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KANTOROVICH-STANCU TYPE OPERATORS

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ABSTRACT. Considering two given real parameters α, β which satisfy the condition $0 \leq \alpha \leq \beta$, D.D. Stancu ([11]) constructed and studied the linear positive operators $P_m^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$, defined for any $f \in C([0,1])$ and any $m \in \mathbb{N}$ by

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{mk}(x) f\left(\frac{k+\alpha}{m+\beta}\right).$$

In this paper, we are dealing with the Kantorovich form of the above operators. We construct the linear positive operators $K_m^{(\alpha,\beta)} : L_1([0,1]) \to C([0,1])$, defined for any $f \in L_1([0,1])$ and any $m \in \mathbb{N}$ by

$$\left(K_m^{(\alpha,\beta)} f\right)(x) = (m+\beta+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(s) ds$$

and we study some approximation properties of the sequence $\left\{K_m^{(\alpha,\beta)}\right\}_{m\in\mathbb{N}}$.

Key words and phrases: Linear positive operators, Bernstein operator, Kantorovich operator, Stancu operator, First order modulus of smoothness, Shisha-Mond theorem.

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1. PRELIMINARIES

Starting with two given real parameters α, β satisfying the conditions $0 \leq \alpha \leq \beta$ in 1968, D.D. Stancu (see [11]) constructed and studied the linear positive operators $P_m^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$ defined for any $f \in C([0,1])$ and any $m \in \mathbb{N}$ by

$$\left(P_m^{(\alpha,\beta)} f\right) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+p}\right), \tag{1.1}$$

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where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$ are the fundamental Bernstein polynomials ([5]).

The operators (1.1) are known in mathematical literature as "the operators of D.D. Stancu" (see ([2])).

Note that for $\alpha = \beta = 0$, the operator $P_m^{(0,0)}$ is the classical Bernstein operator B_m ([5]).

In 1930, L.V. Kantorovich constructed and studied the linear positive operators $K_m : L_1([0,1]) \to C([0,1])$ defined for any $f \in L_1([0,1])$ and any non-negative integer m by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(s)ds. \tag{1.2}$$

The operators (1.2) are known as the Kantorovich operators. These operators are obtained from the classical Bernstein operators (1.1), replacing there the value $f(k/m)$ of the approximated function by the integral of f in a neighborhood of k/m.

Following the ideas of L.V. Kantorovich ([7]), let us consider the operators $K_m^{(\alpha,\beta)} : L_1([0,1]) \to C([0,1])$, defined for any $f \in C([0,1])$ and any $m \in \mathbb{N}$ by

$$\left(K_m^{(\alpha,\beta)} f\right)(x) = (m+\beta+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(s)ds \tag{1.3}$$

obtained from the Stancu type operators (1.1).

Section 2 provides some interesting approximation properties of operators (1.3), called "Kantorovich-Stancu type operators" because they are obtained starting from the Stancu type operators (1.1) following Kantorovich's ideas (see also G.G. Lorentz [9]).

A convergence theorem for the sequence $\left\{K_m^{(\alpha,\beta)} f\right\}_{m\in\mathbb{N}}$ is proved and the rate of convergence under some assumptions on the approximated function f is evaluated.

2. MAIN RESULTS

Lemma 2.1. *The Kantorovich-Stancu type operators (1.3) are linear and positive.*

Proof. The assertion follows from definition (1.3). □

In what follows we will denote by $e_k(s) = s^k, k \in \mathbb{N}$, the test functions.

Lemma 2.2. *The operators (1.3) verify*

$$\left(K_m^{(\alpha,\beta)} e_0\right)(x) = 1, \tag{2.1}$$

$$\left(K_m^{(\alpha,\beta)} e_1\right)(x) = \frac{m}{m+\beta+1}x + \frac{\alpha}{m+\beta+1} + \frac{m+\beta}{2(m+\beta+1)^2}, \tag{2.2}$$

$$\begin{aligned}\left(K_m^{(\alpha,\beta)} e_2\right)(x) &= \frac{1}{(m+\beta+1)^2}\left\{m^2x^2 + mx(1-x) + \frac{2\alpha m^2}{m+\beta} + \frac{\alpha^2(3m+\beta)}{m+\beta}\right\} \\ &\quad + \frac{1}{(m+\beta+1)^2}\{mk+\alpha\} + \frac{1}{3(m+\beta+1)^2}\end{aligned} \tag{2.3}$$

for any $x \in [0,1]$.

Proof. It is well known (see [11]) that the Stancu type operators (1.1) satisfy

$$\left(P_m^{(\alpha,\beta)} e_0\right)(x) = 1$$

$$\left(P_m^{(\alpha,\beta)} e_1\right)(x) = \frac{m}{m+\beta}x + \frac{\alpha}{m+\beta}$$

$$\left(P_m^{(\alpha,\beta)} e_2\right)(x) = \frac{1}{(m+\beta)^2}\left\{m^2x^2 + mx(1-x) + 2\frac{\alpha m^2}{m+\beta}x + \frac{3\alpha^2 m}{m+\beta}\right\}$$

□

Next we apply the definition (2.1).

