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THE ULAM STABILITY PROBLEM IN APPROXIMATION OF APPROXIMATELY QUADRATIC MAPPINGS BY QUADRATIC MAPPINGS

JOHN MICHAEL RASSIAS

Pedagogical Department, E. E.,
National and Capodistrian University of Athens,
Section of Mathematics and Informatics,
4, Agamemnonos Str., Aghia Paraskevi,
Athens 15342, Greece.
EMail: jrassias@primedu.uoa.gr
URL: http://www.primedu.uoa.gr/ jrassias/

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Abstract

S.M. Ulam, 1940, proposed the well-known Ulam stability problem and in 1941, the problem for linear mappings was solved by D.H. Hyers. D.G. Bourgin, 1951, also investigated the Ulam problem for additive mappings. P.M. Gruber, claimed, in 1978, that this kind of stability problem is of particular interest in probability theory and in the case of functional equations of different types. F. Skof, in 1981, was the first author to solve the Ulam problem for quadratic mappings. During the years 1982-1998, the author established the Hyers-Ulam stability for the Ulam problem for different mappings. In this paper we solve the Ulam stability problem by establishing an approximation of approximately quadratic mappings by quadratic mappings. Today there are applications in actuarial and financial mathematics, sociology and psychology, as well as in algebra and geometry.

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Key words: Ulam problem, Ulam type problem, General Ulam problem, Quadratic mapping, Approximately quadratic mapping, Square of the quadratic weighted mean.

Contents

1. Introduction

S.M. Ulam [24] proposed the *general Ulam stability problem*: "When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" D.H. Hyers [13] solved this problem for linear mappings. D.G. Bourgin [3] also investigated the Ulam problem for additive mappings. P.M. Gruber [12] claimed that this kind of stability problem is of particular interest in probability theory and in the case of functional equations of different types. Th.M. Rassias [20] employed Hyers' ideas to new additive mappings, and later I. Fenyö ([7], [8]) established the stability of the Ulam problem for quadratic and other mappings. Z. Gajda and R. Ger [10] showed that one can obtain analogous stability results for subadditive multifunctions. Other interesting stability results have been achieved also by the following authors: J. Aczél [1], C. Borelli and G.L. Forti ([2], [9]), P.W. Cholewa [4], St. Czerwik [5], H. Drljevic [6] and L. Paganoni [14]. F. Skof ([21] – [23]) was the first author to solve the Ulam problem for quadratic mappings. We ([15] – [19]) solved the above Ulam problem for different mappings. P. Găvruţă [11] answered a question of ours [17] concerning the stability of the Cauchy equation. Today there are applications in actuarial and financial mathematics, sociology and psychology, as well as in algebra and geometry.

In this paper we introduce the following quadratic functional equation

$$(*) \qquad Q(a_1x_1 + a_2x_2) + Q(a_2x_1 - a_1x_2) = \left(a_1^2 + a_2^2\right)[Q(x_1) + Q(x_2)]$$

with quadratic mappings $Q : X \to Y$ such that X and Y are real linear spaces.

Denote

$$
\begin{aligned}
K_r &= K_r\left(\|x_1\|, \|x_2\|\right) \\
&= \left|2^{r-1}\left(\|x_1\|^r + \|x_2\|^r\right) - \left(\|x_1+x_2\|^r + \|x_1-x_2\|^r\right)\right| \\
&= \begin{cases} 2^{r-1}\left(\|x_1\|^r + \|x_2\|^r\right) - \left(\|x_1+x_2\|^r + \|x_1-x_2\|^r\right), & \text{if } r > 2 \\ \|x_1+x_2\|^r + \|x_1-x_2\|^r - 2^{r-1}\left(\|x_1\|^r + \|x_2\|^r\right), & \text{if } 1 < r < 2, \end{cases}
\end{aligned}
$$

for every $(x_1, x_2) \in X^2$, where X is a normed linear space. Note that $K_r \geq 0$ for any fixed real $r : 1 < r \neq 2$. Note also that

