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VOLTERRA INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS IN TWO VARIABLES

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ABSTRACT. The aim of this paper is to study the existence, uniqueness and other properties of solutions of certain Volterra integral and integrodifferential equations in two variables. The tools employed in the analysis are based on the applications of the Banach fixed point theorem coupled with Bielecki type norm and certain integral inequalities with explicit estimates.

Key words and phrases: Volterra integral and integrodifferential equations, Banach fixed point theorem, Bielecki type norm, integral inequalities, existence and uniqueness, estimates on the solutions, approximate solutions.

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1. INTRODUCTION

Let $\mathbb{R}^n$ denote the real n-dimensional Euclidean space with appropriate norm denoted by $|\cdot|$. We denote by $I_a = [a, \infty)$, $\mathbb{R}_+ = [0, \infty)$, the given subsets of $\mathbb{R}$, the set of real numbers, $E = \{(x, y, m, n) : a \le m \le x < \infty,\ b \le n \le y < \infty\}$ and $\Delta = I_a \times I_b$. For $x, y \in \mathbb{R}$, the partial derivatives of a function $z(x, y)$ with respect to x, y and xy are denoted by $D_1 z(x, y), D_2 z(x, y)$ and $D_2 D_1 z(x, y) = D_1 D_2 z(x, y)$. Consider the Volterra integral and integrodifferential equations of the forms:

$$u(x, y) = f(x, y, u(x, y), (Ku)(x, y)), \tag{1.1}$$

and

$$D_2 D_1 u(x, y) = f(x, y, u(x, y), (Ku)(x, y)), \tag{1.2}$$

with the given initial boundary conditions

$$u(x, 0) = \sigma(x), \qquad u(0, y) = \tau(y), \qquad u(0, 0) = 0, \tag{1.3}$$

for $(x, y) \in \Delta$, where

$$(Ku)(x, y) = \int_a^x \int_b^y k(x, y, m, n, u(m, n))\, dndm, \tag{1.4}$$

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$k \in C(E\times\mathbb{R}^n,\mathbb{R}^n)$, $f \in C(\Delta\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$, $\sigma \in C(I_a,\mathbb{R}^n)$, $\tau \in C(I_b,\mathbb{R}^n)$. By a solution of equation (1.1) (or equations (1.2) – (1.3)) we mean a function $u \in C(\Delta,\mathbb{R}^n)$ which satisfies the equation (1.1) (or equations (1.2) – (1.3)).

In general, existence theorems for equations of the above forms are proved by the use of one of the three fundamental methods (see [1], [3] – [9], [12] – [16]): the method of successive approximations, the method based on the theory of nonexpansive and monotone mappings and on the theory exploiting the compactness of the operator often by the use of the well known fixed point theorems. The aim of the present paper is to study the existence, uniqueness and other properties of solutions of equations (1.1) and (1.2) – (1.3) under various assumptions on the functions involved therein. The main tools employed in the analysis are based on applications of the well known Banach fixed point theorem (see [3] – [5], [8]) coupled with a Bielecki type norm (see [2]) and the integral inequalities with explicit estimates given in [11]. In fact, our approach here to the study of equations (1.1) and (1.2) – (1.3) leads us to obtain new conditions on the qualitative properties of their solutions and present some useful basic results for future reference, by using elementary analysis.

2. EXISTENCE AND UNIQUENESS

We first construct the appropriate metric space for our analysis. Let $\alpha > 0, \beta > 0$ be constants and consider the space of continuous functions $C(\Delta,\mathbb{R}^n)$ such that $\sup_{(x,y)\in\Delta} \frac{|u(x,y)|}{e^{\alpha(x-a)+\beta(y-b)}} < \infty$ for $u(x,y) \in C(\Delta,\mathbb{R}^n)$ and denote this special space by $C_{\alpha,\beta}(\Delta,\mathbb{R}^n)$ with suitable metric

$$d_{\alpha,\beta}^{\infty}(u,v) = \sup_{(x,y)\in\Delta} \frac{|u(x,y) - v(x,y)|}{e^{\alpha(x-a)+\beta(y-b)}},$$

and a norm defined by

$$|u|_{\alpha,\beta}^{\infty} = \sup_{(x,y)\in\Delta} \frac{|u(x,y)|}{e^{\alpha(x-a)+\beta(y-b)}}.$$

The above definitions of $d_{\alpha,\beta}^{\infty}$ and $|\cdot|_{\alpha,\beta}^{\infty}$ are variants of Bielecki's metric and norm (see [2, 5]).