Lemma 2.3. *The operators (1.3) satisfy*

$$K_m^{(\alpha,\beta)}((e_1-x)^2;x) = \frac{(\beta+1)^2}{(m+\beta+1)^2}\,x^2 + \frac{m}{(m+\beta+1)^2}\,x(1-x) + \frac{m}{(m+\beta+1)^2(m+\beta)}\{m+2\alpha(m-\beta-1)\}x + \frac{3\alpha^2(3m+\beta)+(m+\beta)(1-3m-3\beta)}{3(m+\beta)(m+\beta+1)^2} \tag{2.4}$$

for any $x \in [0,1]$.

Proof. From the linearity of $K_m^{(\alpha,\beta)}$, we get

$$K_m^{(\alpha,\beta)}((e_1-x)^2;x) = \left(K_m^{(\alpha,\beta)}e_2\right)(x) - 2x\,K_m^{(\alpha,\beta)}(e_1;x) + x^2\left(K_m^{(\alpha,\beta)}e_0\right)(x)$$

□

Next, we apply Lemma 2.2.

Theorem 2.4. *The sequence* $\left\{K_m^{(\alpha,\beta)}f\right\}_{m\in N}$ *converges to* f*, uniformly on* $[0,1]$*, for any* $f \in L_1([0,1])$.

Proof. Using Lemma 2.3, we get

$$\lim_{m\to\infty} K_m^{(\alpha,\beta)}((e_1-x)^2;x) = 0$$

uniformly on $[0,1]$. We can then apply the well known Bohman-Korovkin Theorem (see [6] and [8]) to obtain the desired result. □

Next, we deal with the rate of convergence for the sequence $\left\{K_m^{(\alpha,\beta)}f\right\}_{m\in\mathbb{N}}$, under some assumptions on the approximated function f. In this sense, the first order modulus of smoothness will be used.

Let us recall that if $I \subseteq \mathbb{R}$ is an interval of the real axis and f is a real valued function defined on I and bounded on this interval, the first order modulus of smoothness for f is the function $\omega_1 : [0,+\infty) \to \mathbb{R}$, defined for any $\delta \geq 0$ by

$$\omega_1(f;\delta) = \sup\left\{|f(x') - f(x'')| : x', x'' \in I,\ |x'-x''| \leq \delta\right\}. \tag{2.5}$$

For more details, see for example [1].

Theorem 2.5. *For any* $f \in L_1([0,1])$*, any* $\alpha,\beta \geq 0$ *satisfying the condition* $\alpha \leq \beta$ *and each* $x \in [0,1]$ *the Kantorovich-Stancu type operators (1.3) satisfy*

$$\left|\left(K_m^{(\alpha,\beta)}f\right)(x) - f(x)\right| \leq 2\omega_1\left(f;\sqrt{\delta_m^{(\alpha,\beta)}(x)}\right), \tag{2.6}$$

where

$$\delta_m^{(\alpha,\beta)}(x) = K_m^{(\alpha,\beta)}((e_1-x)^2;x) \tag{2.7}$$

Proof. From Lemma 2.2 follows

$$\left|\left(K_m^{(\alpha,\beta)}f\right)(x)-f(x)\right|\le(m+\beta+1)\sum_{k=0}^{m+p}\int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}}|f(s)-f(x)|ds \tag{2.8}$$

On the other hand

$$|f(s)-f(x)|\le\omega_1(f;|s-x|)\le(1+\delta^{-2}(s-x)^2)\omega_1(f;\delta).$$

For $|s-x|<\delta$, the lost increase is clear. For $|s-x|\ge\delta$, we use the following properties

$$\omega_1(f;\lambda\delta)\le(1+\lambda)\omega_1(f;\delta)\le(1+\lambda^2)\omega_1(f;\delta),$$

where we choose $\lambda=\delta^{-1}\cdot|s-x|$.
This way, after some elementary transformation, (2.8) implies

$$\left|\left(K_m^{(\alpha,\beta)}f\right)(x)-f(x)\right|\le\left\{\left(K_m^{(\alpha,\beta)}e_0\right)(x)+\delta^{-2}K_m^{(\alpha,\beta)}((e_1-x)^2;x\right\}\omega_1(f;\delta) \tag{2.9}$$

for any $\delta>0$ and each $x\in[0,1]$.
Using next Lemma 2.2 and Lemma 2.3, from (2.9) one obtains

$$\left|\left(K_m^{(\alpha,\beta)}f\right)(x)-f(x)\right|\le\left(1+\delta^{-2}\delta_m^{(\alpha,\beta)}(x)\right)\omega_1(f;\delta) \tag{2.10}$$

for any $\delta\ge0$ and each $x\in[0,1]$.
Taking into account Lemma 2.1, it follows that $\delta_m^{(\alpha,\beta)}(x)\ge0$ for each $x\in[0,1]$. Consequently, we can take $\delta:=\delta_m^{(\alpha,\beta)}(x)$ in (2.9), arriving at the desired result. □

