math.ubbcluj.ro



volume 6, issue 3, article 73,
2005.

*Received 14 October, 2004;
accepted 13 May, 2005.*

Communicated by: L. Tóth

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

We study certain properties and conjectures on the composition of the arithmetic functions σ , φ , ψ , where σ is the sum of divisors function, φ is Euler's totient, and ψ is Dedekind's function.

2000 Mathematics Subject Classification: 11A25, 11N37.

Key words: Arithmetic functions, Makowski-Schinzel conjecture, Sándor's conjecture, Inequalities.

The author wishes to thank Professors P. Erdős, K. Atanassov, F. Luca and V. Vitek, for their helpful conversations. He is indebted also to Professors A. Makowski, H.-J. Kanold, K. Ford, G.L. Cohen, Ch. Wall, for providing interesting reprints of their works. Finally, the author thanks his colleagues Kovács Lehel István (Cluj, Romania), and László Tóth (Pécs, Hungary) for helpful discussions, which improved the presentation of the paper.

Contents

1	Introduction	3
1.1	Basic symbols and notations	6
2	Basic Lemmas	7
3	Main Results	10
	References	



On the Composition of Some Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 2 of 37

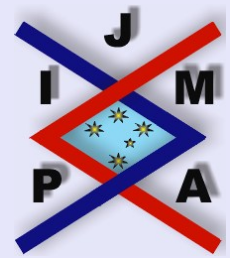
1. Introduction

Let $\sigma(n)$ denote the sum of divisors of the positive integer n , i.e. $\sigma(n) = \sum_{d|n} d$, where by convention $\sigma(1) = 1$. It is well-known that n is called *perfect* if $\sigma(n) = 2n$. Euclid and Euler ([10], [21]) have determined all even perfect numbers, by showing that they are of the form $n = 2^k(2^{k+1} - 1)$, where $2^{k+1} - 1$ is a prime ($k \geq 1$). The primes of the form $2^{k+1} - 1$ are the so-called Mersenne primes, and at this moment there are known exactly 41 such primes (for the recent discovery of the 41th Mersenne prime, see the site www.ams.org). It is possible that there are infinitely many Mersenne primes, but the proof of this result seems unattackable at present. On the other hand, no odd perfect number is known, and the existence of such numbers is one of the most difficult open problems of Mathematics. D. Suryanarayana [23] defined the notion of a *superperfect* number, i.e. a number n with the property $\sigma(\sigma(n)) = 2n$, and he and H.J. Kanold [23], [11] have obtained the general form of even superperfect numbers. These are $n = 2^k$, where $2^{k+1} - 1$ is a prime. Numbers n with the property $\sigma(n) = 2n - 1$ have been called *almost perfect*, while that of $\sigma(n) = 2n + 1$, *quasi-perfect*. For many results and conjectures on this topic, see [9], and the author's book [21] (see Chapter 1).

For an arithmetic function f , the number n is called *f-perfect*, if $f(n) = 2n$. Thus, the superperfect numbers will be in fact the $\sigma \circ \sigma$ -perfect numbers where " \circ " denotes composition.

The Euler totient function, resp. Dedekind's arithmetic function are given by

$$(1.1) \quad \varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right), \quad \psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right),$$



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 3 of 37

where p runs through the distinct prime divisors of n . Following convention we let, $\varphi(1) = 1, \psi(1) = 1$. All these functions are multiplicative, i.e. they satisfy the functional equation $f(mn) = f(m)f(n)$ for $(m, n) = 1$. For results on $\psi \circ \psi$ -perfect, $\psi \circ \sigma$ -perfect, $\sigma \circ \psi$ -perfect, and $\psi \circ \varphi$ -perfect numbers, see the first part of [18]. Let $\sigma^*(n)$ be the sum of unitary divisors of n , given by

$$(1.2) \quad \sigma^*(n) = \prod_{p^\alpha || n} (p^\alpha + 1),$$

where $p^\alpha || n$ means that for the prime power p^α one has $p^\alpha | n$, but $p^{\alpha+1} \nmid n$. By convention, let $\sigma^*(1) = 1$. In [18] almost and quasi $\sigma^* \circ \sigma^*$ -perfect numbers (i.e. satisfying $\sigma^*(\sigma^*(n)) = 2n \mp 1$) are studied, where it is shown that for $n > 3$ there are no such numbers. This result has been rediscovered by V. Sitaramaiah and M.V. Subbarao [22].

In 1964, A. Makowski and A. Schinzel [13] conjectured that

$$(1.3) \quad \sigma(\varphi(n)) \geq \frac{n}{2}, \text{ for all } n \geq 1.$$

The first results after the Makowski and Schinzel paper were proved by J. Sándor [16], [17]. He proved that (1.3) holds if and only if

$$(1.4) \quad \sigma(\varphi(m)) \geq m, \text{ for all odd } m \geq 1$$

and obtained a class of numbers satisfying (1.3) and (1.4). But (1.4) holds iff is it true for squarefree n , see [17], [18]. This has been rediscovered by G.L. Cohen and R. Gupta ([4]). Many other partial results have been discovered by C. Pomerance [14], G.L. Cohen [4], A. Grytczuk, F. Luca and M. Wojtowicz



On the Composition of Some Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 4 of 37

[7], [8], F. Luca and C. Pomerance [12], K. Ford [6]. See also [2], [19], [20]. Kevin Ford proved that

$$(1.5) \quad \sigma(\varphi(n)) \geq \frac{n}{39.4}, \text{ for all } n.$$

In 1988 J. Sándor [15], [16] conjectured that

$$(1.6) \quad \psi(\varphi(m)) \geq m, \text{ for all odd } m.$$

He showed that (1.6) is equivalent to

$$(1.7) \quad \psi(\varphi(n)) \geq \frac{n}{2}$$

for all n , and obtained a class of numbers satisfying these inequalities. In 1988 J. Sándor [15] conjectured also that

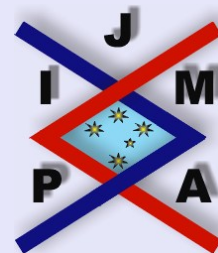
$$(1.8) \quad \varphi(\psi(n)) \leq n, \text{ for any } n \geq 2$$

and V. Vitek [24] of Praha verified this conjecture for $n \leq 10^4$.

In 1990 P. Erdős [5] expressed his opinion that this new conjecture could be as difficult as the Makowski-Schinzel conjecture (1.3). In 1992 K. Atanassov [3] believed that he obtained a proof of (1.8), but his proof was valid only for certain special values of n .

Nonetheless, as we will see, conjectures (1.6), (1.7) and (1.8) are not generally true, and it will be interesting to study the classes of numbers for which this is valid.

The aim of this paper is to study this conjecture and certain new properties of the above – and related – composite functions.



On the Composition of Some Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

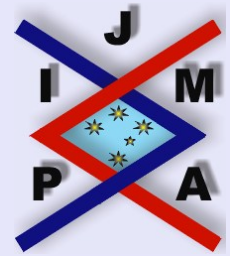
Close

Quit

Page 5 of 37

1.1. Basic symbols and notations

- $\sigma(n)$ = sum of divisors of n ,
- $\sigma^*(n)$ = sum of unitary divisors of n ,
- $\varphi(n)$ = Euler's totient function,
- $\psi(n)$ = Dedekind's arithmetic function,
- $[x]$ = integer part of x ,
- $\omega(n)$ = number of distinct divisors of n ,
- $a|b = a$ divides b ,
- $a \nmid b = a$ does not divide b ,
- $pr\{n\}$ = set of distinct prime divisors of n ,
- $f \circ g$ = composition of f and g .



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 6 of 37

2. Basic Lemmas

Lemma 2.1.

$$(2.1) \quad \varphi(ab) \leq a\varphi(b), \text{ for any } a, b \geq 2$$

with equality only if $pr\{a\} \subset pr\{b\}$, where $pr\{a\}$ denotes the set of distinct prime factors of a .