The following variant of the lemma proved in [5] holds.

Lemma 2.1. *If $\alpha > 0, \beta > 0$ are constants, then $\left(C_{\alpha,\beta}(\Delta,\mathbb{R}^n), |\cdot|_{\alpha,\beta}^{\infty}\right)$ is a Banach space.*

Our main results concerning the existence and uniqueness of solutions of equations (1.1) and (1.2) – (1.3) are given in the following theorems.

Theorem 2.2. *Let $\alpha > 0, \beta > 0, L > 0, M \geq 0, \gamma > 1$ be constants with $\alpha\beta = L\gamma$. Suppose that the functions f, k in equation (1.1) satisfy the conditions*

$$|f(x,y,u,v) - f(x,y,\bar{u},\bar{v})| \leq M\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{2.1}$$

$$|k(x,y,m,n,u) - k(x,y,m,n,v)| \leq L|u-v|, \tag{2.2}$$

and

$$d_1 = \sup_{(x,y)\in\Delta} \frac{1}{e^{\alpha(x-a)+\beta(y-b)}} |f(x,y,0,(K0)(x,y))| < \infty. \tag{2.3}$$

If $M\left(1+\frac{1}{\gamma}\right) < 1$, then the equation (1.1) has a unique solution $u \in C_{\alpha,\beta}(\Delta,\mathbb{R}^n)$.

Proof. Let $u \in C_{\alpha,\beta}(\Delta,\mathbb{R}^n)$ and define the operator T by

$$(Tu)(x,y) = f(x,y,u(x,y),(Ku)(x,y)). \tag{2.4}$$

Now we shall show that T maps $C_{\alpha,\beta}(\Delta,\mathbb{R}^n)$ into itself. From (2.4) and using the hypotheses, we have

$$\begin{aligned}
|Tu|_{\alpha,\beta}^{\infty} &\le \sup_{(x,y)\in\Delta} \frac{1}{e^{\alpha(x-a)+\beta(y-b)}} |f(x,y,0,(K0)(x,y))| \\
&\quad + \sup_{(x,y)\in\Delta} \frac{1}{e^{\alpha(x-a)+\beta(y-b)}} |f(x,y,u(x,y),(Ku)(x,y)) - f(x,y,0,(K0)(x,y))| \\
&\le d_1 + \sup_{(x,y)\in\Delta} \frac{1}{e^{\alpha(x-a)+\beta(y-b)}} M \left[|u(x,y)| + \int_a^x \int_b^y L|u(m,n)|\,dndm\right] \\
&= d_1 + M \left[\sup_{(x,y)\in\Delta} \frac{|u(x,y)|}{e^{\alpha(x-a)+\beta(y-b)}}\right. \\
&\quad \left. + \sup_{(x,y)\in\Delta} \frac{1}{e^{\alpha(x-a)+\beta(y-b)}} \int_a^x \int_b^y L e^{\alpha(m-a)+\beta(n-b)} \frac{|u(m,n)|}{e^{\alpha(m-a)+\beta(n-b)}}\,dndm\right] \\
&\le d_1 + M|u|_{\alpha,\beta}^{\infty} \left[1 + L \sup_{(x,y)\in\Delta} \frac{1}{e^{\alpha(x-a)+\beta(y-b)}}\right. \\
&\quad \left. \times \int_a^x \int_b^y e^{\alpha(m-a)+\beta(n-b)}\,dndm\right] \\
&\le d_1 + M|u|_{\alpha,\beta}^{\infty} \left[1 + \frac{L}{\alpha\beta}\right] \\
&= d_1 + M|u|_{\alpha,\beta}^{\infty} \left[1 + \frac{1}{\gamma}\right] < \infty.
\end{aligned}$$

This proves that the operator T maps $C_{\alpha,\beta}(\Delta,\mathbb{R}^n)$ into itself.

