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APPLICATIONS OF NUNOKAWA'S THEOREM

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Abstract

The object of the present paper is to give applications of the Nunokawa Theorem [Proc. Japan Acad. Ser. A Math. Sci. 69 (1993), 234-237]. Our results have some interesting examples as special cases .

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1. Introduction

Let \mathcal{A} be the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. It is known that the class

$$(1.2) \quad B(\mu) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} \left\{ f'(z) \left\{ \frac{f(z)}{z} \right\}^{\mu-1} \right\} > 0, \mu > 0, z \in U \right\}$$

is the class of univalent functions in U ([3]).

To derive our main theorem, we need the following lemma due to Nunokawa [2].

Lemma 1.1. *Let $p(z)$ be analytic in U , with $p(0) = 1$ and $p(z) \neq 0$ ($z \in U$). If there exists a point $z_0 \in U$, such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

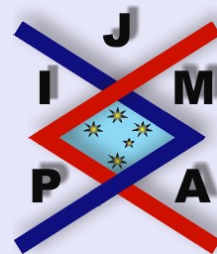
and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha \quad (\alpha > 0),$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where $k \geq 1$ when $\arg p(z_0) = \frac{\pi}{2}\alpha$ and $k \leq -1$ when $\arg p(z_0) = -\frac{\pi}{2}\alpha$



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In [1], Miller and Mocanu proved the following theorem.

Theorem A. Let $\beta_0 = 1.21872\dots$, be the solution of

$$\beta\pi = \frac{3}{2}\pi - \tan^{-1} \beta$$

and let

$$\alpha = \alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \beta$$

for $0 < \beta < \beta_0$.

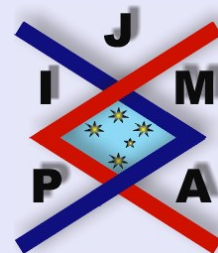
If $p(z)$ is analytic in U , with $p(0) = 1$, then

$$p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^\alpha \Rightarrow p(z) \prec \left(\frac{1+z}{1-z}\right)^\beta$$

or

$$|\arg(p(z) + zp'(z))| < \frac{\pi}{2}\alpha \Rightarrow |\arg p(z)| < \frac{\pi}{2}\beta.$$

Corresponding to Theorem A, we will obtain a result which is useful in obtaining applications of analytic function theory.



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2. Main Results

Now we derive:

Theorem 2.1. *Let $p(z)$ be analytic in U , with $p(0) = 1$ and $p(z) \neq 0$ ($z \in U$) and suppose that*

$$|\arg(p(z) + \beta zp'(z))| < \frac{\pi}{2} \left(\alpha + \frac{2}{\pi} \tan^{-1} \beta \alpha \right) \quad (\alpha > 0, \beta > 0),$$

then we have

$$|\arg p(z)| < \frac{\pi}{2} \alpha \quad \text{for } z \in U.$$

Proof. If there exists a point $z_0 \in U$, such that

$$|\arg p(z)| < \frac{\pi}{2} \alpha \quad \text{for } |z| < |z_0|$$

and

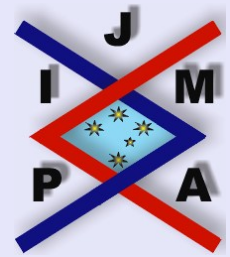
$$|\arg p(z_0)| = \frac{\pi}{2} \alpha \quad (\alpha > 0),$$

then from Lemma 1.1, we have

(i) for the case $\arg p(z_0) = \frac{\pi}{2} \alpha$,

$$\begin{aligned} \arg(p(z) + \beta zp'(z)) &= \arg p(z_0) \left\{ 1 + \beta \frac{z_0 p'(z_0)}{p(z_0)} \right\} \\ &= \frac{\pi}{2} \alpha + \arg(1 + i\beta \alpha k) \geq \frac{\pi}{2} \alpha + \tan^{-1} \beta \alpha. \end{aligned}$$

This contradicts our condition in the theorem.