Proof. We have

$$ab = \prod_{p|a, p \nmid b} p^\alpha \cdot \prod_{q|a, q|b} q^\beta \cdot \prod_{r|b, r \nmid a} r^\gamma,$$

so

$$\begin{aligned} \frac{\varphi(ab)}{ab} &= \prod \left(1 - \frac{1}{p}\right) \cdot \prod \left(1 - \frac{1}{q}\right) \cdot \prod \left(1 - \frac{1}{r}\right) \\ &\leq \prod \left(1 - \frac{1}{q}\right) \cdot \prod \left(1 - \frac{1}{r}\right) = \frac{\varphi(b)}{b}, \end{aligned}$$

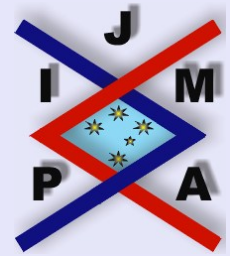
so $\varphi(ab) \leq a\varphi(b)$, with equality if " p does not exist", i.e. p with the property $p|a, p \nmid b$. Thus for all $p|a$ one has also $p|b$. \square

Lemma 2.2. *If $pr\{a\} \not\subset pr\{b\}$, then for any $a, b \geq 2$ one has*

$$(2.2) \quad \varphi(ab) \leq (a-1)\varphi(b),$$

and

$$(2.3) \quad \psi(ab) \geq (a+1)\psi(b).$$



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 7 of 37

Proof. We give only the proof of (2.2).

Let $a = \prod p^\alpha \cdot \prod q^\beta$, $b = \prod r^\gamma \cdot \prod q^{\beta'}$, where the q are the common prime factors, and the $p \in pr\{a\}$ are such that $p \notin pr\{b\}$, i.e. suppose that $\alpha \geq 1$. Clearly $\beta, \beta', \gamma \geq 0$. Then

$$\frac{\varphi(ab)}{\varphi(b)} = a \cdot \prod \left(1 - \frac{1}{p}\right) \leq a - 1$$

iff

$$\prod \left(1 - \frac{1}{p}\right) \leq 1 - \frac{1}{a} = 1 - \frac{1}{\prod p^\alpha \cdot \prod q^\beta}.$$

Now,

$$1 - \frac{1}{\prod p^\alpha \cdot \prod q^\beta} \geq 1 - \frac{1}{\prod p^\alpha} \geq 1 - \frac{1}{\prod p}$$

by $\alpha \geq 1$. The inequality

$$1 - \frac{1}{\prod p} \geq \prod \left(1 - \frac{1}{p}\right)$$

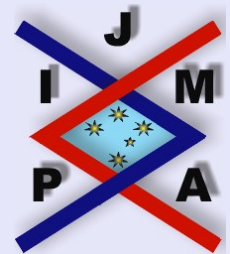
is trivial, since by putting e.g. $p - 1 = u$, one gets

$$\prod (u + 1) \geq 1 + \prod u,$$

and this is clear, since $u > 0$. There is equality only when there is a single u , i.e. if the set of p such that $pr\{a\} \not\subset pr\{b\}$ has a single element, at the first power, and all $\beta = 0$, i.e. when $a = p \nmid b$. Indeed:

$$\varphi(pb) = \varphi(p)\varphi(b) = (p - 1)\varphi(b).$$

□



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 8 of 37

Lemma 2.3. For all $a, b \geq 1$,

$$(2.4) \quad \sigma(ab) \geq a\sigma(b),$$

and

$$(2.5) \quad \psi(ab) \geq a\psi(b).$$

Proof. (2.4) is well-known, see e.g. [16], [18]. There is equality here, only for $a = 1$.

For (2.5), let $u|v$. Then

$$\frac{\psi(u)}{u} = \prod_{p|u} \left(1 + \frac{1}{p}\right) \leq \prod_{p|v, p|u} \left(1 + \frac{1}{p}\right) \cdot \prod_{q|v, q \nmid u} \left(1 + \frac{1}{q}\right) = \frac{\psi(v)}{v},$$

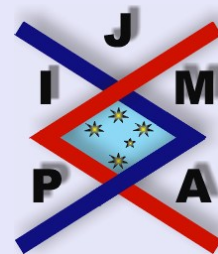
with equality if q does not exist with $q|v, q \nmid u$. Put $v = ab$ and $u = b$. Then $\frac{\psi(u)}{u} \leq \frac{\psi(v)}{v}$ becomes exactly (2.5). There is equality if for each $p|a$ one also has $p|b$, i.e. $pr\{a\} \subset pr\{b\}$. \square

Remark 1. Therefore, there is a similarity between the inequalities (2.1) and (2.5).

Lemma 2.4. If $pr\{a\} \not\subset pr\{b\}$, then for any $a, b \geq 2$ one has

$$(2.6) \quad \sigma(ab) \geq \psi(a) \cdot \sigma(b).$$

Proof. This is given in [16]. \square



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 9 of 37

3. Main Results

Theorem 3.1. *There are infinitely many n such that*

$$(3.1) \quad \psi(\varphi(n)) < \varphi(\psi(n)) < n.$$

For infinitely many m one has

$$(3.2) \quad \varphi(\psi(m)) < \psi(\varphi(m)) < m.$$

There are infinitely many k such that

$$(3.3) \quad \varphi(\psi(k)) = \frac{1}{2}\psi(\varphi(k)).$$

Proof. We prove that (3.1) is valid for $n = 3 \cdot 2^a$ for any $a \geq 1$. This follows from $\varphi(3 \cdot 2^a) = 2^a$, $\psi(2^a) = 3 \cdot 2^{a-1}$, $\psi(3 \cdot 2^a) = 3 \cdot 2^{a+1}$, $\varphi(3 \cdot 2^{a+1}) = 2^{a+1}$, so

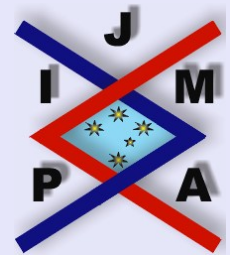
$$3 \cdot 2^a > \varphi(\psi(3 \cdot 2^a)) > \psi(\varphi(3 \cdot 2^a)).$$

For the proof of (3.2), put $m = 2^a \cdot 5^b$ ($b \geq 2$). Then an easy computation shows that $\psi(\varphi(m)) = 2^{a+1} \cdot 3^2 \cdot 5^{b-2}$, and $\varphi(\psi(m)) = 2^{a+2} \cdot 3 \cdot 5^{b-2}$ and the inequalities (3.2) will follow.

For $h = 3^s$ remark that $\varphi(\psi(h)) = \frac{4}{9} \cdot h$ and $\psi(\varphi(h)) = \frac{4}{3} \cdot h$, so

$$(3.4) \quad \varphi(\psi(h)) < h < \psi(\varphi(h)),$$

which complete (3.1) and (3.2), in a certain sense.



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 10 of 37

Finally, for $k = 2^a \cdot 7^b$ ($b \geq 2$) one can deduce $\psi(\varphi(k)) = \frac{48}{49} \cdot k$, $\varphi(\psi(k)) = \frac{24}{49} \cdot k$, so (3.3) follows. We remark that since

$$(3.5) \quad \psi(\varphi(k)) < k,$$

by (3.3) and (3.5) one can say that

$$(3.6) \quad \varphi(\psi(k)) < \frac{k}{2},$$

for the above values of k . Remark also that for h in (3.4) one has

$$(3.7) \quad \varphi(\psi(h)) = \frac{1}{3}\psi(\varphi(h)).$$

For the values m given by (3.2) one has

$$(3.8) \quad \varphi(\psi(m)) = \frac{2}{3}\psi(\varphi(m)).$$

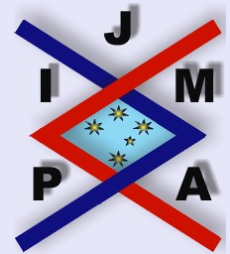
For $n = 2^a \cdot 3^b$ ($b \geq 2$) one can remark that $\varphi(\psi(n)) = \psi(\varphi(n))$. □

More generally, one can prove:

Theorem 3.2. *Let $1 < n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ the prime factorization of n and suppose that the odd part of n is squarefull, i.e. $\alpha_i \geq 2$ for all i with $p_i \geq 3$.*

Then $\varphi(\psi(n)) = \psi(\varphi(n))$ is true if and only if

$$(3.9) \quad pr\{(p_1 - 1) \cdots (p_r - 1)\} \subset pr\{p_1, \dots, p_r\} \quad \text{and} \\ pr\{(p_1 + 1) \cdots (p_r + 1)\} \subset pr\{p_1, \dots, p_r\}.$$



On the Composition of Some Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 11 of 37