Now we verify that the operator T is a contraction map. Let $u, v \in C_{\alpha,\beta}(\Delta,\mathbb{R}^n)$. From (2.4) and using the hypotheses, we have

$$\begin{aligned}
&d_{\alpha,\beta}^{\infty}(Tu,Tv) \\
&= \sup_{(x,y)\in\Delta} \frac{1}{e^{\alpha(x-a)+\beta(y-b)}} |f(x,y,u(x,y),(Ku)(x,y)) \\
&\qquad - f(x,y,v(x,y),(Kv)(x,y))| \\
&\le \sup_{(x,y)\in\Delta} \frac{1}{e^{\alpha(x-a)+\beta(y-b)}} M \left[|u(x,y) - v(x,y)| + \int_a^x \int_b^y L|u(m,n) - v(m,n)|\,dndm\right] \\
&= M \left[\sup_{(x,y)\in\Delta} \frac{|u(x,y) - v(x,y)|}{e^{\alpha(x-a)+\beta(y-b)}}\right. \\
&\qquad \left. + L \sup_{(x,y)\in\Delta} \frac{1}{e^{\alpha(x-a)+\beta(y-b)}} \int_a^x \int_b^y e^{\alpha(m-a)+\beta(n-b)} \frac{|u(m,n) - v(m,n)|}{e^{\alpha(m-a)+\beta(n-b)}}\,dndm\right]
\end{aligned}$$

$$\leq Md_{\alpha,\beta}^{\infty}(u,v)\left[1+L\sup_{(x,y)\in\Delta}\frac{1}{e^{\alpha(x-a)+\beta(y-b)}}\times\int_a^x\int_b^y e^{\alpha(m-a)+\beta(n-b)}dndm\right]$$
$$=Md_{\alpha,\beta}^{\infty}(u,v)\left[1+\frac{L}{\alpha\beta}\right]$$
$$=M\left(1+\frac{1}{\gamma}\right)d_{\alpha,\beta}^{\infty}(u,v).$$

Since $M\left(1+\frac{1}{\gamma}\right)<1$, it follows from the Banach fixed point theorem (see [3] – [5], [8]) that T has a unique fixed point in $C_{\alpha,\beta}(\Delta,\mathbb{R}^n)$. The fixed point of T is however a solution of equation (1.1). The proof is complete. ☐

Theorem 2.3. *Let $M, L, \alpha, \beta, \gamma$ be as in Theorem 2.2. Suppose that the functions f, k in equation (1.2) satisfy the conditions (2.1), (2.2) and*

$$d_2=\sup_{(x,y)\in\Delta}\frac{1}{e^{\alpha(x-a)+\beta(y-b)}}\left|\sigma(x)+\tau(y)+\int_a^x\int_b^y f(s,t,0,(K0)(s,t))\,dtds\right|<\infty, \tag{2.5}$$

where σ, τ are as in (1.3). If $\frac{M}{\alpha\beta}\left(1+\frac{1}{\gamma}\right)<1$, then the equations (1.2) – (1.3) have a unique solution $u\in C_{\alpha,\beta}(\Delta,\mathbb{R}^n)$.