$$
K_r = K_r\left(\|x\|, \|x\|\right) = 0,
$$

$$
\begin{aligned}
K_r\left(|a_1|\,\|x\|, |a_2|\,\|x\|\right) &= \beta_1 \|x\|^r, \\
K_r\left(m^{-1}|a_1|\,\|x\|, m^{-1}|a_2|\,\|x\|\right) &= \beta_1 m^{-r} \|x\|^r, \\
K_r\left(\|x\|, 0\right) &= \beta_2 \|x\|^r \quad \text{and} \\
K_r\left(m^{-1}\|x\|, 0\right) &= \beta_3 \|x\|^r,
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_1 &= K_r\left(|a_1|, |a_2|\right) \\
&= \left|2^{r-1}\left(|a_1|^r + |a_2|^r\right) - \left(|a_1+a_2|^r + |a_1-a_2|^r\right)\right| \\
&= \begin{cases} 2^{r-1}\left(|a_1|^r + |a_2|^r\right) - \left(|a_1+a_2|^r + |a_1-a_2|^r\right), & \text{if } r > 2 \\ |a_1+a_2|^r + |a_1-a_2|^r - 2^{r-1}\left(|a_1|^r + |a_2|^r\right), & \text{if } 1 < r < 2, \end{cases}
\end{aligned}
$$

$$
\beta_2 = K_r(1,0) = \left|2^{r-1} - 2\right| = \begin{cases} 2^{r-1} - 2, & \text{if } r > 2 \\ 2 - 2^{r-1}, & \text{if } 1 < r < 2, \end{cases}
$$

$$\beta_3 = K_r\left(m^{-1}, 0\right) = \beta_2 m^{-r},$$

Note that $a_1 \neq a_2$, and $1 \neq m = a_1^2 + a_2^2 > 0$.

If X and Y are normed linear spaces and Y complete, then we establish an approximation of approximately quadratic mappings $f : X \to Y$ by quadratic mappings $Q : X \to Y$, such that the corresponding approximately quadratic functional inequality

$$(**)\quad \left\| f(a_1x_1 + a_2x_2) + f(a_2x_1 - a_1x_2) - \left(a_1^2 + a_2^2\right)\left[f\left(x_1\right) + f\left(x_2\right)\right]\right\| \\ \leq cK_r\left(\|x_1\|, \|x_2\|\right)$$

holds with a constant $c \geq 0$ (independent of x_1,$x_2 \in X$), and any fixed pair $a = (a_1, a_2) \in \mathbb{R}^2 - \{(0,0)\}$ and any fixed real $r > 1$:

$$I_1 = \{(r,m) \in \mathbb{R}^2 : 1 < r < 2, m > 1 \text{ and } r > 2, 0 < m < 1\}, \text{or}$$
$$I_2 = \{(r,m) \in \mathbb{R}^2 : 1 < r < 2, 0 < m < 1 \text{ and } r > 2, m > 1\},$$

hold, where $1 \neq m = a_1^2 + a_2^2 = |a|^2 > 0$ and $a_1 \neq a_2$. Note that $m^{r-2} < 1$ if $(r,m) \in I_1$, and $m^{2-r} < 1$ if $(r,m) \in I_2$.

It is useful for the following, to observe that, from $(*)$ with $x_1 = x_2 = 0$, and $0 < m \neq 1$ we get

$$2(m-1)Q(0) = 0,$$

or

$$Q(0) = 0. \tag{1.1}$$

Definition 1.1. *Let* X*and* Y *be real linear spaces. Let* $a = (a_1, a_2) \in \mathbb{R}^2 - \{(0,0)\} : 0 < m = a_1^2 + a_2^2 \neq 1$ *and* $a_1 \neq a_2$*. Then a mapping* $Q : X \to Y$ *is called quadratic with respect to* a*, if* ($*$) *holds for every vector* $(x_1, x_2) \in X^2$*.*