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(ii) for the case $\arg p(z_0) = -\frac{\pi}{2}\alpha$, the application of the same method as in (i) shows that

$$\arg(p(z) + \beta z_0 p'(z_0)) \leq -\left(\frac{\pi}{2}\alpha + \tan^{-1} \beta\alpha\right).$$

This also contradicts the assumption of the theorem, hence the theorem is proved. \square

Making $p(z) = f'(z)$ for $f(z) \in \mathcal{A}$ in Theorem 2.1, we have

Example 2.1. If $f(z) \in \mathcal{A}$ satisfies

$$|\arg(f'(z) + \beta z f''(z))| < \frac{\pi}{2} \left(\alpha + \frac{2}{\pi} \tan^{-1} \beta\alpha \right)$$

then we have

$$|\arg f'(z)| < \frac{\pi}{2}\alpha,$$

where $\alpha > 0$, $\beta > 0$ and $z \in U$.

Further, taking $p(z) = \frac{f(z)}{z}$ for $f(z) \in \mathcal{A}$ in Theorem 2.1, we have

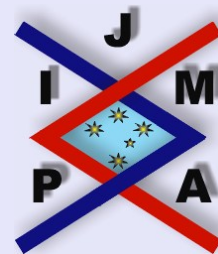
Example 2.2. If $f(z) \in \mathcal{A}$ satisfies

$$\left| \arg\left\{ (1 - \beta) \frac{f(z)}{z} + \beta f'(z) \right\} \right| < \frac{\pi}{2} \left(\alpha + \frac{2}{\pi} \tan^{-1} \beta\alpha \right),$$

then we have

$$\left| \arg \frac{f(z)}{z} \right| < \frac{\pi}{2}\alpha,$$

where $\alpha > 0$, $0 < \beta \leq 1$ and $z \in U$.



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Theorem 2.2. If $f(z) \in \mathcal{A}$ satisfies

$$\left| \arg f'(z) \left\{ \frac{f(z)}{z} \right\}^{\mu-1} \right| < \frac{\pi}{2} \left(\alpha + \frac{2}{\pi} \tan^{-1} \frac{\alpha}{\mu} \right),$$

then we have

$$\left| \arg \left\{ \frac{f(z)}{z} \right\}^{\mu} \right| < \frac{\pi}{2} \alpha,$$

where $\alpha > 0$, $\mu > 0$ and $z \in U$.

Proof. Let $p(z) = \left\{ \frac{f(z)}{z} \right\}^{\mu}$, $\mu > 0$, then we have

$$p(z) + \frac{1}{\mu} z p'(z) = f'(z) \left\{ \frac{f(z)}{z} \right\}^{\mu-1}$$

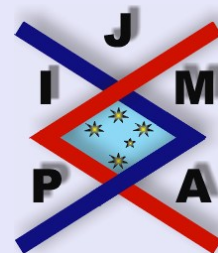
and the statements of the theorem directly follow from Theorem 2.1. □

Theorem 2.3. Let $\mu > 0$, $c + \mu > 0$ and $\alpha > 0$. If $f(z) \in \mathcal{A}$ satisfies

$$\left| \arg f'(z) \left\{ \frac{f(z)}{z} \right\}^{\mu-1} \right| < \frac{\pi}{2} \left(\alpha + \frac{2}{\pi} \tan^{-1} \frac{\alpha}{\mu + c} \right), \quad (z \in U)$$

then $F(z) = [I_{\mu,c}(f)](z)$ defined by

$$I_{\mu,c} f(z) = \left[\frac{\mu + c}{z^c} \int_0^z f^{\mu}(t) t^{c-1} dt \right]^{\frac{1}{\mu}}, \quad ([I_{\mu,c}(f)](z)/z \neq 0 \text{ in } U)$$



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satisfies

$$\left| \arg F'(z) \left\{ \frac{F(z)}{z} \right\}^{\mu-1} \right| < \frac{\pi}{2} \alpha.$$

Proof. Consider the function p defined by

$$p(z) = F'(z) \left\{ \frac{F(z)}{z} \right\}^{\mu-1} \quad (z \in U).$$

Then we easily see that

$$p(z) + \frac{1}{\mu + c} z p'(z) = f'(z) \left\{ \frac{f(z)}{z} \right\}^{\mu-1},$$