Proof. Let $u\in C_{\alpha,\beta}(\Delta,\mathbb{R}^n)$ and define the operator S by

$$(Su)(x,y)=\sigma(x)+\tau(y)+\int_a^x\int_b^y f(s,t,u(s,t),(Ku)(s,t))\,dtds,$$

for $(x,y)\in\Delta$. The proof that S maps $C_{\alpha,\beta}(\Delta,\mathbb{R}^n)$ into itself and is a contraction map can be completed by closely looking at the proof of Theorem 2.2 given above with suitable modifications. Here, we leave the details to the reader. ☐

Remark 1. We note that the problems of existence and uniqueness of solutions of special forms of equations (1.1) and (1.2) – (1.3) have been studied under a variety of hypotheses in [16]. In [7] the authors have obtained existence and uniqueness of solutions to general integral-functional equations involving n variables by using the comparative method (see also [1], [6], [12] – [15]). The approach here in the treatment of existence and uniqueness problems for equations (1.1) and (1.2) – (1.3) is fundamental and our results do not seem to be covered by the existing theorems. Furthermore, the ideas used here can be extended to n dimensional versions of equations (1.1) and (1.2) – (1.3).

3. Estimates on the Solutions

In this section we obtain estimates on the solutions of equations (1.1) and (1.2) – (1.3) under some suitable assumptions on the functions involved therein.

We need the following versions of the inequalities given in [11, Remark 2.2.1, p. 66 and p. 86]. For similar results, see [10].

Lemma 3.1. *Let $u\in C(\Delta,\mathbb{R}_+)$, r, D_1r, D_2r, $D_2D_1r\in C(E,\mathbb{R}_+)$ and $c\geq 0$ be a constant. If*

$$u(x,y)\leq c+\int_a^x\int_b^y r(x,y,\xi,\eta)\,u(\xi,\eta)\,d\eta d\xi, \tag{3.1}$$

for $(x,y)\in\Delta$, then

$$u(x,y)\leq c\exp\left(\int_a^x\int_b^y A(s,t)\,dtds\right), \tag{3.2}$$

for $(x,y) \in \Delta$, *where*

$$A(x,y) = r(x,y,x,y) + \int_a^x D_1 r(x,y,\xi,y)\, d\xi + \int_b^y D_2 r(x,y,x,\eta)\, d\eta + \int_a^x \int_b^y D_2 D_1 r(x,y,\xi,\eta)\, d\eta d\xi. \tag{3.3}$$

Lemma 3.2. *Let* $u,e,p \in C(\Delta,\mathbb{R}_+)$ *and* $r,\ D_1 r,\ D_2 r,\ D_2 D_1 r \in C(E,\mathbb{R}_+)$. *If* $e(x,y)$ *is nondecreasing in each variable* $(x,y) \in \Delta$ *and*

$$u(x,y) \le e(x,y) + \int_a^x \int_b^y p(s,t) \times \left[u(s,t) + \int_a^s \int_b^t r(s,t,m,n)\, u(m,n)\, dndm \right] dtds, \tag{3.4}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \le e(x,y) \left[1 + \int_a^x \int_b^y p(s,t) \times \exp\left(\int_a^s \int_b^t [p(m,n) + A(m,n)]\, dndm \right) dtds \right], \tag{3.5}$$

for $(x,y) \in \Delta$, *where* $A(x,y)$ *is defined by (3.3).*

First, we shall give the following theorem concerning an estimate on the solution of equation (1.1).

Theorem 3.3. *Suppose that the functions* f,k *in equation (1.1) satisfy the conditions*

$$|f(x,y,u,v) - f(x,y,\bar{u},\bar{v})| \le N[|u-\bar{u}| + |v-\bar{v}|], \tag{3.6}$$

$$|k(x,y,m,n,u) - k(x,y,m,n,v)| \le r(x,y,m,n)\, |u-v|, \tag{3.7}$$

where $0 \le N < 1$ *is a constant and* r, $D_1 r$, $D_2 r$, $D_2 D_1 r \in C(E,\mathbb{R}_+)$. *Let*

$$c_1 = \sup_{(x,y)\in\Delta} |f(x,y,0,(K0)(x,y))| < \infty. \tag{3.8}$$

If $u(x,y)$, $(x,y) \in \Delta$ *is any solution of equation (1.1), then*

$$|u(x,y)| \le \left(\frac{c_1}{1-N} \right) \exp\left(\int_a^x \int_b^y B(s,t)\, dtds \right), \tag{3.9}$$