Definition 1.2. *Let* X*and* Y *be real linear spaces. Let* $a = (a_1, a_2) \in \mathbb{R}^2 - \{(0,0)\} : 0 < m = a_1^2 + a_2^2 \neq 1$ *and* $a_1 \neq a_2$*. Then a mapping* $\bar{Q} : X \to Y$ *is called the square of the quadratic weighted mean of* Q *with respect to* $a = (a_1, a_2)$*, if*

$$\bar{Q}(x) = \begin{cases} \frac{Q(a_1 x) + Q(a_2 x)}{a_1^2 + a_2^2}, & \text{if } (r, m = a_1^2 + a_2^2) \in I_1 \\ (a_1^2 + a_2^2)\left[Q\left(\frac{a_1}{a_1^2 + a_2^2} x\right) + Q\left(\frac{a_2}{a_1^2 + a_2^2} x\right)\right], & \text{if } (r, m = a_1^2 + a_2^2) \in I_2 \end{cases} \tag{1.2}$$

for all $x \in X$*.*

For every $x \in \mathbb{R}$ set $Q(x) = x^2$. Then the mapping $\bar{Q} : \mathbb{R} \to \mathbb{R}$ is quadratic, such that $\bar{Q}(x) = x^2$. Denoting by $\sqrt{\overline{x_w^2}}$ *the quadratic weighted mean*, we note that the above-mentioned mapping $\bar{Q}$ is an analogous case to *the square of the quadratic weighted mean* employed in mathematical statistics: $\overline{x_w^2} = \frac{a_1^2 x_1^2 + a_2^2 x_2^2}{a_1^2 + a_2^2}$ with weights $w_1 = a_1^2$ and $w_2 = a_2^2$, data $x_1 = x_2 = x$, and $Q(a_i x) = (a_i x)^2$, $(i = 1, 2)$.

Now, claim that for $n \in N_0 = \{0, 1, 2, \dots\}$ that

$$Q(x) = \begin{cases} m^{-2n} Q(m^n x), & \text{if } (r, m) \in I_1 \\ m^{2n} Q(m^{-n} x), & \text{if } (r, m) \in I_2, \end{cases} \tag{1.3}$$

for all $x \in X$ and $n \in N_0$.

For $n = 0$, it is trivial. From (1.1), (1.2) and ($*$), with $x_i = a_i x \quad (i = 1, 2)$, we obtain

$$Q(mx) = m\left[Q(a_1 x) + Q(a_2 x)\right],$$

or

$$\bar{Q}(x) = m^{-2} Q(mx), \tag{1.4}$$

if I_1 holds. Besides from (1.1), (1.2) and ($*$), with $x_1 = x$, $x_2 = 0$, we get

$$Q(a_1 x) + Q(a_2 x) = mQ(x),$$

or

$$\bar{Q}(x) = Q(x), \tag{1.5}$$

if I_1 holds. Therefore from (1.4) and (1.5) we have

$$Q(x) = m^{-2} Q(mx), \tag{1.6}$$

which is (1.3) for $n = 1$, if I_1 holds. Similarly, from (1.1), (1.2) and ($*$), with $x_i = \frac{a_i}{m} x \quad (i = 1, 2)$, we obtain

$$Q(x) = \bar{Q}(x) \tag{1.7}$$

if I_2 holds. Besides from (1.1), (1.2) and ($*$), with $x_1 = \frac{x}{m}$, $x_2 = 0$, we get

$$Q\left(\frac{a_1}{m} x\right) + Q\left(\frac{a_2}{m} x\right) = mQ(m^{-1} x),$$

or

$$\bar{Q}(x) = m^2 Q(m^{-1}x) \tag{1.8}$$

if I_2 holds. Therefore from (1.7) and (1.8) we have

$$Q(x) = m^2 Q(m^{-1}x), \tag{1.9}$$

which is (1.3) for $n = 1$, if I_2 holds.

Assume (1.3) is true and from (1.6), with $m^n x$ in place of x, we get:

$$Q\left(m^{n+1}x\right) = m^2 Q(m^n x) = m^2 (m^n)^2 Q(x) = \left(m^{n+1}\right)^2 Q(x). \tag{1.10}$$

Similarly, with $m^{-n}x$ in place of x, we get:

$$\begin{aligned} Q\left(m^{-(n+1)}x\right) &= m^{-2} Q(m^{-n}x) \\ &= m^{-2}(m^{-n})^2 Q(x) = \left(m^{-(n+1)}\right)^2 Q(x). \end{aligned} \tag{1.11}$$

These formulas (1.10) and (1.11) by induction, prove formula (1.3).

