

# Journal of Inequalities in Pure and Applied Mathematics

## ON THE HEISENBERG-PAULI-WEYL INEQUALITY

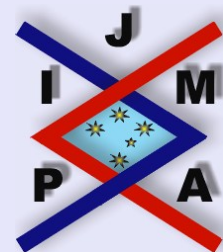
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ISSN (electronic): 1443-5756  
163-03



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volume 5, issue 1, article 8,  
2004.

*Received 17 November, 2003;  
accepted 23 January, 2004.*

*Communicated by: H.M. Srivastava*

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Abstract

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## Abstract

In this paper, we obtain a subordination result for a class of meromorphic functions.

*2000 Mathematics Subject Classification:* Primary 30C45, Secondary 30C80.

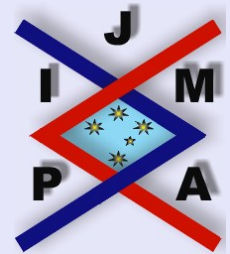
*Key words:* Meromorphic convex function, Meromorphic starlike function, Subordination.

The authors are thankful to the referee for his comments for improving the presentation.

The work presented here was supported by IRPA grant 09-02-02-0080-EA208, Malaysia.

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# 1. Introduction and Main Result

Let  $\Sigma$  be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are *analytic* in the punctured unit disc  $E = \{z : 0 < |z| < 1\}$ . Let  $\mathcal{A}$  be the class of all functions  $p(z) = 1 + p_1 z + p_2 z^2 + \dots$  which are analytic in  $\Delta = \{z : |z| < 1\}$ . The class  $\mathcal{P}$  of *Caratheodory functions* consists of functions  $p(z) \in \mathcal{A}$  having positive real part. A function  $f(z) \in \Sigma$  is *meromorphic starlike of order  $\alpha$*  if  $f(z) \neq 0$  and

$$-\Re \frac{z f'(z)}{f(z)} > \alpha, \quad (\alpha < 1; z \in \Delta).$$

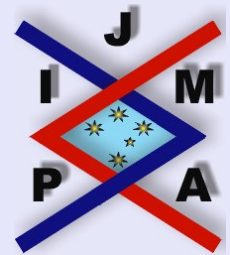
Similarly the function  $f(z)$  is *meromorphic convex of order  $\alpha$*  if  $f'(z) \neq 0$  and

$$-\Re \left( 1 + \frac{z f''(z)}{f'(z)} \right) > \alpha, \quad (\alpha < 1; z \in \Delta).$$

The classes of these functions are denoted by  $MS^*(\alpha)$  and  $MC(\alpha)$  respectively. The class  $\Sigma_{\gamma}^*(\alpha)$  of  $\gamma$ -*meromorphic convex of order  $\alpha$*  consists of functions  $f(z)$  with  $f(z) f'(z) \neq 0$  satisfying

$$-\Re \left[ (1 - \gamma) \frac{z f'(z)}{f(z)} + \gamma \left( 1 + \frac{z f''(z)}{f'(z)} \right) \right] > \alpha, \quad (\alpha < 1; z \in \Delta).$$

Nunokawa and Ahuja [2] have proved the following:



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**Theorem 1.1.** Let  $\alpha < 0$ . If

$$f \in MC \left( \frac{\alpha(3 - 2\alpha)}{2(1 - \alpha)} \right),$$

then  $f \in MS^*(\alpha)$ .

**Theorem 1.2.** Let  $\alpha < 0$  and  $\gamma \geq 0$ . If

$$f \in \Sigma_\gamma^* \left( \frac{2\alpha - 2\alpha^2 + \gamma\alpha}{2(1 - \alpha)} \right),$$

then  $f \in MS^*(\alpha)$ .

Our main result is the following generalization of Theorem 1.1 and Theorem 1.2:

**Theorem 1.3.** Let  $q(z)$  be univalent and  $q(z) \neq 0$  in  $\Delta$  and

1.  $zq'(z)/q(z)$  is starlike univalent in  $\Delta$ , and
2.  $\Re \left[ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} - \frac{q(z)}{\gamma} \right] > 0$  for  $z \in \Delta, \gamma \neq 0$ .

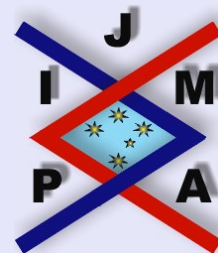
If  $f(z) \in \Sigma$  and

$$- \left[ (1 - \gamma) \frac{zf'(z)}{f(z)} + \gamma \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] \prec q(z) - \gamma \frac{zq'(z)}{q(z)},$$

then

$$- \frac{zf'(z)}{f(z)} \prec q(z)$$

and  $q(z)$  is the best dominant.