for $(x,y) \in \Delta$, *where*

$$B(x,y) = \frac{N}{1-N} A(x,y), \tag{3.10}$$

in which $A(x,y)$ *is defined by (3.3).*

Proof. By using the fact that $u(x,y)$ is a solution of equation (1.1) and the hypotheses, we have

$$\begin{aligned} |u(x,y)| &\le |f(x,y,u(x,y),(Ku)(x,y)) - f(x,y,0,(K0)(x,y))| \\ &\quad + |f(x,y,0,(K0)(x,y))| \\ &\le c_1 + N\left[|u(x,y)| + \int_a^x \int_b^y r(x,y,m,n)\, |u(m,n)|\, dndm \right]. \end{aligned} \tag{3.11}$$

From (3.11) and using the assumption $0 \leq N < 1$, we observe that

$$|u(x,y)| \leq \left(\frac{c_1}{1-N}\right) + \frac{N}{1-N} \int_a^x \int_b^y r(x,y,m,n) |u(m,n)| \, dndm. \tag{3.12}$$

Now a suitable application of Lemma 3.1 to (3.12) yields (3.9). □

Next, we shall obtain an estimate on the solution of equations (1.2) – (1.3).

Theorem 3.4. *Suppose that the function f in equation (1.2) satisfies the condition*

$$|f(x,y,u,v) - f(x,y,\bar{u},\bar{v})| \leq p(x,y) [|u-\bar{u}| + |v-\bar{v}|], \tag{3.13}$$

where $p \in C(\Delta,\mathbb{R}_+)$ and the function k in equation (1.2) satisfies the condition (3.7). Let

$$c_2 = \sup_{(x,y)\in\Delta} \left| \sigma(x) + \tau(y) + \int_a^x \int_b^y f(s,t,0,(K0)(s,t)) \, dtds \right| < \infty. \tag{3.14}$$

If $u(x,y)$, $(x,y) \in \Delta$ is any solution of equations (1.2) – (1.3), then

$$|u(x,y)| \leq c_2 \left[1 + \int_a^x \int_b^y p(s,t) \exp\left(\int_a^s \int_b^t [p(m,n) + A(m,n)] \, dndm\right) dtds\right], \tag{3.15}$$

for $(x,y) \in \Delta$, where $A(x,y)$ is defined by (3.3).

Proof. Using the fact that $u(x,y)$ is a solution of equations (1.2) – (1.3) and the hypotheses, we have

$$\begin{aligned} &|u(x,y)| \\ &\leq \left| \sigma(x) + \tau(y) + \int_a^x \int_b^y f(s,t,0,(K0)(s,t)) \, dtds \right| \\ &\quad + \int_a^x \int_b^y |f(s,t,u(s,t),(Ku)(s,t)) - f(s,t,0,(K0)(s,t))| \, dtds \\ &\leq c_2 + \int_a^x \int_b^y p(s,t) \left[|u(s,t)| + \int_a^s \int_b^t r(s,t,m,n) |u(m,n)| \, dndm\right] dtds. \end{aligned} \tag{3.16}$$

Now a suitable application of Lemma 3.2 to (3.16) yields (3.15). □

Remark 2. We note that the results in Theorems 3.3 and 3.4 provide explicit estimates on the solutions of equations (1.1) and (1.2) – (1.3) and are obtained by simple applications of the inequalities in Lemmas 3.1 and 3.2. If the estimates on the right hand sides in (3.9) and (3.15) are bounded, then the solutions of equations (1.1) and (1.2) – (1.3) are bounded.

4. Approximate Solutions

In this section we shall deal with the approximation of solutions of equations (1.1) and (1.2) – (1.3). We obtain conditions under which we can estimate the error between the solutions and approximate solutions.