Proof. Since

$$\varphi(n) = p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1} \cdot (p_1 - 1) \cdots (p_r - 1)$$

and

$$\psi(n) = p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1} \cdot (p_1 + 1) \cdots (p_r + 1),$$

one can write

$$\psi(\varphi(n)) = p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1} \cdot (p_1-1) \cdots (p_r-1) \cdot \prod_{t|(p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1} \cdot (p_1-1) \cdots (p_r-1))} 1 + \frac{1}{t}$$

and

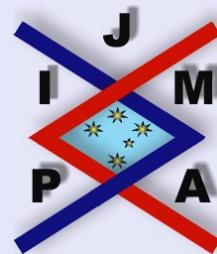
$$\varphi(\psi(n)) = p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1} \cdot (p_1 + 1) \cdots (p_r + 1) \cdot \prod_{q|(p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1} \cdot (p_1+1) \cdots (p_r+1))} \left(1 - \frac{1}{q}\right).$$

Since $\alpha_i - 1 \geq 1$ when $p_i \geq 3$, the equality $\psi(\varphi(n)) = \varphi(\psi(n))$, by

$$\begin{aligned} (p_1 - 1) \cdots (p_r - 1) \cdot \left(1 + \frac{1}{p_1}\right) \cdots \left(1 + \frac{1}{p_r}\right) \\ = (p_1 + 1) \cdots (p_r + 1) \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right), \end{aligned}$$

can also be written as

$$\prod_{t|(p_1-1) \cdots (p_r-1)} \left(1 + \frac{1}{t}\right) = \prod_{q|(p_1+1) \cdots (p_r+1)} \left(1 - \frac{1}{q}\right).$$



**On the Composition of Some
Arithmetic Functions, II**

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 12 of 37

Since $1 + \frac{1}{t} > 1$ and $1 - \frac{1}{q} < 1$, this is impossible in general. It is possible only if all prime factors of $(p_1 + 1) \cdots (p_r - 1)$ are among p_1, \dots, p_r , and also the same for the prime factors of $(p_1 + 1) \cdots (p_r + 1)$. \square

Remark 2. For example, $n = 2^a \cdot 3^b \cdot 5^c$ with $a \geq 1, b \geq 2, c \geq 2$ satisfy (3.9).
Indeed

$$pr\{(2-1)(3-1)(5-1)\} = \{2\}, pr\{(2+1)(3+1)(5+1)\} = \{2, 3\}.$$

Similar examples are $n = 2^a \cdot 3^b \cdot 5^c \cdot 7^d, n = 2^a \cdot 3^b \cdot 5^c \cdot 11^d, n = 2^a \cdot 3^b \cdot 7^c \cdot 13^d,$
 $n = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e \cdot 13^f, n = 2^a \cdot 3^b \cdot 17^c$, etc.

Theorem 3.3. Let n be squarefull. Then inequality (1.8) is true.

Proof. Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $\alpha_i \geq 2$ for all $i = \overline{1, r}$. Then

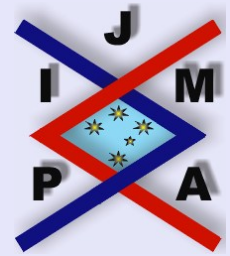
$$\begin{aligned} \varphi(\psi(n)) &= \varphi(p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1} \cdot (p_1 + 1) \cdots (p_r + 1)) \\ &\leq (p_1 + 1) \cdots (p_r + 1) \cdot \varphi(p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1}), \end{aligned}$$

by Lemma 2.1. But

$$\varphi(p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1}) = p_1^{\alpha_1-2} \cdots p_r^{\alpha_r-2} \cdot (p_1 - 1) \cdots (p_r - 1),$$

since $\alpha \geq 2$. Then

$$\begin{aligned} \varphi(\psi(n)) &\leq (p_1^2 - 1) \cdots (p_r^2 - 1) \cdot p_1^{\alpha_1-2} \cdots p_r^{\alpha_r-2} \\ &= p_1^{\alpha_1} \cdots p_r^{\alpha_r} \cdot \left(1 - \frac{1}{p_1^2}\right) \cdots \left(1 - \frac{1}{p_r^2}\right), \end{aligned}$$



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 13 of 37

so

$$(3.10) \quad \varphi(\psi(n)) \leq n \cdot \left(1 - \frac{1}{p_1^2}\right) \cdots \left(1 - \frac{1}{p_r^2}\right).$$

There is equality in (3.10) if

$$pr\{(p_1 + 1) \cdots (p_r + 1)\} \subset \{p_1, \dots, p_r\}.$$

Clearly, inequality (3.10) is best possible, and by

$$\left(1 - \frac{1}{p_1^2}\right) \cdots \left(1 - \frac{1}{p_r^2}\right) < 1,$$

it implies inequality (1.8). □

Theorem 3.4. For any $n \geq 2$ one has

$$(3.11) \quad \varphi\left(n \left[\frac{\psi(n)}{n}\right]\right) < n,$$

where $[x]$ denotes the integer part of x .

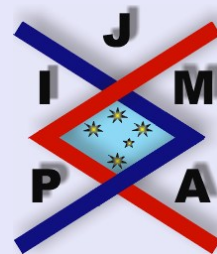
Proof. It is immediate that

$$\frac{\varphi(n)\psi(n)}{n^2} = \prod_{p|n} \left(1 - \frac{1}{p^2}\right) < 1,$$

so $\varphi(n)\psi(n) < n^2$ for any $n \geq 2$. Now, by (2.1) one can write

$$\varphi\left(n \left[\frac{\psi(n)}{n}\right]\right) \leq \left[\frac{\psi(n)}{n}\right] \varphi(n) \leq \frac{\psi(n)}{n} \cdot \varphi(n) < n,$$

by the relation proved above. □



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 14 of 37

Remark 3. If $n|\psi(n)$, i.e., when $\left[\frac{\psi(n)}{n}\right] = \frac{\psi(n)}{n}$, relation (3.11) gives inequality (1.8), i.e. $\varphi(\psi(n)) < n$. For the study of an equation

$$(3.12) \quad \psi(n) = k \cdot n$$

we shall use a notion and a method of Ch. Wall [25]. We say that n is ω -multiple of m if $m|n$ and $pr\{m\} = pr\{n\}$.

We need a simple result, stated as:

Lemma 3.5. If m and n are squarefree, and $\frac{\psi(n)}{n} = \frac{\psi(m)}{m}$, then $n = m$.

Proof. Without loss of generality we may suppose

$$(m, n) = 1; \quad m, n > 1, \quad m = q_1 \cdots q_j \quad (q_1 < \cdots < q_j)$$

and

$$n = p_1 \cdots p_k \quad (p_1 < \cdots < p_k).$$

Then the assumed equality has the form

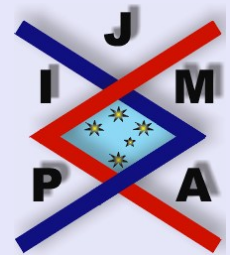
$$n(1 + q_1) \cdots (1 + q_j) = m(1 + p_1) \cdots (1 + p_k).$$

Since $p_k|n$, the relation

$$p_k|(1 + p_1) \cdots (1 + p_{k-1})(1 + p_k)$$

implies $p_k|(1 + p_k)$ for some $i \in \{1, 2, \dots, k\}$. Here

$$1 + p_1 < \cdots < 1 + p_{k-1} < 1 + p_k,$$



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 15 of 37

so we must have $p_k | (1 + p_{k-1})$. This may happen only when $k = 2$, $p_1 = 2$, $p_2 = 3$; $j = 2$, $q_1 = 2$, $q_3 = 3$ (since for $k \geq 3$, $p_k - p_{k-1} \geq 2$, so $p_k \nmid (1 + p_{k-1})$). In this case $(n, m) = 6 > 1$, a contradiction. Thus $k = j$ and $p_k = q_j$. \square

Theorem 3.6. *Assume that the least solution n_k of (3.12) is a squarefree number. Then all solutions of (3.12) are given by the ω -multiples of n_k .*

Proof. If n is ω -multiple of n_k , then clearly

$$\frac{\psi(n)}{n} = \frac{\psi(n_k)}{n_k} = k,$$

by (1.1). Conversely, if n is a solution, set $m =$ greatest squarefree divisor of n . Then

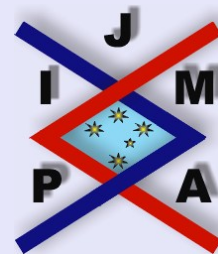
$$\frac{\psi(n)}{n} = \frac{\psi(m)}{m} = k = \frac{\psi(n_k)}{n_k}.$$

By Lemma 3.5, $m = n_k$, i.e. n is an ω -multiple of n_k . \square

Theorem 3.7. *Let $n \geq 3$, and suppose that n is ψ -deficient, i.e. $\psi(n) < 2n$. Then inequality (1.8) holds.*

Proof. First remark that for any $n \geq 3$, $\psi(n)$ is an even number. Indeed, if $n = 2^a$, then $\psi(n) = 2^{a-1} \cdot 3$, which is odd only for $a = 1$, i.e. $n = 2$. If n has at least one odd prime factor p , then by (1.1), $\psi(n)$ will be even.

