

SIMULTANEOUS APPROXIMATION BY LUPAŞ MODIFIED OPERATORS WITH WEIGHTED FUNCTION OF SZASZ OPERATORS

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Abstract

In the present paper, we consider a new modification of the Lupaş operators with the weight function of Szasz operators and study simultaneous approximation. Here we obtain a Voronovskaja type asymptotic formula and an estimate of error in simultaneous approximation for these Lupaş-Szasz operators.

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1. Introduction

Lupaş proposed a family of linear positive operators mapping $C[0, \infty)$ into $C[0, \infty)$, the class of all bounded and continuous functions on $[0, \infty)$, namely,

$$(L_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right),$$

where $x \in [0, \infty)$.

Motivated by the integral modification of Bernstein polynomials by Derriennic [1], Sahai and Prasad [3] modified the operators L_n for functions integrable on $C[0, \infty)$ as

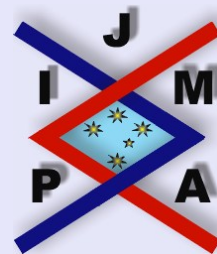
$$(M_n f)(x) = (n-1) \sum_{k=0}^{\infty} P_{n,k}(x) \int_0^{\infty} P_{n,k}(y) f(y) dy,$$

where

$$P_{n,k}(t) = \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}}.$$

Integral modification of Szasz-Mirakyan operators were studied by Gupta [2]. Now we consider another modification of Lupaş operators with the weight function of Szasz operators, which are defined as

$$(1.1) \quad (B_n f)(x) = n \sum_{k=0}^{\infty} P_{n,k}(x) \int_0^{\infty} S_{n,k}(y) f(y) dy$$



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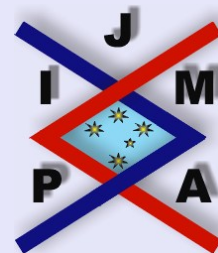
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where

$$P_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$$

and

$$S_{n,k}(y) = \frac{e^{-ny} (ny)^k}{k!}.$$



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2. Basic Results

The following lemmas are useful for proving the main results.

Lemma 2.1. Let $m \in \mathbb{N}^0$, $n \in \mathbb{N}$, if we define

$$T_{n,m}(x) = n \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y)(y-x)^m dy$$

then

(i) $T_{n,0}(x) = 1, T_{n,1}(x) = \frac{1+r(1+x)}{n}$, and

$$(2.1) \quad T_{n,2}(x) = \frac{rx(1+x) + 1 + [1 + r(1+x)]^2 + nx(2+x)}{n^2}$$

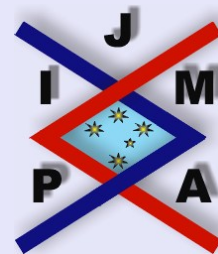
(ii) For all $x \geq 0$,

$$T_{n,m}(x) = O\left(\frac{1}{n^{\lfloor \frac{m+1}{2} \rfloor}}\right).$$

(iii)

$$nT_{n,m+1}(x) = x(1+x)T_{n,m+1}^{(1)}(x) + [m+1+r(1+x)]T_{n,m}(x) + mxT_{n,m-1}(x)$$

where $m \geq 2$.



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Proof. The value of $T_{n,0}(x), T_{n,1}(x)$ easily follows from the definition, we give the proof of (iii) as follows.

$$\begin{aligned} & x(x+1)T_{n,m}^{(1)}(x) \\ &= n \sum_{k=0}^{\infty} x(1+x)P_{n+r,k}^{(1)}(x) \int_0^{\infty} S_{n,k+r}(y)(y-x)^m dy \\ &\quad - mn \sum_{k=0}^{\infty} x(1+x)P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y)(y-x)^{m-1} dy. \end{aligned}$$

Now using the identities

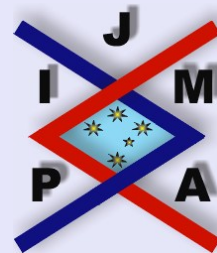
$$yS_{n,k}^{(1)}(y) = (k - ny)S_{n,k}(y),$$

and $x(1+x)P_{n,k}^{(1)}(x) = (k - nx)P_{n,k}(x)$, we get

$$\begin{aligned} & x(1+x)T_{n,m}^{(1)}(x) \\ &= n \sum_{k=0}^{\infty} [k - (n+r)x]P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y)(y-x)^m dy \\ &\quad - mx(1+x)T_{n,m-1}(x). \end{aligned}$$

Therefore,

$$\begin{aligned} & x(1+x)[T_{n,m}^{(1)}(x) + mT_{n,m-1}(x)] \\ &= n \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} [(k+r-ny) + n(y-x) - r(1+x)]S_{n,k+r}(y)(y-x)^m dy \end{aligned}$$