We call a function $u \in C(\Delta,\mathbb{R}^n)$ an ε-approximate solution of equation (1.1) if there exists a constant $\varepsilon \geq 0$ such that

$$|u(x,y) - f(x,y,u(x,y),(Ku)(x,y))| \leq \varepsilon,$$

for all $(x,y) \in \Delta$. Let $u \in C(\Delta,\mathbb{R}^n)$, D_2D_1u exists and satisfies the inequality

$$|D_2D_1u(x,y) - f(x,y,u(x,y),(Ku)(x,y))| \leq \varepsilon,$$

for a given constant $\varepsilon \geq 0$, where it is supposed that (1.3) holds. Then we call $u(x,y)$ an ε-approximate solution of equation (1.2) with (1.3).

The following theorems deal with estimates on the difference between the two approximate solutions of equations (1.1) and (1.2) with (1.3).

Theorem 4.1. *Suppose that the functions f and k in equation (1.1) satisfy the conditions (3.6) and (3.7). For $i = 1, 2$, let $u_i(x, y)$ be respectively ε_i-approximate solutions of equation (1.1) on Δ. Then*

$$|u_1(x, y) - u_2(x, y)| \leq \left(\frac{\varepsilon_1 + \varepsilon_2}{1 - N}\right) \exp\left(\int_a^x \int_b^y B(s, t)\, dtds\right), \tag{4.1}$$

for $(x, y) \in \Delta$, where $B(x, y)$ is given by (3.10).

Proof. Since $u_i(x, y)$ $(i = 1, 2)$ for $(x, y) \in \Delta$ are respectively ε_i-approximate solutions to equation (1.1), we have

$$|u_i(x, y) - f(x, y, u_i(x, y), (Ku_i)(x, y))| \leq \varepsilon_i. \tag{4.2}$$

From (4.2) and using the elementary inequalities $|v - z| \leq |v| + |z|$ and $|v| - |z| \leq |v - z|$, we observe that

$$\begin{aligned} \varepsilon_1 + \varepsilon_2 &\geq |u_1(x, y) - f(x, y, u_1(x, y), (Ku_1)(x, y))| \\ &\quad + |u_2(x, y) - f(x, y, u_2(x, y), (Ku_2)(x, y))| \\ &\geq |\{u_1(x, y) - f(x, y, u_1(x, y), (Ku_1)(x, y))\} \\ &\quad - \{u_2(x, y) - f(x, y, u_2(x, y), (Ku_2)(x, y))\}| \\ &\geq |u_1(x, y) - u_2(x, y)| - |f(x, y, u_1(x, y), (Ku_1)(x, y)) \\ &\quad - f(x, y, u_2(x, y), (Ku_2)(x, y))|. \end{aligned} \tag{4.3}$$

Let $w(x, y) = |u_1(x, y) - u_2(x, y)|$, $(x, y) \in \Delta$. From (4.3) and using the hypotheses, we observe that

$$w(x, y) \leq \varepsilon_1 + \varepsilon_2 + N\left[w(x, y) + \int_a^x \int_b^y r(x, y, m, n)\, w(m, n)\, dndm\right]. \tag{4.4}$$

From (4.4) and using the assumption that $0 \leq N < 1$, we observe that

$$w(x, y) \leq \left(\frac{\varepsilon_1 + \varepsilon_2}{1 - N}\right) + \frac{N}{1 - N} \int_a^x \int_b^y r(x, y, m, n)\, w(m, n)\, dndm. \tag{4.5}$$