2. Quadratic Functional Stability

Theorem 2.1. *Let X and Y be normed linear spaces. Assume that Y is complete. Assume in addition that mapping $f : X \to Y$ satisfies the functional inequality (**). Define $I_1 = \{(r,m) \in \mathbb{R}^2 : 1 < r < 2,\ m > 1$, or $r > 2,\ 0 < m < 1\}$, and $I_2 = \{(r,m) \in \mathbb{R}^2 : 1 < r < 2,\ 0 < m < 1$, or $r > 2,\ m > 1\}$ for any fixed pair $a = (a_1, a_2)$ of reals $a_i \neq 0\ \ (i = 1, 2)$ and any fixed real $r > 1 : 1 \neq m = a_1^2 + a_2^2 = |a|^2 > 0$, $a_1 \neq a_2$. Besides define*

$$\begin{aligned} 0 < \beta_1 &= K_r\left(|a_1|, |a_2|\right) \\ &= \left|2^{r-1}\left(|a_1|^r + |a_2|^r\right) - \left(|a_1 + a_2|^r + |a_1 - a_2|^r\right)\right| \\ &= \begin{cases} 2^{r-1}\left(|a_1|^r + |a_2|^r\right) - \left(|a_1 + a_2|^r + |a_1 - a_2|^r\right), & \text{if } r > 2 \\ |a_1 + a_2|^r + |a_1 - a_2|^r - 2^{r-1}\left(|a_1|^r + |a_2|^r\right), & \text{if } 1 < r < 2, \end{cases} \end{aligned}$$

$\beta_2 = K_r(1,0) = |2^{r-1} - 2|$, and $\sigma = \beta_1 + m\beta_2 > 0$. Also define

$$f_n(x) = \begin{cases} m^{-2n} f(m^n x), & \text{if } (r,m) \in I_1 \\ m^{2n} f(m^{-n} x), & \text{if } (r,m) \in I_2 \end{cases}$$

for all $x \in X$ and $n \in N_0 = \{0, 1, 2, \dots\}$.

Then the limit

$$Q(x) = \lim_{n\to\infty} f_n(x) \tag{2.1}$$

exists for all $x \in X$ and $Q : X \to Y$ is the unique quadratic mapping with *respect to* $a = (a_1, a_2)$, *such that*

$$\|f(x) - Q(x)\| \leq \frac{\sigma c}{|m^2 - m^r|} \|x\|^r = \|x\|^r \begin{cases} \sigma c/(m^2 - m^r), & \text{if } (r,m) \in I_1 \\ \sigma c/(m^r - m^2), & \text{if } (r,m) \in I_2 \end{cases} \tag{2.2}$$

holds for all $x \in X$ *and* $n \in N_0$ *and* $c \geq 0$ *(constant independent of* $x \in X$*).*

Existence.

Proof. It is useful for the following, to observe that, from $(**)$ with $x_1 = x_2 = 0$ and $0 < m \neq 1$, we get

$$2\,|m - 1|\,\|f(0)\| \leq 0,$$

or

$$f(0) = 0. \tag{2.3}$$

Now claim that for $n \in N_0$

$$\begin{aligned} \|f(x) - f_n(x)\| &\leq \frac{\sigma c}{|m^2 - m^r|}\left(1 - m^{n|r-2|}\right)\|x\|^r \\ &= \|x\|^r \begin{cases} \frac{\sigma c}{m^2 - m^r}\left(1 - m^{n(r-2)}\right), & \text{if } (r,m) \in I_1 : m^{r-2} < 1 \\ \frac{\sigma c}{m^r - m^2}\left(1 - m^{n(2-r)}\right), & \text{if } (r,m) \in I_2 : m^{2-r} < 1. \end{cases} \end{aligned} \tag{2.4}$$

For $n = 0$, it is trivial.

Define $\bar{f} : X \to Y$, *the square of the quadratic weighted mean of* f *with respect to* $a = (a_1, a_2)$ by replacing Q, $\bar{Q}$ of (1.2) with f, $\bar{f}$, respectively, as follows:

$$(2.5)\quad \bar{f}(x) = \begin{cases} \frac{f(a_1x)+f(a_2x)}{a_1^2+a_2^2}, & \text{if } (r, m = a_1^2 + a_2^2 = |a|^2) \in I_1 \\ (a_1^2 + a_2^2)\left[f\left(\frac{a_1}{a_1^2+a_2^2}x\right) + f\left(\frac{a_2}{a_1^2+a_2^2}x\right)\right], & \text{if } (r, m = a_1^2 + a_2^2 = |a|^2) \in I_2 \end{cases}$$

for all $x \in X$.