Now, applying Lemma 2.1 for $b = 2$, one obtains $\varphi(2a) \leq a$, i.e. $\varphi(u) \leq \frac{u}{2}$ for $u = 2a$ (even). Here equality occurs only when $u = 2^k$ ($k \geq 1$). Now, $\varphi(\psi(n)) \leq \frac{\psi(n)}{2}$, $\psi(n)$ being even, and since n is ψ -deficient, the theorem follows. \square



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 16 of 37

Remark 4. *The inequality*

$$(3.13) \quad \varphi(\psi(n)) \leq \frac{\psi(n)}{2}$$

is best possible, since we have equality for $\psi(n) = 2^k$. Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$; then $p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1} \cdot (p_1+1) \cdots (p_r+1) = 2^k$ is possible only if $\alpha_1 = \cdots = \alpha_r = 1$, and $p_1+1 = 2_1^a, \dots, p_r+1 = 2_r^a$; i.e. when $p_1 = 2_1^a - 1, \dots, p_r = 2_r^a - 1$ are distinct Mersenne primes, and $n = p_1 \cdots p_r$. So, there is equality in (3.13) iff n is a product of distinct Mersenne primes. Since by Theorem 3.6 one has $\psi(n) = 2n$ iff $n = 2^a \cdot 3^b$ ($a, b \geq 1$), if one assumes $\psi(n) \leq 2n$, then by (3.13), inequality (1.8) follows again. Therefore, in Theorem 3.7 one may assume $\psi(n) \leq 2n$.

Let $\omega(n)$ denote the number of distinct prime factors of n . Theorem 3.7 and the above remark implies that when n is even, and $\omega(n) \leq 2$, (1.8) is true. Indeed, $1 + \frac{1}{2} = \frac{3}{2} < 2$, and $(1 + \frac{1}{2})(1 + \frac{1}{3}) = 2$. So e.g. when $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2}$, then

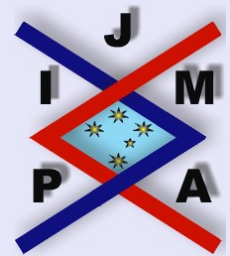
$$\frac{\psi(n)}{n} = \left(1 + \frac{1}{p_1}\right) \cdot \left(1 + \frac{1}{p_2}\right) \leq \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) = 2.$$

On the other hand, if n is odd, and $\omega(n) \leq 4$, then (1.8) is valid. Indeed,

$$\left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{5}\right) \left(1 + \frac{1}{7}\right) \left(1 + \frac{1}{11}\right) = \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} = \frac{2304}{1155} < 2.$$

Another remark is the following:

If 2 and 3 do not divide n , and n has at most six prime factors, then $\varphi(\psi(n)) < n$. If 2, 3 and 5 do not divide n , and n has at most 12 prime factors, then the



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 17 of 37

same result holds true. If 2, 3, 5 and 7 do not divide n , and n has at most 21 prime factors, then the inequality is true.

If 2 and 3 do not divide n , we prove that $\psi(n) < 2n$, and by the presented method the results will follow. E.g., when n is not divisible by 2 and 3, then the least prime factor of n could be 5, so

$$\frac{\psi(n)}{n} < \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} \cdot \frac{24}{23} \cdot \frac{30}{29} \cdot \frac{32}{31} < 2,$$

and the first result follows. The other affirmations can be proved in a similar way.

In [16] it is proved that

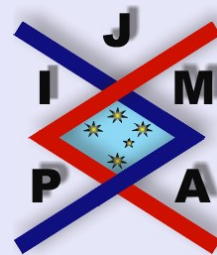
$$(3.14) \quad \psi(n) \leq \begin{cases} 3^{\omega(n)} \cdot \varphi(n), & \text{if } n \text{ is even} \\ 2^{\omega(n)} \cdot \varphi(n), & \text{if } n \text{ is odd} \end{cases}.$$

Thus, as a corollary of (3.13) and (3.14) one can state that if $\frac{3^{\omega(n)} \cdot \varphi(n)}{2} < n$ (or $\leq n$), for n even; and $2^{\omega(n)-1} \cdot \varphi(n)$ (or $\leq n$) for n odd, then relation (1.8) is valid.

By (3.13), if n is a product of distinct Mersenne primes, then $\varphi(\psi(n)) = \frac{\psi(n)}{2}$. We will prove that $\psi(n) < 2n$ for such n , thus obtaining:

Theorem 3.8. *If n is a product of distinct Mersenne primes, then inequality (1.8) is valid.*

Proof. Let $n = M_1 \cdots M_s$, where $M_i = 2^{p_i} - 1$ (p_i primes, $i = 1, 2, \dots, s$) are distinct Mersenne primes. We have to prove that $(2^{p_1} - 1) \cdots (2^{p_s} - 1) >$



On the Composition of Some Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 18 of 37

$2^{p_1+\dots+p_s-1}$, or equivalently, $(1 - \frac{1}{2^{p_1}}) \cdots (1 - \frac{1}{2^{p_s}}) > \frac{1}{2}$. Clearly $p_1 \geq 2$, $p_2 \geq 3$, \dots , $p_s \geq s + 1$, so it is sufficient to prove that

$$(3.15) \quad \left(1 - \frac{1}{2^2}\right) \cdots \left(1 - \frac{1}{2^{s+1}}\right) > \frac{1}{2}.$$

In the proof of (3.15) we will use the classical Weierstrass inequality

$$(3.16) \quad \prod_{k=1}^s (1 - a_k) > 1 - \sum_{k=1}^s a_k,$$

where $a_k \in (0, 1)$ (see e.g. D.S. Mitrinović: *Analytic inequalities*, Springer-Verlag, 1970).

Put $a_k = \frac{1}{2^{k+1}}$ in (3.16). Since

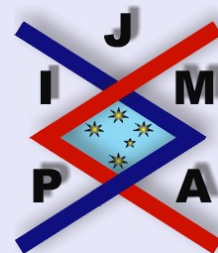
$$\sum_{k=1}^s \frac{1}{2^{k+1}} = \frac{1}{4} \cdot \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{s-1}}\right) = \frac{1}{4} \cdot \left(\frac{1 - \frac{1}{2^s}}{1 - \frac{1}{2}}\right) = \frac{2^s - 1}{2^{s+1}},$$

(3.15) becomes equivalent to $1 - \frac{2^s - 1}{2^{s+1}} > \frac{1}{2}$, or $\frac{1}{2} > \frac{2^s - 1}{2^{s+1}}$, i.e. $2^s > 2^s - 1$, which is true. Therefore, (3.15) follows, and the theorem is proved. \square

Remark 5. By Theorem 3.15 (see relation (3.29)), if $n = M_1^{a_1} \cdots M_s^{s_s}$ (with arbitrary $a_i \geq 1$), the inequality (1.8) holds true.

Related to the above theorems is the following result:

Theorem 3.9. Let n be even, and suppose that the greatest odd part m of n is ψ -deficient, and that $3 \nmid \psi(m)$. Then (1.8) is true.



On the Composition of Some Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 19 of 37

Proof. Let $n = 2^k \cdot m$, when

$$\varphi(\psi(n)) = \varphi(2^{k-1} \cdot 3\psi(m)) = 2 \cdot \varphi(2^{k-1} \cdot \psi(m))$$

since $(3, 2^{k-1} \cdot \psi(m)) = 1$. But

$$\varphi(2^{k-1} \cdot \psi(m)) \leq 2^{k-2} \cdot \psi(m) < 2^{k-1} \cdot m,$$

so $\varphi(\psi(n)) < 2^k \cdot m = n$. □

Remark 6. In [18] it is proved that for all $n \geq 2$ even, one has

$$(3.17) \quad \varphi(\sigma(n)) \geq 2n,$$

with equality only if $n = 2^k$, where $2^{k+1} - 1 = \text{prime}$. The proof is based on Lemma 2.3. Since $\sigma(m) \geq \psi(m)$, clearly this implies

$$(3.18) \quad \sigma(\sigma(n)) \geq 2n,$$

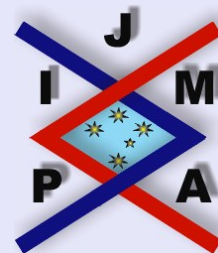
with the above equalities. So, the Surayanarayana-Kanved theorem is reobtained, in an improved form.