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$$\begin{aligned}
&= n \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} y S_{n,k+r}^{(1)}(y)(y-x)^m dy + nT_{n,m+1}(x) - r(1+x)T_{n,m}(x) \\
&= n \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} y S_{n,k+r}^{(1)}(y)(y-x)^{m+1} dy \\
&\quad + nx \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}^{(1)}(y)(y-x)^m dy \\
&\quad + nT_{n,m+1}(x) - r(1+x)T_{n,m}(x) \\
&= -(m+1)T_{n,m}(x) - mxT_{n,m-1}(x) + nT_{n,m+1}(x) - r(1+x)T_{n,m}(x)
\end{aligned}$$

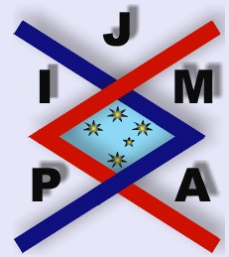
This leads to proof of (iii). □

Corollary 2.2. *Let α and δ be positive numbers, then for every $m \in \mathbb{N}$ and $x \in [0, \infty)$, there exists a positive constant $C_{m,x}$ depending on m and x such that*

$$n \sum_{k=0}^{\infty} P_{n,k}(x) \int_{|t-x| \geq \delta} S_{n,k}(t) e^{\alpha t} dt \leq C_{m,x} n^{-m}.$$

Lemma 2.3. *If f is differentiable r times ($r = 1, 2, 3, \dots$) on $[0, \infty)$, then we have*

$$(2.2) \quad (B_n^{(r)} f)(x) = \frac{(n+r-1)!}{n^{r-1}(n-1)!} \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y) f^{(r)}(y) dy.$$



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Proof. Applying Leibniz's theorem in (1.1) we have

$$\begin{aligned}
 & (B_n^{(r)} f)(x) \\
 &= n \sum_{i=0}^r \sum_{k=i}^{\infty} \binom{r}{i} \frac{(n+k+r-i-1)!}{(n-1)!k!} (-1)^{r-i} x^{k-i} (1+x)^{-n-k-r+i} \\
 & \quad \times \int_0^{\infty} S_{n,k}(y) f(y) dy \\
 &= n \sum_{k=0}^{\infty} \frac{(n+k+r-1)!}{(n-1)!k!} \cdot \frac{x^k}{(1+x)^{n+k+r}} \int_0^{\infty} \sum_{i=0}^r (-1)^{r-i} \binom{r}{i} S_{n,k+i}(y) f(y) dy \\
 &= n \frac{(n+r-1)!}{(n-1)!} \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} \sum_{i=0}^r (-1)^{r-i} \binom{r}{i} S_{n,k+i}(y) f(y) dy.
 \end{aligned}$$

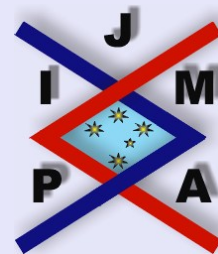
Again using Leibniz's theorem,

$$\begin{aligned}
 S_{n,k+r}^{(r)}(y) &= \sum_{i=0}^r \binom{r}{i} (-1)^i n^r \frac{e^{-ny} (ny)^{k+i}}{(k+i)!} \\
 &= n^r \sum_{i=0}^r (-1)^i \binom{r}{i} S_{n,k+i}(y).
 \end{aligned}$$

Hence

$$(B_n^{(r)} f)(x) = \frac{(n+r-1)!}{n^{r-1}(n-1)!} \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}^{(r)}(y) (-1)^r f(y) dy$$

and integrating by parts r times, we get the required result. \square



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3. Main Results

Theorem 3.1. Let f be integrable in $[0, \infty)$, admitting a derivative of order $(r + 2)$ at a point $x \in [0, \infty)$. Also suppose $f^{(r)}(x) = o(e^{\alpha x})$ as $x \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} n[(B_n^{(r)} f)(x) - f^{(r)}(x)] = [1 + r(1 + x)]f^{(r+1)}(x) + x(2 + x)f^{(r+2)}(x).$$

Proof. By Taylor's formula, we get

$$(3.1) \quad f^{(r)}(y) - f^{(r)}(x) = (y - x)f^{(r+1)}(x) + \frac{(y - x)^2}{2}f^{(r+2)}(x) + \frac{(y - x)^2}{2}\eta(y, x),$$

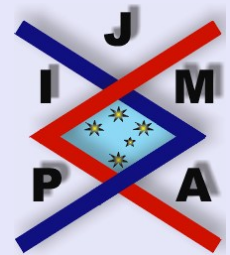
where

$$\eta(y, x) = \frac{f^{(r)}(y) - f^{(r)}(x) - (y - x)f^{(r+1)}(x) - \frac{(y-x)^2}{2}f^{(r+2)}(x)}{\frac{(y-x)^2}{2}} \quad \text{if } x \neq y$$

$$= 0 \quad \text{if } x = y.$$

Now, for arbitrary $\varepsilon > 0, A > 0 \exists a\delta > 0$ s. t.

$$(3.2) \quad |\eta(y, x)| \leq \varepsilon \quad \text{for } |y - x| < \delta, x \leq A.$$