From (2.3), (2.5) and ($**$), with $x_i = a_i x$ $(i = 1, 2)$, we obtain

$$\|f(mx)-m\,[f(a_1x)+f(a_2x)]\| \le \sigma c\,\|x\|^r\,,$$

or

$$(2.6)\qquad \left\|m^{-2}f(mx)-\bar{f}(x)\right\| \le \frac{\beta_1 c}{m^2}\,\|x\|^r\,,$$

if I_1 holds. Besides from (2.3), (2.5) and ($**$), with $x_1 = x$, $x_2 = 0$, we get

$$\|f(a_1x)+f(a_2x)-mf(x)\| \le cK_r\left(\|x\|\,,0\right) = \beta_2 c\,\|x\|^r\,,$$

or

$$(2.7)\qquad \left\|\bar{f}(x)-f(x)\right\| \le \frac{\beta_2 c}{m}\,\|x\|^r\,,$$

if I_1 holds. Therefore from (2.6) and (2.7) we have

$$\left\|f(x) - m^{-2}f(mx)\right\| \leq \frac{\sigma c}{m^2}\left\|x\right\|^r = \frac{\sigma c}{m^2 - m^r}\left(1 - m^{r-2}\right)\left\|x\right\|^r, \tag{2.8}$$

which is (2.4) for $n = 1$, if I_1 holds.

Similarly, from (2.3), (2.5) and ($**$), with $x_i = \frac{a_i}{m}x$ $(i = 1, 2)$, we obtain

$$\left\|f(x) - \bar{f}(x)\right\| \leq \frac{\beta_1 c}{m^r}\left\|x\right\|^r, \tag{2.9}$$

if I_2 holds. Besides from (2.3), (2.5) and ($**$), with $x_1 = \frac{x}{m}$, $x_2 = 0$, we get

$$\left\|f\left(\frac{a_1}{m}x\right) + f\left(\frac{a_2}{m}x\right) - mf(m^{-1}x)\right\| \leq cK_r\left(m^{-1}\left\|x\right\|, 0\right) = \beta_3 c\left\|x\right\|^r,$$

or

$$\left\|\bar{f}(x) - m^2 f(m^{-1}x)\right\| \leq m\beta_3 c\left\|x\right\|^r = \frac{m\beta_2 c}{m^r}\left\|x\right\|^r, \tag{2.10}$$

if I_2 holds. Therefore from (2.9) and (2.10) we have

$$\left\|f(x) - m^2 f(m^{-1}x)\right\| \leq \frac{\sigma c}{m^r}\left\|x\right\|^r = \frac{\sigma c}{m^r - m^2}\left(1 - m^{2-r}\right)\left\|x\right\|^r, \tag{2.11}$$

which is (2.4) for $n = 1$, if I_2 holds.

Assume (2.4) is true if $(r, m) \in I_1$. From (2.8), with $m^n x$ in place of x, and the triangle inequality, we have

$$\begin{aligned}(2.12)\qquad &\|f(x)-f_{n+1}(x)\| \\ &= \left\|f(x)-m^{-2(n+1)}f\left(m^{n+1}x\right)\right\| \\ &\le \left\|f(x)-m^{-2n}f\left(m^{n}x\right)\right\| \\ &\quad + \left\|m^{-2n}f(m^{n}x)-m^{-2(n+1)}f\left(m^{n+1}x\right)\right\| \\ &\le \frac{\sigma c}{m^2-m^r}\left[\left(1-m^{n(r-2)}\right)+m^{-2n}\left(1-m^{r-2}\right)m^{nr}\right]\|x\|^r \\ &= \frac{\sigma c}{m^2-m^r}\left(1-m^{(n+1)(r-2)}\right)\|x\|^r,\end{aligned}$$

if I_1 holds.