In [18] it is proved also that for all $n \geq 2$ even, one has

$$(3.19) \quad \sigma(\psi(n)) \geq 2n,$$

with equality only for $n = 2$. What are the odd solutions of $\sigma(\psi(n)) = 2n$?

We now prove:



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 20 of 37

Theorem 3.10. Let $n = 2^k \cdot m$ be even ($k \geq 1, m > 1$ odd), and suppose that m is not a product of distinct Fermat primes, and that m satisfies (1.6). Then

$$(3.20) \quad \sigma(\varphi(n)) \geq n - m \geq \frac{n}{2}.$$

Proof. First remark that if m is not a product of distinct Fermat primes, then $\varphi(m)$ is not a power of 2. Indeed, if $m = p_1^{a_1} \cdots p_r^{a_r}$, then

$$\varphi(m) = p_1^{a_1-1} \cdots p_r^{a_r-1} (p_1 - 1) \cdots (p_r - 1) = 2^s$$

iff (since $p_i \geq 3$),

$$a_1 - 1 = \cdots = a_r - 1 = 0$$

and

$$p_1 - 1 = 2^{s_1}, \dots, p_r - 1 = 2^{s_r},$$

i.e.

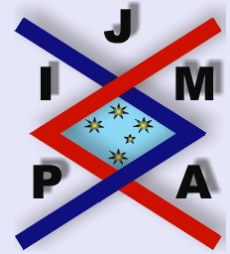
$$p_1 = 2^{s_1} + 1, \dots, p_r = 2^{s_r} + 1$$

are distinct Fermat primes. Thus there exists at least an odd prime divisor of $\varphi(m)$. Now, by Lemma 2.4,

$$\sigma(\varphi(2^k \cdot m)) = \sigma(2^{k-1} \cdot \varphi(m)) \geq \psi(\varphi(m)) \cdot \sigma(2^{k-1}) \geq m \cdot (2^k - 1) = n - m,$$

by relation (1.6). The last inequality of (3.20) is trivial, since $m \leq \frac{n}{2} = 2^{k-1} \cdot m$, where $k - 1 \geq 0$. \square

Remark 7. Relation (3.17) gives an improvement of (1.3) for certain values of n .



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 21 of 37

Theorem 3.11. *Let p be an odd prime. Then*

$$(3.21) \quad \varphi(\psi(p)) \leq \frac{p+1}{2},$$

with equality only if p is a Mersenne prime, and $\psi(\varphi(p)) \geq \frac{3}{2} \cdot (p-1)$, with equality only if p is a Fermat prime.

Proof. $\psi(p) = p+1$ and $p+1$ being even, $\varphi(p+1) \leq \frac{p+1}{2}$, with equality only if $p+1 = 2^k$, i.e. when $p = 2^k - 1 =$ Mersenne prime. Since $\frac{3}{2} \cdot (p-1) \geq p$, this inequality is better than (1.6) for $n = p$. Similarly, $\varphi(p) = p-1 =$ even, so $\psi(p-1) \geq \frac{3}{2} \cdot (p-1)$, on base of the following: \square

Lemma 3.12. *If $n \geq 2$ is even, then*

$$(3.22) \quad \psi(n) \geq \frac{3}{2} \cdot n,$$

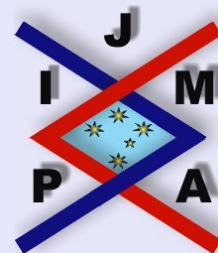
with equality only if $n = 2^a$ (power of 2).

Proof. If $n = 2^a \cdot N$, with N odd,

$$\psi(n) = \psi(2^a) \cdot \psi(N) = 2^{a-1} \cdot 3 \cdot \psi(N) \geq 2^{a-1} \cdot 3 \cdot N = \frac{3}{2} \cdot n.$$

Equality occurs only, when $N = 1$, i.e. when $n = 2^a$. \square

Since $p-1 = 2^a$ implies $p = 2^a + 1 =$ Fermat prime, (3.21) is completely proved. Since $\frac{3}{2} \cdot (p-1) \geq p$, this inequality is better than (1.6) for $n = p$.



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 22 of 37

Remark 8. For $p \geq 5$ one has $\frac{p+1}{2} < p < \frac{3}{2} \cdot (p-1)$, so (3.21) implies, as a corollary that

$$(3.23) \quad \varphi(\psi(p)) < p < \psi(\varphi(p)),$$

for $p \geq 5$, prime.

This is related to relation (3.4). If n is even, and $n \neq 2^a$ (power of 2), then since $\psi(N) \geq N + 1$, with equality only when N is a prime, (3.22) can be improved to

$$(3.24) \quad \psi(n) \geq \frac{3}{2} \cdot \left(n + \frac{n}{N} \right),$$

with equality only for $n = 2^a \cdot N$, where $N = \text{prime}$.

Theorem 3.13. Let $a, b \geq 1$ and suppose that $a|b$. Then $\varphi(\psi(a))|\varphi(\psi(b))$ and $\psi(\varphi(a))|\psi(\varphi(b))$. In particular, if $a|b$, then

$$(3.25) \quad \varphi(\psi(a)) \leq \varphi(\psi(b)); \quad \psi(\varphi(a)) \leq \psi(\varphi(b)).$$

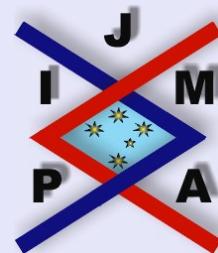
Proof. The proof follows at once from the following:

Lemma 3.14. If $a|b$, then

$$(3.26) \quad \varphi(a)|\varphi(b),$$

and

$$(3.27) \quad \psi(a)|\psi(b),$$



On the Composition of Some Arithmetic Functions, II

József Sándor

Title Page

Contents

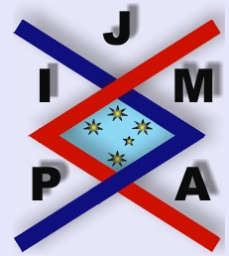


Go Back

Close

Quit

Page 23 of 37



□
□

Proof. This follows by (1.1), see e.g. [16], [18].

Now, if $a|b$, then $\psi(a)|\psi(b)$ by (3.27), so by (3.26), $\varphi(\psi(a))|\varphi(\psi(b))$. Similarly, $a|b$ implies $\varphi(a)|\varphi(b)$ by (3.26), so by (3.27), $\psi(\varphi(a))|\psi(\varphi(b))$. The inequalities in (3.22) are trivial consequences.

Remark 9. Let $a = p$ be a prime such that $p \nmid k$, and put $b = k^{p-1} - 1$.

By Fermat's little theorem one has $a|b$, so all results of (3.25) are correct in this case. For example, $\psi(\varphi(a)) \leq \psi(\varphi(b))$ gives, in the case of (3.25), and Theorem 3.10:

$$(3.28) \quad \psi(\varphi(k^{p-1} - 1)) \geq \psi(\varphi(p)) \geq \frac{3}{2} \cdot (p - 1),$$

for any prime $p \nmid k$, and any positive integer $k > 1$.

Let $(n, k) = 1$. Then by Euler's divisibility theorem, one has similarly:

$$(3.29) \quad \psi(\varphi(k^{\varphi(n)} - 1)) \geq \psi(\varphi(n)),$$

for any positive integers $n, k > 1$ such that $(n, k) = 1$.