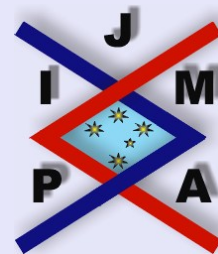
and the statements of the theorem directly follow from Theorem 2.1. □

Theorem 2.4. *Let a function $f(z) \in \mathcal{A}$ satisfy the following inequalities*

$$(2.1) \quad \left| \arg f'(z) \left\{ \frac{z}{f(z)} \right\}^{\mu+1} \right| < \frac{\pi}{2} \left(-\alpha + \frac{2}{\pi} \tan^{-1} \frac{\alpha}{\mu} \right), \quad (z \in U)$$

for some α ($0 < \alpha \leq 1$), ($0 < \mu < 1$). Then

$$\left| \arg \left\{ \frac{f(z)}{z} \right\}^{\mu} \right| < \frac{\pi}{2} \alpha.$$



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Proof. Let us define the function $p(z)$ by $p(z) = \left(\frac{f(z)}{z}\right)^\mu$, ($0 < \mu < 1$). Then $p(z)$ satisfies

$$f'(z) \left\{ \frac{z}{f(z)} \right\}^{\mu+1} = \frac{1}{p(z)} \left(1 + \frac{1}{\mu} \frac{zp'(z)}{p(z)} \right).$$

If there exists a point $z_0 \in U$, such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha,$$

then from Lemma 1.1, we have:

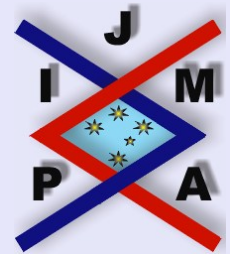
(i) for the case $\arg p(z_0) = \frac{\pi}{2}\alpha$,

$$\begin{aligned} \arg f'(z_0) \left\{ \frac{z}{f(z_0)} \right\}^{\mu+1} &= \arg \left\{ \frac{1}{p(z_0)} \left(1 + \frac{1}{\mu} \frac{zp'(z_0)}{p(z_0)} \right) \right\} \\ &= -\frac{\pi}{2}\alpha + \arg \left(1 + \frac{i\alpha k}{\mu} \right) \\ &\geq -\frac{\pi}{2}\alpha + \tan^{-1} \frac{\alpha}{\mu}. \end{aligned}$$

This contradicts our condition in the theorem.

(ii) for the case $\arg p(z_0) = -\frac{\pi}{2}\alpha$, the application of the same method as in (i) shows that

$$\arg f'(z_0) \left\{ \frac{z}{f(z_0)} \right\}^{\mu+1} \leq -\left(-\frac{\pi}{2}\alpha + \tan^{-1} \frac{\alpha}{\mu} \right).$$



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This also contradicts the assumption of the theorem, hence the theorem is proved. \square

Theorem 2.5. Let $f(z) \in \mathcal{A}$ satisfy the condition (2.1) and let

$$(2.2) \quad F(z) = \left[\frac{c - \mu}{z^{c-\mu}} \int_0^z \left\{ \frac{t}{f(t)} \right\}^\mu dt \right]^{-\frac{1}{\mu}},$$

where $c - \mu > 0$. Then

$$\left| \arg F'(z) \left\{ \frac{z}{F(z)} \right\}^{\mu+1} \right| < \frac{\pi}{2} \alpha.$$

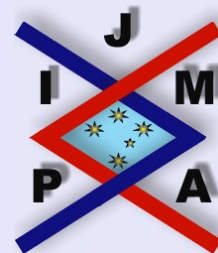
Proof. If we put

$$p(z) = F'(z) \left\{ \frac{z}{F(z)} \right\}^{\mu+1},$$

then from (2.2) we have

$$p(z) + \frac{1}{c - \mu} zp'(z) = f'(z) \left\{ \frac{z}{f(z)} \right\}^{\mu+1}.$$

The statements of the theorem then directly follow from Theorem 2.1. \square



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