Now a suitable application of Lemma 3.1 to (4.5) yields (4.1). □

Theorem 4.2. *Suppose that the functions f and k in equation (1.2) satisfy the conditions (3.13) and (3.7). For $i = 1, 2$, let $u_i(x, y)$ be respectively ε_i-approximate solutions of equation (1.2) on Δ with*

$$u_i(x, 0) = \alpha_i(x), \qquad u_i(0, y) = \beta_i(y), \qquad u_i(0, 0) = 0, \tag{4.6}$$

where $\alpha_i \in C(I_a, \mathbb{R}^n)$, $\beta_i \in C(I_b, \mathbb{R}^n)$ such that

$$|\alpha_1(x) - \alpha_2(x) + \beta_1(y) - \beta_2(y)| \leq \delta, \tag{4.7}$$

where $\delta \geq 0$ is a constant. Then

$$\begin{aligned} |u_1(x, y) - u_2(x, y)| \leq e(x, y) \Bigg[1 + \int_a^x \int_b^y p(s, t) \\ \times \exp\left(\int_a^s \int_b^t [p(m, n) + A(m, n)]\, dmdn\right) dtds\Bigg], \end{aligned} \tag{4.8}$$

for $(x, y) \in \Delta$, where

$$e(x, y) = (\varepsilon_1 + \varepsilon_2)(x - a)(y - b) + \delta. \tag{4.9}$$

Proof. Since $u_i(x,y)$ $(i=1,2)$ for $(x,y)\in\Delta$ are respectively ε_i-approximate solutions of equation (1.2) with (4.6), we have

$$|D_2D_1u_i(x,y)-f(x,y,u_i(x,y),(Ku_i)(x,y))|\le\varepsilon_i. \tag{4.10}$$

First keeping x fixed in (4.10), setting $y=t$ and integrating both sides over t from b to y, then keeping y fixed in the resulting inequality and setting $x=s$ and integrating both sides over s from a to x and using (4.6), we observe that

$$\begin{aligned}
&\varepsilon_i(x-a)(y-b)\\
&\ge\int_a^x\int_b^y|D_2D_1u_i(s,t)-f(s,t,u_i(s,t),(Ku_i)(s,t))|\,dtds\\
&\ge\left|\int_a^x\int_b^y\{D_2D_1u_i(s,t)-f(s,t,u_i(s,t),(Ku_i)(s,t))\}\,dtds\right|\\
&=\left|\left\{u_i(x,y)-[\alpha_i(x)+\beta_i(y)]-\int_a^x\int_b^y f(s,t,u_i(s,t),(Ku_i)(s,t))\right\}\right|.
\end{aligned}$$

The rest of the proof can be completed by closely looking at the proof of Theorem 4.1 and using the inequality in Lemma 3.2. Here, we omit the details. □

Remark 3. When $u_1(x,y)$ is a solution of equation (1.1) (respectively equations (1.2) – (1.3)), then we have $\varepsilon_1=0$ and from (4.1) (respectively (4.8)) we see that $u_2(x,y)\to u_1(x,y)$ as $\varepsilon_2\to0$ (respectively $\varepsilon_2\to0$ and $\delta\to0$). Furthermore, if we put $\varepsilon_1=\varepsilon_2=0$ in (4.1) (respectively $\varepsilon_1=\varepsilon_2=0$, $\alpha_1(x)=\alpha_2(x)$, $\beta_1(y)=\beta_2(y)$, i.e. $\delta=0$ in (4.8)), then the uniqueness of solutions of equation (1.1) (respectively equations (1.2) – (1.3)) is established.

Consider the equations (1.1), (1.2) – (1.3) together with the following Volterra integral and integrodifferential equations

$$v(x,y)=\bar f(x,y,v(x,y),(Kv)(x,y)), \tag{4.11}$$

and

$$D_2D_1v(x,y)=\bar f(x,y,v(x,y),(Kv)(x,y)), \tag{4.12}$$

with the given initial boundary conditions

$$v(x,0)=\bar\alpha(x),\qquad v(0,y)=\bar\beta(y),\qquad v(0,0)=0, \tag{4.13}$$

for $(x,y)\in\Delta$, where K is given by (1.4) and $\bar f\in C(\Delta\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$, $\bar\alpha\in C(I_a,\mathbb{R}^n)$, $\bar\beta\in C(I_b,\mathbb{R}^n)$.

The following theorems show the closeness of the solutions to equations (1.1), (4.11) and (1.2) – (1.3), (4.12) – (4.13).