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Using (2.2) in (3.1)

$$\begin{aligned}
 & \frac{n^r(n-1)!}{(n+r-1)!} (B_n^{(r)} f)(x) - f^{(r)}(x) \\
 &= n \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y) f^{(r)}(y) dy - f^{(r)}(x) \\
 &= n \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y) \{f^{(r)}(y) - f^{(r)}(x)\} dy \\
 &= T_{n,1} f^{(r+1)}(x) + T_{n,2} f^{(r+2)}(x) + E_{n,r}(x),
 \end{aligned}$$

where

$$E_{n,r}(x) = \frac{n}{2} \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y) (y-x)^2 \eta(y,x) dy.$$

In order to completely prove the theorem it is sufficient to show that

$$nE_{n,r}(x) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now

$$nE_{n,r}(x) = R_{n,r,1}(x) + R_{n,r,2}(x),$$

where

$$R_{n,r,1}(x) = \frac{n^2}{2} \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x|<\delta} S_{n,k+r}(y) (y-x)^2 \eta(x,y) dy$$



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and

$$R_{n,r,2}(x) = \frac{n^2}{2} \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x|>\delta} S_{n,k+r}(y)(y-x)^2 \eta(y,x) dy$$

By (3.2) and (2.1)

$$(3.3) \quad |R_{n,r,1}(x)| < \frac{n\varepsilon}{2} \left[n \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x|\leq\delta} S_{n,k+r}(y)(y-x)^2 dy \right] \\ \leq \varepsilon x(2+x)$$

as $n \rightarrow \infty$.

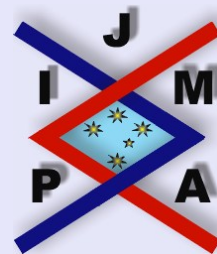
Finally we estimate $R_{n,r,2}(x)$. Using Corollary 2.2 we have

$$(3.4) \quad R_{n,r,2}(x) = \frac{n^2}{2} \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x|>\delta} S_{n,k+r}(y) e^{\alpha y} dy \\ = \frac{n}{2} M_{m,x} n^{-m} = 0$$

as $n \rightarrow \infty$. □

Theorem 3.2. Let $f \in C^{(r+1)}[0, a]$ and let $w(f^{(r+1)}; \cdot)$ be the modulus of continuity of $f^{(r+1)}$, then $r = 0, 1, 2, \dots$

$$\| (B_n^{(r)} f)(x) - f^{(r)}(x) \| \leq \frac{[1 + r(1+a)]}{n} \| f^{(r+1)}(x) \| \\ + \frac{1}{n^2} \left(\sqrt{T_{n,2}(a)} + \frac{T_{n,2}(a)}{2} \right) w(f^{(r+1)}; n^{-2})$$



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where $\|\cdot\|$ is the sup norm $[0, a]$.

Proof. We have by Taylor's expansion

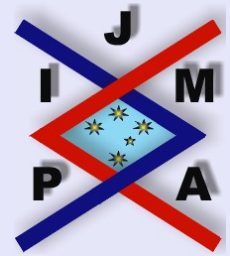
$$\begin{aligned}
 f^{(r)}(y) - f^{(r)}(x) &= (y - x)f^{(r+1)}(x) + \int_x^y [f^{(r+1)}(t) - f^{(r+1)}(x)]dt \\
 &\quad \times \frac{n^r(n-1)!}{(n+r-1)!} (B_n^{(r)}f)(x) - f^{(r)}(x) \\
 &= n \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y) \{f^{(r)}(y) - f^{(r)}(x)\} dy \\
 &= n \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y) \left((y-x)f^{(r+1)}(x) \right. \\
 &\quad \left. + \int_x^y [f^{(r+1)}(t) - f^{(r+1)}(x)]dt \right) dy.
 \end{aligned}$$

Also

$$|f^{(r+1)}(t) - f^{(r+1)}(x)| \leq \left(1 + \frac{|t-x|}{\delta}\right) w(f^{(r+1)}; \delta)$$

Hence

$$\begin{aligned}
 &\left| \frac{n^r(n-1)!}{(n+r-1)!} (B_n^{(r)}f)(x) - f^{(r)}(x) \right| \\
 &\leq |T_{n,1}| \cdot |f^{(r+1)}(x)| + \left(\left| \sqrt{T_{n,2}} \right| + \frac{|T_{n,2}|}{2\delta} \right) \cdot w(f^{(r+1)}; \delta).
 \end{aligned}$$



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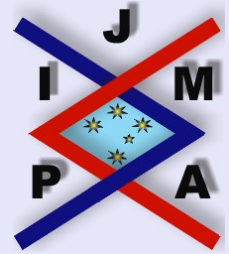
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By Schwarz's inequality. Choosing $\delta = \frac{1}{n^2}$ and using (i) and (2.1) we obtain the required result. \square



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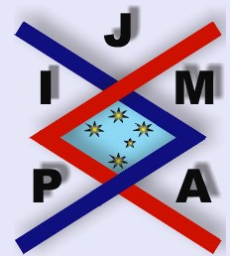
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