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## 2. Proof of Theorem 1.3

To prove our Theorem 1.3, we need the following result of Miller and Mocanu:

**Lemma 2.1.** [1, p. 132, Theorem 3.4h] Let  $q(z)$  be univalent in the unit disk  $\Delta$  and  $\theta$  and  $\phi$  be analytic in a domain  $D$  containing  $q(\Delta)$  with  $\phi(w) \neq 0$  when  $w \in q(\Delta)$ . Set

$$Q(z) := zq'(z)\phi(q(z)), \quad h(z) := \theta(q(z)) + Q(z).$$

Suppose that either  $h(z)$  is convex, or  $Q(z)$  is starlike univalent in  $\Delta$ . In addition, assume that

$$\Re \frac{zh'(z)}{Q(z)} > 0 \quad (z \in \Delta).$$

If  $p(z)$  is analytic in  $\Delta$ , with  $p(0) = q(0)$ ,  $p(\Delta) \subseteq D$  and

$$(2.1) \quad \theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then  $p(z) \prec q(z)$  and  $q(z)$  is the best dominant.

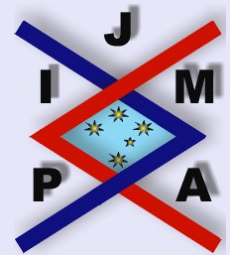
By applying Lemma 2.1, we first prove the following:

**Lemma 2.2.** Let  $q(z) \in \mathcal{A}$  satisfy  $q(z) \neq 0$  and the conditions (1) and (2) in Theorem 1.3.

If  $p(z) \in \mathcal{A}$  satisfies  $p(z) \neq 0$  and

$$(2.2) \quad p(z) - \gamma \frac{zp'(z)}{p(z)} \prec q(z) - \gamma \frac{zq'(z)}{q(z)},$$

then  $p(z) \prec q(z)$  and  $q(z)$  is a best dominant.



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*Proof.* Define the functions  $\theta$  and  $\phi$  by

$$\theta(w) := w \quad \text{and} \quad \phi(w) := -\frac{\gamma}{w}.$$

Then  $\theta$  and  $\phi$  are analytic in  $\mathbb{C} \setminus \{0\}$  and  $\phi(w) \neq 0$ . Define the functions  $Q(z)$  and  $h(z)$  by

$$Q(z) := zq'(z)\phi(q(z)) = -\gamma\frac{zq'(z)}{q(z)}$$

and

$$h(z) := \theta(q(z)) + Q(z) = q(z) - \gamma\frac{zq'(z)}{q(z)}.$$

In view of our assumptions, the functions  $Q(z)$  and  $h(z)$  satisfy the conditions of Lemma 2.1. Since the subordination (2.2) can be written as the subordination (2.1), the result now follows by an application of Lemma 1.1.  $\square$

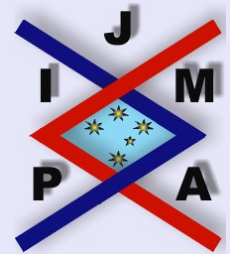
*Proof of Theorem 1.3.* Define the function  $p(z)$  by

$$p(z) := -\frac{zf'(z)}{f(z)} \quad (z \in \Delta).$$

Then a computation shows that

$$p(z) - \gamma\frac{zp'(z)}{p(z)} = -\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right].$$

The result of Theorem 1.3 now follows from Lemma 2.2.  $\square$



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### 3. A Special Case

By setting

$$q(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}$$

in Theorem 1.3, we have the following:

**Corollary 3.1.** *Let  $\alpha < 0$ ,  $\gamma \neq 0$ . If  $f(z) \in \Sigma$  and*

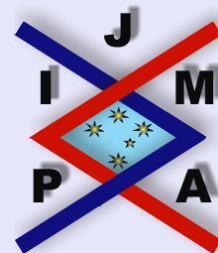
$$\begin{aligned} - \left[ (1 - \gamma) \frac{zf'(z)}{f(z)} + \gamma \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] \\ \prec \frac{1 + 2[1 - \gamma + (\alpha - 1)\gamma]z + (1 - 2\alpha)^2 z^2}{1 - 2\alpha z - (1 - 2\alpha)z^2}, \end{aligned}$$

then  $-\Re \frac{zf'(z)}{f(z)} > \alpha$ .

**Remark 3.1.** *Fairly straightforward calculation shows that the image of  $|z| < 1$  under*

$$w(z) := \frac{1 + 2[1 - \gamma + (\alpha - 1)\gamma]z + (1 - 2\alpha)^2 z^2}{1 - 2\alpha z - (1 - 2\alpha)z^2}$$

*contains the half plane  $\Re w(z) > \frac{2\alpha - 2\alpha^2 + \gamma\alpha}{2(1 - \alpha)}$ . Therefore we see that Theorem 1.2 now follows from Corollary 3.1. Theorem 1.1 is indeed a special case of Theorem 1.2 when  $\gamma = 1$ .*



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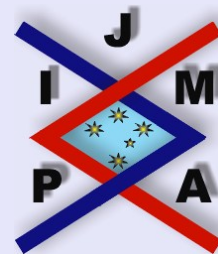
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