Theorem 4.3. *Suppose that the functions f,k in equation (1.1) satisfy the conditions (3.6), (3.7) and there exists a constant $\varepsilon\ge0$, such that*

$$\left|f(x,y,u,w)-\bar f(x,y,u,w)\right|\le\varepsilon, \tag{4.14}$$

where $f,\bar f$ are as given in (1.1) and (4.11). Let $u(x,y)$ and $v(x,y)$ be respectively the solutions of equations (1.1) and (4.11) for $(x,y)\in\Delta$. Then

$$|u(x,y)-v(x,y)|\le\left(\frac{\varepsilon}{1-N}\right)\exp\left(\int_a^x\int_b^y B(s,t)\,dtds\right), \tag{4.15}$$

for $(x,y)\in\Delta$, where $B(x,y)$ is given by (3.10).

Proof. Let $z(x,y) = |u(x,y) - v(x,y)|$ for $(x,y) \in \Delta$. Using the facts that $u(x,y)$ and $v(x,y)$ are the solutions of equations (1.1) and (4.11) and the hypotheses, we observe that

$$\begin{aligned} z(x,y) &\le |f(x,y,u(x,y),(Ku)(x,y)) - f(x,y,v(x,y),(Kv)(x,y))| \\ &\quad + \left|f(x,y,v(x,y),(Kv)(x,y)) - \bar{f}(x,y,v(x,y),(Kv)(x,y))\right| \\ &\le \varepsilon + N\left[z(x,y) + \int_a^x \int_b^y r(s,t,m,n)\, z(m,n)\, dndm\right]. \end{aligned} \tag{4.16}$$

From (4.16) and using the assumption that $0 \le N < 1$, we observe that

$$z(x,y) \le \frac{\varepsilon}{1-N} + \frac{N}{1-N}\int_a^x \int_b^y r(s,t,m,n)\, z(m,n)\, dndm. \tag{4.17}$$

Now a suitable application of Lemma 3.1 to (4.17) yields (4.15). □

Theorem 4.4. *Suppose that the functions f, k in equation (1.2) are as in Theorem 4.2 and there exist constants $\varepsilon \ge 0, \delta \ge 0$ such that the condition (4.14) holds and*

$$\left|\alpha(x) - \bar{\alpha}(x) + \beta(y) - \bar{\beta}(y)\right| \le \delta, \tag{4.18}$$

where α, β and $\bar{\alpha}, \bar{\beta}$ are as in (1.3) and (4.13). Let $u(x,y)$ and $v(x,y)$ be respectively the solutions of equations (1.2) – (1.3) and (4.12) – (4.13) for $(x,y) \in \Delta$. Then

$$\begin{aligned} |u(x,y) - v(x,y)| \le \bar{e}(x,y) \Bigg[1 + \int_a^x \int_b^y p(s,t) \\ \times \exp\left(\int_a^s \int_b^t [p(m,n) + A(m,n)]\, dndm\right) dtds\Bigg], \end{aligned} \tag{4.19}$$

for $(x,y) \in \Delta$, where

$$\bar{e}(x,y) = \varepsilon(x-a)(y-b) + \delta, \tag{4.20}$$

and $A(x,y)$ is given by (3.3).

The proof can be completed by rewriting the equivalent integral equations corresponding to the equations (1.2) – (1.3) and (4.12) – (4.13) and by following the proof of Theorem 4.3 and using Lemma 3.2. We leave the details to the reader.

Remark 4. It is interesting to note that Theorem 4.3 (respectively Theorem 4.4) relates the solutions of equations (1.1) and (4.11) (respectively equations (1.2) – (1.3) and (4.12) – (4.13)) in the sense that if f is close to $\bar{f}$, (respectively f is close to $\bar{f}$, α is close to $\bar{\alpha}$, β is close to $\bar{\beta}$), then the solutions of equations (1.1) and (4.11) (respectively solutions of equations (1.2) – (1.3) and (4.12) – (4.13)) are also close together.

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