Let $n > 1$ be a positive integer, having as distinct prime factors p_1, \dots, p_r . Then, using (1.1) it is immediate that

$$(3.30) \quad \varphi(n)|\psi(n)$$

iff $(p_1 - 1) \cdots (p_r - 1) | (p_1 + 1) \cdots (p_r + 1)$. For example, (3.30) is true for $n = 2^m, n = 2^m \cdot 5^s$ ($m, s \geq 1$), etc. Now assuming (3.30), by (3.26) one can write the following inequalities:

$$(3.31) \quad \varphi(\psi(\varphi(n))) \leq \varphi(\psi(\psi(n))) \text{ and } \psi(\varphi(\varphi(n))) \leq \psi(\varphi(\psi(n))).$$

By studying the first 100 values of n with the property (3.30), the following interesting example may be remarked: $\varphi(15) = \varphi(16) = 8, \psi(15) = \psi(16) = 24$ and $\varphi(15) | \psi(15)$. Similarly $\varphi(70) = \varphi(72) = 24, \psi(70) = \psi(72) = 144$, with $\varphi(70) | \psi(70)$.

Are there infinitely many such examples? Are there infinitely many n such that $\varphi(n) = \varphi(n + 1)$ and $\psi(n) = \psi(n + 1)$? Or $\varphi(n) = \varphi(n + 2)$ and $\psi(n) = \psi(n + 2)$?

Let $a = 8, b = \sigma(8k - 1)$. Then $a | b$ (see e.g. [18] for such relations), and since $\psi(\varphi(8)) = 6, \varphi(\psi(8)) = 12$, by (3.25) we obtain the divisibility relations

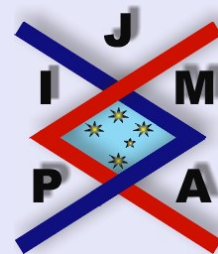
$$(3.32) \quad 6 | \psi(\varphi(\sigma(8k - 1))) \text{ and } 12 | \varphi(\psi(\sigma(8k - 1)))$$

for $k \geq 1$.

The second relation implies e.g. that if $\varphi(\psi(\sigma(n))) = 2n$, then $n \not\equiv -1 \pmod{8}$ and if $\varphi(\psi(\sigma(n))) = 4n$, then $n \not\equiv -1 \pmod{24}$.

Theorem 3.15. *Inequality (1.8) is true for an $n \geq 2$ if it is true for the square-free part of $n \geq 2$. Inequality (1.6) is true for an odd $m \geq 3$ if it is true for the squarefree part of $m \geq 3$.*

Proof. As we have stated in the Introduction, such results were first proved by the author. We give here the proof for the sake of completeness.



On the Composition of Some Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 25 of 37

Let n' be the squarefree part of n , i.e. if $n = p_1^{a_1} \cdots p_r^{a_r}$, then $n' = p_1 \cdots p_r$.
Then

$$\begin{aligned}\varphi(\psi(n)) &= \varphi(p_1^{a_1-1} \cdots p_r^{a_r-1} \cdot (p_1 + 1) \cdots (p_r + 1)) \\ &\leq p_1^{a_1-1} \cdots p_r^{a_r-1} \cdot \varphi((p_1 + 1) \cdots (p_r + 1)) \\ &= \frac{n}{n'} \cdot \varphi(\psi(n'))\end{aligned}$$

by inequality (2.1).

Thus

$$(3.33) \quad \frac{\varphi(\psi(n))}{n} \leq \frac{\varphi(\psi(n'))}{n'}.$$

Therefore, if $\frac{\varphi(\psi(n'))}{n'} < 1$, then $\frac{\varphi(\psi(n))}{n} < 1$. Similarly one can prove that

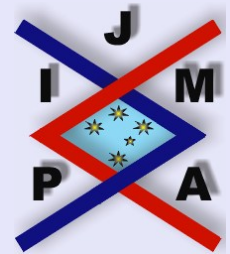
$$(3.34) \quad \frac{\psi(\varphi(m))}{m} \geq \frac{\psi(\varphi(m'))}{m'},$$

so if (1.6) is true for the squarefree part m' of m , then (1.6) is true also for m .

As a consequence, (1.8) is true for all n if and only if it is true for all square-free n .

As we have stated in the introduction, (1.6) is not generally true for all m . Let e.g. $m = 3 \cdot F$, where $F > 3$ is a Fermat prime. Indeed, put $F = 2^k + 1$. Then $\varphi(m) = 2^{k+1}$, so

$$\psi(\varphi(m)) = 2^k \cdot 3 < 3 \cdot (2^k + 1) = 3 \cdot F = m,$$



**On the Composition of Some
Arithmetic Functions, II**

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 26 of 37

contradicting (1.6). However, if m has the form $m = 5 \cdot F$, where $F > 5$ is again a Fermat prime, then (1.6) is valid, since in this case

$$\psi(\varphi(m)) = 6 \cdot 2^k > 5 \cdot (2^k + 1) = m.$$

□

More generally, we will prove now:

Theorem 3.16. *Let $5 \leq F_1 < \dots < F_s$ be Fermat primes. Then inequality (1.6) is valid (with strict inequality) for $m = F_1^{a_1} \dots F_s^{a_s}$, with arbitrary $a_i \geq 1$ ($i = \overline{1, s}$).*

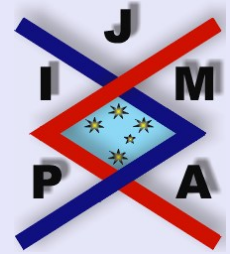
Proof. Let $F_i = 1 + 2^{2^{b_i}}$ ($i \geq 1$) be Fermat primes, where $b_1 \geq 1$. Since $b_1 < b_2 < \dots < b_s$, clearly $b_i \geq i$ for any $i = 1, 2, \dots, s$. By (3.34) it is sufficient to prove the result for $m' = F_1 \dots F_s$, when (1.6) becomes, after some elementary computations:

$$(3.35) \quad \left(1 + \frac{1}{2^{2^{b_1}}}\right) \dots \left(1 + \frac{1}{2^{2^{b_s}}}\right) \leq \frac{3}{2}.$$

We will prove that (3.35) holds with strict inequality. By the classical Weierstrass inequalities one has

$$\prod_{k=1}^s (1 + a_k) < \frac{1}{1 - \sum_{k=1}^s a_k},$$

where $a_k \in (0, 1)$.



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 27 of 37

Since $b_i \geq 1$, it is sufficient to prove that

$$(3.36) \quad \left(1 + \frac{1}{2^2}\right) \cdots \left(1 + \frac{1}{2^{2^s}}\right) \leq \frac{3}{2}.$$

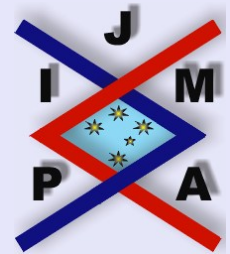
Put $a_k = 2^{2^k}$ ($k \geq 1$), so by the above inequality, it is sufficient to prove that

$$(3.37) \quad \sum = \frac{1}{2^{2^1}} + \frac{1}{2^{2^2}} + \cdots + \frac{1}{2^{2^s}} < \frac{1}{3}.$$

Clearly (3.37) is true for $s = 1, 2$, since $\frac{1}{4} < \frac{1}{3}$, $\frac{1}{4} + \frac{1}{16} = \frac{5}{16} < \frac{1}{3}$. Let $s \geq 3$. Then, since $2^s \geq s + 5$ for $s \geq 3$, we can write

$$\begin{aligned} \sum &\leq \frac{1}{4} + \frac{1}{16} + \frac{1}{2^8} \cdot \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{s-3}}\right) \\ &= \frac{5}{16} + \frac{1}{128} \cdot \left(1 - \frac{1}{2^{s-2}}\right) \\ &< \frac{5}{16} + \frac{1}{128} = \frac{41}{128} < \frac{1}{3}, \end{aligned}$$

and the assertion is proved. □



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 28 of 37

Remark 10. By Lemma 2.2, relation (2.2) one can write successively

$$\begin{aligned} \varphi((p_1 + 1)(p_2 + 1)) &\leq p_2\varphi(p_1 + 1) < p_1p_2, \\ &\text{if } pr\{p_2 + 1\} \not\subset pr\{p_1 + 1\} \\ \varphi((p_1 + 1)(p_2 + 1)(p_3 + 1)) &\leq p_3\varphi(p_1 + 1)(p_2 + 1) < p_1p_2p_3, \\ &\text{if in addition } pr\{p_3 + 1\} \not\subset pr\{(p_1 + 1)(p_2 + 1)\} \\ (3.38) \quad &\dots \end{aligned}$$

$$\begin{aligned} \varphi((p_1 + 1) \cdots (p_{r-1} + 1)(p_r + 1)) &\leq p_r\varphi((p_1 + 1) \cdots (p_{r-1} + 1)) \\ &< p_1 \cdots p_r, \\ &\text{if } pr\{p_r + 1\} \not\subset pr\{(p_1 + 1) \cdots (p_{r-1} + 1)\} \end{aligned}$$

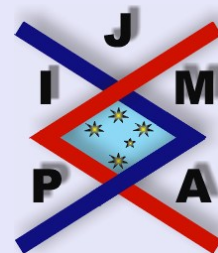
is satisfied, then by Theorem 3.15, inequality (1.8) is valid.