Similarly assume (2.4) is true if $(r,m)\in I_2$. From (2.11), with $m^{-n}x$ in place of x, and the triangle inequality, we have

$$\begin{aligned}(2.13)\qquad &\|f(x)-f_{n+1}(x)\| \\ &= \left\|f(x)-m^{2(n+1)}f\left(m^{-(n+1)}x\right)\right\| \\ &\le \left\|f(x)-m^{2n}f\left(m^{-n}x\right)\right\| \\ &\quad + \left\|m^{2n}f(m^{-n}x)-m^{2(n+1)}f\left(m^{-(n+1)}x\right)\right\| \\ &\le \frac{\sigma c}{m^r-m^2}\left[\left(1-m^{n(2-r)}\right)+m^{2n}\left(1-m^{2-r}\right)m^{-nr}\right]\|x\|^r \\ &= \frac{\sigma c}{m^r-m^2}\left(1-m^{(n+1)(2-r)}\right)\|x\|^r,\end{aligned}$$

if I_2 holds.

Therefore inequalities (2.12) and (2.13) prove inequality (2.4) for any $n\in N_0$.

Claim now that the sequence $\{f_n(x)\}$ *converges*. To do this it suffices to prove that it is a Cauchy sequence. Inequality (2.4) is involved if $(r, m) \in I_1$. In fact , if $i > j > 0$, and $h_1 = m^j x$, we have:

$$\begin{aligned} \|f_i(x) - f_j(x)\| &= \left\|m^{-2i} f(m^i x) - m^{-2j} f(m^j x)\right\| \\ &= m^{-2j} \left\|m^{-2(i-j)} f\left(m^{i-j} h_1\right) - f(h_1)\right\| \\ &\leq m^{-2j} \frac{\sigma c}{m^2 - m^r} \left(1 - m^{(i-j)(r-2)}\right) \|x\|^r \\ &< \frac{\sigma c}{m^2 - m^r} m^{-2j} \|x\|^r \xrightarrow[j\to\infty]{} 0, \end{aligned} \tag{2.14}$$

if I_1 holds: $m^{r-2} < 1$.

Similarly, if $h_2 = m^{-j} x$ in I_2, we have:

$$\begin{aligned} \|f_i(x) - f_j(x)\| &= \left\|m^{2i} f(m^{-i} x) - m^{2j} f(m^{-j} x)\right\| \\ &= m^{2j} \left\|m^{2(i-j)} f\left(m^{-(i-j)} h_2\right) - f(h_2)\right\| \\ &\leq m^{2j} \frac{\sigma c}{m^r - m^2} \left(1 - m^{(i-j)(2-r)}\right) \|x\|^r \\ &< \frac{\sigma c}{m^r - m^2} m^{2j} \|x\|^r \xrightarrow[j\to\infty]{} 0, \end{aligned} \tag{2.15}$$

if I_2 holds: $m^{2-r} < 1$.

Then inequalities (2.14) and (2.15) *define* a mapping $Q : X \to Y$, given by (2.1).

Claim that from $(**)$ and (2.1) we can get $(*)$, or equivalently that the aforementioned well-defined mapping $Q : X \to Y$ is *quadratic*.

In fact, it is clear from the functional inequality (∗∗) and the limit (2.1) for $(r,m) \in I_1$ that the following functional inequality

$$m^{-2n} \left\| f(a_1 m^n x_1 + a_2 m^n x_2) + f(a_2 m^n x_1 - a_1 m^n x_2) \right. \\ \left. -(a_1^2 + a_2^2)\left[f(m^n x_1) + f(m^n x_2)\right]\right\| \\ \leq m^{-2n} c K_r \left(m^n \|x_1\|, m^n \|x_2\|\right),$$

holds for all vectors $(x_1, x_2) \in X^2$, and all $n \in \mathbb{N}$ with $f_n(x) = m^{-2n} f(m^n x) : I_1$ holds. Therefore

$$\left\| \lim_{n\to\infty} f_n(a_1 x_1 + a_2 x_2) + \lim_{n\to\infty} f_n(a_2 x_1 - a_1 x_2) \right. \\ \left. - (a_1^2 + a_2^2)\left[\lim_{n\to\infty} f_n(x_1) + \lim_{n\to\infty} f_n(x_2)\right]\right\| \\ \leq \left(\lim_{n\to\infty} m^{n(r-2)}\right) c K_r \left(\|x_1\|, \|x_2\|\right) = 0,$$

because $m^{r-2} < 1$ or

$$\left\|Q(a_1x_1+a_2x_2)+Q(a_2x_1-a_1x_2)-\left(a_1^2+a_2^2\right)\left[Q(x_1)+Q(x_2)\right]\right\|=0, \tag{2.16}$$

or mapping Q satisfies the quadratic equation (∗).