Theorem 2.6. *For any $f\in L_1([0,1])$ and any $x\in[0,1]$ the following*

$$\left|\left(K_m^{(\alpha,\beta)}f\right)(x)-f(x)\right|\le2\omega_1\left(f;\sqrt{\delta_m^{(\alpha,\beta)}},1\right) \tag{2.11}$$

holds, where

$$\delta_{m,1}^{(\alpha,\beta)}=\frac{(\beta+1)^2}{(m+\beta+1)^2}+\frac{m^2(2\alpha+1)}{(m+\beta)(m+\beta+1)^2}+\frac{m}{4(m+\beta+1)^2}+\frac{3\alpha^2(3m+\beta)+(m+\beta)(1-3m-3\beta)}{3(m+\beta)(m+\beta+1)^2}. \tag{2.12}$$

Proof. For any $x\in[0,1]$, the inequality

$$K_m^{(\alpha,\beta)}((e_1-x)^2;x)\le\delta_{m,1}^{(\alpha,\beta)}$$

holds. Consequently, applying Theorem 2.5 we get (2.11). □

Remark 2.7. Theorem 2.5 gives us the order of local approximation (in each point $x\in[0,1]$), while Theorem 2.6 contains an evaluation for the global order of approximation (in any point $x\in[0,1]$).
Because the maximum of $\delta_m^{(\alpha,\beta)}(x)$ from (2.6) depends on the relations between α and β, it follows that it can be refined further.
Taking into account the inclusion $C([0,1])\subset L_1([0,1])$, as consequences of Theorem 2.5 and Theorem 2.6, follows the following two results.

Corollary 2.8. *For any $f\in C([0,1])$, any $\alpha,\beta\ge0$ satisfying the condition $\alpha\le\beta$ and each $x\in[0,1]$, the inequality (2.6) holds.*

Corollary 2.9. *For any $f\in C([0,1])$, any $\alpha,\beta\ge0$ satisfying the condition $\alpha\le\beta$ and any $x\in[0,1]$, the inequality (2.11) holds.*

Further, we estimate the rate of convergence for smooth functions.

Theorem 2.10. *For any* $f \in C^1([0,1])$ *and each* $x \in [0,1]$ *the operators (1.3) verify*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq |f'(x)| \cdot \left|\frac{m+\beta}{2(m+\beta+1)^2} - \frac{\beta+1}{(m+\beta+1)^2}\,x\right| + 2\sqrt{2\delta_m^{(\alpha,\beta)}(x)}\,\omega_1\left(f'; \sqrt{\delta_m^{(\alpha,\beta)}(x)}\right), \tag{2.13}$$

where $\delta_m^{(\alpha,\beta)}(x)$ *is given in (2.7).*

Proof. Applying a well known result due to O. Shisha and B. Mond (see [10]), it follows that

$$\begin{aligned}\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| &\leq |f(x)| \cdot \left|\left(K_m^{(\alpha,\beta)} e_0\right)(x) - 1\right| \\ &+ |f'(x)| \cdot \left|\left(K_m^{(\alpha,\beta)} e_1\right)(x) - x\left(K_m^{(\alpha,\beta)} e_0\right)(x)\right| + \sqrt{K_m^{(\alpha,\beta)}((e_1-x)^2;x)} \\ &\times \left\{\sqrt{\left(K_m^{(\alpha,\beta)} e_0\right)(x)} + \delta^{-1}\sqrt{K_m^{(\alpha,\beta)}((e_1-x)^2;x)}\right\}\omega_1(f';\delta).\end{aligned} \tag{2.14}$$

From (2.14), using Lemma 2.2 and Lemma 2.3, we get

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq |f'(x)| \cdot \left|\frac{m+\beta}{(m+\beta+1)^2} - \frac{\beta+1}{(m+\beta+1)^2\,x}\right| + \sqrt{\delta_m^{(\alpha,\beta)}(x)}\left\{1 + \delta^{-1}\sqrt{\delta_m^{(\alpha,\beta)}(x)}\right\}\omega_1(f';\delta). \tag{2.15}$$

Choosing $\delta = \sqrt{\delta_m^{(\alpha,\beta)}(x)}$ in (2.15), we arrive at the desired result. □

Theorem 2.11. *For any* $f \in C^1([0,1])$ *and any* $x \in [0,1]$ *the operators (1.3) verify*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq \frac{m+\beta}{(m+\beta+1)^2} M_1 + 2\sqrt{\delta}\,\omega_1\left(f'; \sqrt{\delta}\right), \tag{2.16}$$

where

$$M_1 = \max_{x\in[0,1]} |f'(x)|, \quad \delta = \max_{x\in[0,1]} \delta_m^{(\alpha,\beta)}(x).$$

Proof. The assertion follows from Theorem 2.10. □

Remark 2.12. Because δ depends on the relation between α and β, (2.16) can be further refined, following the ideas of D.D. Stancu [11, 12].

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