Similarly, by using Lemma 2.2, (2.3), and Theorem 3.15, we can state that if

$$\begin{aligned} pr\{p_2 - 1\} &\not\subset pr\{q_1 - 1\}, \\ pr\{q_3 - 1\} &\not\subset pr\{(p_1 - 1)(p_2 - 1)\}, \\ (3.39) \quad &\dots, \\ pr\{q_r - 1\} &\not\subset pr\{(p_1 - 1) \cdots (q_{r-1} - 1)\}, \end{aligned}$$

then inequality (1.6) is valid. (Here q_1, q_2, \dots, q_r are the prime divisors of the odd number $m \geq 3$.)

Remark 11. Inequality (1.8) is not generally true. Indeed, for $n = 39270$, $n = 82110$, or $n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 23 \cdot M$, where M is a Mersenne prime, greater or equal than 31, then (1.8) is not true. This has been communicated



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 29 of 37

to the author by Professor L. Tóth. Prof. Kovács Lehel István found recently the counterexamples: 53130, 71610, 78540, 106260, 108570, 117810, 122430, 143220, 157080, 159390, 164010, 164220, 212520, 214830, 217140, 235620, 244860, 246330, 247170, 286440, 293370, 314160, 318780, 325710, 328440, 353430 and 367290.

Now by using a method of L. Alaoglu and P. Erdős [1], we will prove that:

Theorem 3.17. For any $\delta > 0$, the inequality

$$(3.40) \quad \varphi(\psi(n)) < \delta \cdot n$$

is valid, excepting perhaps $n \in S$, where S has asymptotic density zero.

Proof. We prove first that for any given prime p , the set of n such that $p|\psi(n)$, has density 1. This is similar to the proof given in [1].

On the other hand, since $\sum_{n \leq x} \psi(n) \approx \frac{15}{2\pi^2} \cdot x^2$ as $x \rightarrow \infty$ (see e.g. [16]), we can say that excepting at most a number of $\epsilon \cdot x$ integers $n < x$, one has $\psi(n) < c(\epsilon) \cdot n$, where $c(\epsilon) > 0$.

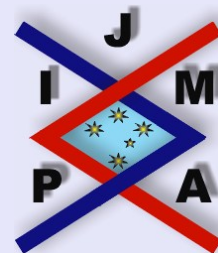
Let now p be a prime such that

$$\prod_{q \leq p} \left(1 - \frac{1}{q}\right) < \frac{\delta}{c(\epsilon)}$$

(this is possible, since $\prod_{q \leq p} \left(1 - \frac{1}{q}\right) \rightarrow 0$ as $p \rightarrow \infty$).

Then, if x is large, then for all $n < x$, excepting perhaps a number of $\eta \cdot x + \epsilon \cdot x$ integers one has $\psi(n) < c(\epsilon) \cdot n$ and $\psi(n) \equiv 0 \pmod{q}$ for any $q \leq p$, ($\eta > 0$).

But for these exceptions one has $\varphi(\psi(n)) < \delta \cdot n$, and this finishes the proof; $\eta, \epsilon > 0$ being arbitrary. \square



On the Composition of Some Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 30 of 37

Remark 12. *It can be proved similarly that*

$$(3.41) \quad \psi(\varphi(n)) > \delta \cdot n,$$

excepting perhaps a set of density zero.

Theorem 3.17 implies that $\liminf_{n \rightarrow \infty} \frac{\psi(\varphi(n))}{n} = 0$, and so, one has $\limsup_{n \rightarrow \infty} \frac{\psi(\varphi(n))}{n} = +\infty$. For other proof of these results, see [16]. We cannot determine the following values: $\liminf_{n \rightarrow \infty} \frac{\psi(\varphi(n))}{n} = ?$, $\limsup_{n \rightarrow \infty} \frac{\psi(\varphi(n))}{n} = ?$

However, we can prove that:

Theorem 3.18.

$$(3.42) \quad \liminf_{n \rightarrow \infty} \frac{\psi(\varphi(n))}{n} \leq \inf \left\{ \frac{\psi(\varphi(k))}{k} : k \text{ is a multiple of } 4 \right\} < \frac{1}{2}.$$

Proof. Let k be a multiple of 4, and $p > \frac{k}{2}$. Then

$$\varphi\left(\frac{1}{2}kp\right) = \varphi\left(\frac{k}{2}\right)\varphi(p) = 2\varphi\left(\frac{k}{2}\right) \cdot \frac{p-1}{2} = \varphi(k) \cdot \frac{p-1}{2},$$

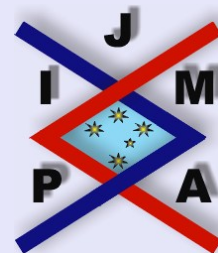
since $2| \frac{k}{2}$. Now by $\psi(ab) \leq \psi(a)\psi(b)$ one can write

$$\psi\left(\varphi\left(\frac{1}{2}kp\right)\right) \leq \psi(\varphi(k))\psi\left(\frac{p-1}{2}\right).$$

Since $\psi\left(\frac{p-1}{2}\right) \leq \sigma\left(\frac{p-1}{2}\right)$, and by the known result of Makowski and Schinzel:

$\liminf_{p \rightarrow \infty} \frac{\sigma\left(\frac{p-1}{2}\right)}{\frac{p-1}{2}} = 1$, from the above one can write:

$$\liminf_{p \rightarrow \infty} \frac{\psi\left(\varphi\left(\frac{1}{2}kp\right)\right)}{\frac{1}{2}kp} \leq \frac{\psi(\varphi(k))}{k} \cdot \liminf_{p \rightarrow \infty} \frac{\psi\left(\frac{p-1}{2}\right)}{\frac{p-1}{2}} \leq \frac{\psi(\varphi(k))}{k},$$



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 31 of 37

and now relation (3.42) follows, by taking inf after k .

Since

$$2^{32} - 1 = F_0 \cdot F_1 \cdot F_2 \cdot F_3 \cdot F_4,$$

where $F_k = 2^{2^k} + 1$, and all F_i ($0 \leq i \leq 4$) are primes, it follows, that

$$\varphi(2^{32} - 1) = 2^1 \cdot 2^2 \cdot 2^4 \cdot 2^8 \cdot 2^{16} = 2^{31}.$$

Thus $\varphi(4(2^{32} - 1)) = 2^{32}$, by $\varphi(4) = 2$. Since $\psi(2^{32}) = 2^{31} \cdot 3$, by letting in (3.42) $k = 4 \cdot (2^{32} - 1)$, we get the inf $\leq \frac{2^{31} \cdot 3}{4 \cdot (2^{32} - 1)} < \frac{1}{2 \cdot (\frac{4}{3} - \theta)}$, where $\theta > \frac{1}{3 \cdot 2^{30}}$.