Similarly, from (∗∗) and (2.1) for $(r,m) \in I_2$ we get that

$$m^{2n} \left\| f(a_1 m^{-n} x_1 + a_2 m^{-n} x_2) + f(a_2 m^{-n} x_1 - a_1 m^{-n} x_2) \right. \\ \left. -(a_1^2 + a_2^2)\left[f(m^{-n} x_1) + f(m^{-n} x_2)\right]\right\| \\ \leq m^{2n} c K_r \left(m^{-n} \|x_1\|, m^{-n} \|x_2\|\right),$$

holds for all vectors $(x_1, x_2) \in X^2$, and all $n \in \mathbb{N}$ with $f_n(x) = m^{2n} f(m^{-n}x)$: I_2 holds. Thus

$$\begin{aligned}\Big\| \lim_{n\to\infty} f_n(a_1x_1 + a_2x_2) &+ \lim_{n\to\infty} f_n(a_2x_1 - a_1x_2) \\ &- (a_1^2 + a_2^2) \left[\lim_{n\to\infty} f_n(x_1) + \lim_{n\to\infty} f_n(x_2)\right]\Big\| \\ &\leq \left(\lim_{n\to\infty} m^{n(2-r)}\right) cK_r\left(\|x_1\|, \|x_2\|\right) = 0,\end{aligned}$$

because $m^{2-r} < 1$, or (2.16) holds or mapping Q satisfies ($*$).

Therefore (2.16) holds if I_j $(j = 1, 2)$ hold or mapping Q satisfies ($*$), completing the proof that Q *is a quadratic mapping* in X.

It is now clear from (2.4) with $n \to \infty$, as well as formula (2.1) that (2.2) holds in X. This completes *the existence proof* of the above Theorem 2.1. □

Uniqueness

Let $Q' : X \to Y$ *be a quadratic mapping satisfying* (2.2), *as well as* Q. *Then* $Q' = Q$.

Proof. Remember both Q and Q' satisfy (1.3) for $(r, m) \in I_1$, too. Then for every $x \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned}(2.17)\qquad \|Q(x) - Q'(x)\| &\\ = \left\|m^{-2n}Q(m^n x) - m^{-2n}Q'(m^n x)\right\| &\\ \leq m^{-2n}\left\{\|Q(m^n x) - f(m^n x)\| + \|Q'(m^n x) - f(m^n x)\|\right\}&\end{aligned}$$

$$\leq m^{-2n} \frac{2\sigma c}{m^2 - m^r} \|m^n x\|^r$$
$$= m^{n(r-2)} \frac{2\sigma c}{m^2 - m^r} \|x\|^r \to 0, \text{ as } n \to \infty,$$

if I_1 holds: $m^{r-2} < 1$.

Similarly for $(r, m) \in I_2$, we establish

$$\begin{aligned}
(2.18)\quad &\|Q(x) - Q'(x)\| \\
&= \left\|m^{2n} Q(m^{-n}x) - m^{2n} Q'(m^{-n}x)\right\| \\
&\leq m^{2n} \left\{\left\|Q(m^{-n}x) - f(m^{-n}x)\right\| + \left\|Q'(m^{-n}x) - f(m^{-n}x)\right\|\right\} \\
&\leq m^{2n} \frac{2\sigma c}{m^r - m^2} \left\|m^{-n}x\right\|^r \\
&= m^{n(2-r)} \frac{2\sigma c}{m^r - m^2} \|x\|^r \to 0, \text{ as } n \to \infty,
\end{aligned}$$

if I_2 holds: $m^{2-r} < 1$.

Thus from (2.17), and (2.18) we find $Q(x) = Q'(x)$ for all $x \in X$.

This completes the proof of the *uniqueness* and *stability* of equation ($*$). □

Open Problem. *What is the situation in the above Theorem 2.1 in case* $r = 2$?

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