In any case we get in (3.42) that $\liminf < \frac{1}{2}$, and fact a value slightly greater than $\frac{1}{2 \cdot \frac{4}{3}} = \frac{3}{8}$. \square

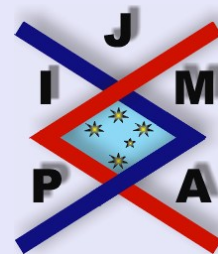
In [16] it is asked the value of $\liminf \frac{\psi(\sigma(n))}{n} \leq 1$. We now prove that this value is 1:

Theorem 3.19.

$$(3.43) \quad \liminf \frac{\psi(\sigma(n))}{n} = 1.$$

Proof. Since $\frac{\psi(\sigma(n))}{n} \geq \frac{\sigma(n)}{n} \geq 1$, clearly this limit is ≥ 1 . By the above inequality, the result follows. However, we give here a new proof of this fact. We remark that, since $\varphi(N) \leq \psi(N) \leq \sigma(N)$, and by the known result

$$\lim_{p \rightarrow \infty} \frac{\varphi(N(a, p))}{N(a, p)} = \lim_{p \rightarrow \infty} \frac{\sigma(N(a, p))}{N(a, p)} = 1,$$



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 32 of 37

where $N(a, p) = \frac{a^p - 1}{p - 1}$, ($a > 1, p$ prime) we easily get

$$(3.44) \quad \lim_{p \rightarrow \infty} \frac{\varphi(N(a, p))}{N(a, p)} = 1.$$

Now let $a = q$ an arbitrary prime in (3.44). We remark that $N(q, p) = \frac{q^p - 1}{q - 1} = \sigma(q^{p-1})$. Now, by

$$\frac{\sigma(q^{p-1})}{q^{p-1}} = \frac{q^p - 1}{(q - 1) \cdot q^{p-1}} \rightarrow \frac{q}{q - 1},$$

as $p \rightarrow \infty$, from (3.44) we can write:

$$(3.45) \quad \lim_{p \rightarrow \infty} \frac{\psi(\sigma(q^{p-1}))}{q^{p-1}} = \frac{q}{q - 1} < 1 + \epsilon,$$

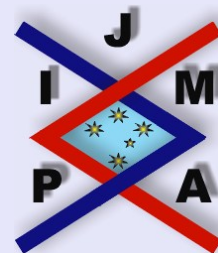
for $q \geq q(\epsilon)$, $\epsilon > 0$. Now by (3.45), (3.43) follows. \square

Remark 13. In [16] it is proved, by assuming the infinitude of Mersenne primes, that

$$(3.46) \quad \liminf_{n \rightarrow \infty} \frac{\psi(\psi(n))}{n} = \frac{3}{2}.$$

Can we prove (3.46) without any assumption?

We have conjectured in [16] that the following limit is true, but in the proof we have used the fact that there are infinitely many Mersenne primes. Now we prove this result without any assumptions:



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 33 of 37

Theorem 3.20. *We have*

$$(3.47) \quad \liminf \frac{\psi(\psi(n))}{n} = \frac{3}{2}.$$

Proof. Since $\psi(n) \geq \frac{3}{2}n$ for all even n , and $\psi(n) \geq n$ for all n , clearly $\psi(\psi(n)) \geq \frac{3}{2} \cdot n$ for all n , therefore it will be sufficient to find a sequence with limit $\frac{3}{2}$. By using deep theorems on primes in arithmetical progressions, it can be proved, as in Makowski-Schinzel [13] that

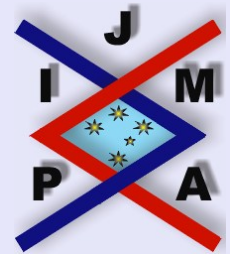
$$\limsup \frac{\varphi(a)}{a} = \liminf \frac{\sigma(a)}{a} = 1$$

as p tends to infinity, where $a = \frac{(p+1)}{2}$, and $p \equiv 1 \pmod{4}$.

Since $\frac{(p+1)}{2}$ is odd, we get

$$\sigma(p+1) = \sigma\left(2 \cdot \frac{(p+1)}{2}\right) = 3 \cdot \sigma\left(\frac{(p+1)}{2}\right),$$

implying that $\liminf \frac{(\sigma(p+1))}{p} = \frac{3}{2}$. Since $\psi(n) \leq \sigma(n)$, we can write that $\liminf \frac{(\psi(p+1))}{p} \leq \frac{3}{2}$. By $\frac{(\psi(p+1))}{p} > \frac{3}{2}$, this yields $\liminf \frac{(\psi(p+1))}{p} = \frac{3}{2}$, completing the proof of the theorem. \square



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 34 of 37

References

- [1] L. ALAOGU AND P. ERDŐS, A conjecture in elementary number theory, *Bull. Amer. Math. Soc.*, **50** (1944), 881–882.
- [2] K. ATANASSOV AND J. SÁNDOR, On some modifications of the φ and σ functions, *C.R. Acad. Bulg. Sci.*, **42** (1989), 55–58.
- [3] K. ATANASSOV, Proof of a conjecture of Sándor, unpublished manuscript (Sofia, 1992).
- [4] G.L. COHEN, On a conjecture of Makowski and Schinzel, *Colloq. Math.*, **74** (1997), 1–8.
- [5] P. ERDŐS, Personal communication to the author, 1990, *Math. Inst. Hungarian Acad. Sci.*
- [6] K. FORD, An explicit sieve bound and small values of $\sigma(\varphi(m))$, *Periodica Math. Hungar.*, **43**(1-2), (2001), 15–29.
- [7] A. GRYTCZUK, F. LUCA AND M. WÓJTOWICZ, On a conjecture of Makowski and Schinzel concerning the composition of arithmetic functions σ and φ , *Colloq. Math.*, **86** (2000), 31–36.
- [8] A. GRYTCZUK, F. LUCA AND M. WÓJTOWICZ, Some results on $\sigma(\varphi(n))$, *Indian J. Math.*, **43** (2001), 263–275.
- [9] R.K. GUY, *Unsolved Problems in Number Theory*, Third ed., Springer Verlag, 2004.



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



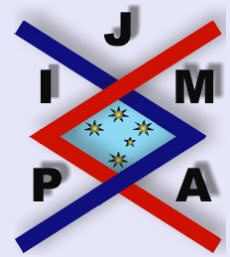
Go Back

Close

Quit

Page 35 of 37

- [10] G.H. HARDY AND E.M. WRIGHT, An Introduction to the Theory of Numbers, 4th ed., Oxford Univ. Press, 1960.
- [11] H.-J. KANOLD, Über Superperfect numbers, *Elem. Math.*, **24** (1969), 61–62.
- [12] F. LUCA AND C. POMERANCE, On some conjectures of Makowski-Schinzel and Erdős concerning the arithmetical functions phi and sigma, *Colloq. Math.*, **92** (2002), 111–130.
- [13] A. MAKOWSKI AND A. SCHINZEL, On the functions $\varphi(n)$ and $\sigma(n)$, *Colloq. Math.*, **13** (1964-65), 95–99.
- [14] C. POMERANCE, On the composition of arithmetic functions σ and φ , *Colloq. Math.*, **58** (1989), 11–15.
- [15] J. SÁNDOR, Notes on the inequality $\varphi(\psi(n)) < n$, 1988, unpublished manuscript.
- [16] J. SÁNDOR, On Dedekind’s arithmetical function, *Seminarul de Teoria Structurilor*, Univ. Timișoara, No. 51, 1988, 1–15.
- [17] J. SÁNDOR, Remarks on the functions $\varphi(n)$ and $\sigma(n)$, *Seminar on Math. Analysis*, Preprint Nr. 7, Babeș-Bolyai University, 1989, 7–12.
- [18] J. SÁNDOR, On the composition of some arithmetic functions, *Studia Babeș-Bolyai University, Math.*, **34** (1989), 7–14.
- [19] J. SÁNDOR, A note on the functions $\varphi_k(n)$ and $\sigma_k(n)$, *Studia Babeș-Bolyai University, Math.*, **35** (1990), 3–6.



On the Composition of Some
Arithmetic Functions, II

József Sándor

Title Page

Contents



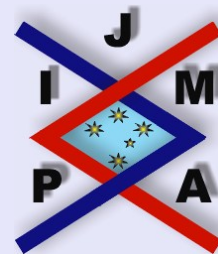
Go Back

Close

Quit

Page 36 of 37

- [20] J. SÁNDOR AND R. SIVARAMAKRISHNAN, The many facets of Euler's totient, III: An assortment of miscellaneous topics, *Nieuw Arch. Wiskunde*, **11** (1993), 97–130.
- [21] J. SÁNDOR, *Handbook of Number Theory, II.*, Springer Verlag, 2004.
- [22] V. SITARAMAIAH AND M.V. SUBBARAO, On the equation $\sigma^*(\sigma^*(n)) = 2n$, *Util. Math.*, **53** (1998), 101–124.
- [23] D. SURYANARAYANA, Superperfect numbers, *Elem. Math.*, **24** (1969), 16–17.
- [24] V. VITEK, Verifying $\varphi(\sigma(n)) < n$ for $n < 10^4$, Praha, 1990; particular letter dated August 14, 1990.
- [25] Ch. R. WALL, Topics related to the sum of unitary divisors of an integer, Ph.D. Thesis, Univ. of Tennessee, March, 1970.



On the Composition of Some Arithmetic Functions, II

József Sándor

Title Page

Contents



Go Back

Close

Quit

Page 37 of 37