

SUBORDINATION AND SUPERORDINATION RESULTS FOR Φ -LIKE FUNCTIONS

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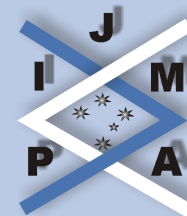
Received: 24 June, 2006

Accepted: 29 December, 2006

Communicated by: N.E. Cho

2000 AMS Sub. Class.: 30C45.

Key words: Differential subordination, Differential superordination, Convolution, Subordinant.



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Superordination**
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vol. 8, iss. 1, art. 20, 2007

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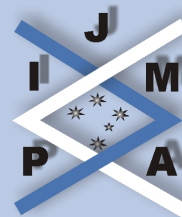
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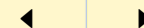
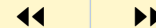
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Abstract:

Let q_1 be convex univalent and q_2 be univalent in $\Delta := \{z : |z| < 1\}$ with $q_1(0) = q_2(0) = 1$. Let f be a normalized analytic function in the open unit disk Δ . Let Φ be an analytic function in a domain containing $f(\Delta)$, $\Phi(0) = 0$ and $\Phi'(0) = 1$. We give some applications of first order differential subordination and superordination to obtain sufficient conditions for the function f to satisfy

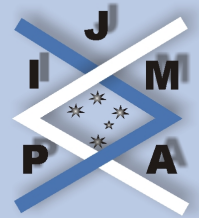
$$q_1(z) \prec \frac{z(f * g)'(z)}{\Phi(f * g)(z)} \prec q_2(z)$$

where g is a fixed function.

Acknowledgements: We would like to thank the referee for his insightful suggestions.

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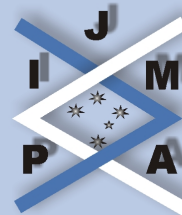
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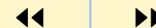
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1. Introduction and Motivations

Let \mathcal{A} be the class of all normalized analytic functions $f(z)$ in the open unit disk $\Delta := \{z : |z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$. Let \mathcal{H} be the class of functions analytic in Δ and for any $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $\mathcal{H}[a, n]$ be the subclass of \mathcal{H} consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$. Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \rightarrow \mathbb{C}$. If p and $\phi(p(z), zp'^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

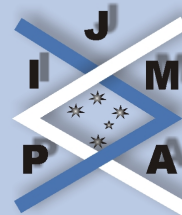
$$(1.1) \quad h(z) \prec \phi(p(z), zp'^2 p''(z); z),$$

then p is a solution of the differential superordination (1.1). If f is subordinate to F , then F is called a superordinate of f . An analytic function q is called a subordinated if $q \prec p$ for all p satisfying (1.1). A univalent subordinated \bar{q} that satisfies $q \prec \bar{q}$ for all subordinated q of (1.1) is said to be the best subordinated. Recently Miller and Mocanu [5] obtained conditions on h, q and ϕ for which the following implication holds:

$$(1.2) \quad h(z) \prec \phi(p(z), zp'^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

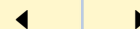
Using the results of Miller and Mocanu [4], Bulboacă [2] considered certain classes of first order differential subordinations as well as superordination-preserving integral operators [1]. In an earlier investigation, Shanmugam et al. [8] obtained sufficient conditions for a normalized analytic function $f(z)$ to satisfy $q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$ and $q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z)$ where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$. A systematic study of the subordination and superordination has been studied very recently by Shanmugam *et al.* in [9] and [10] (see also the references cited by them).

Let Φ be an analytic function in a domain containing $f(\Delta)$ with $\Phi(0) = 0$ and $\Phi'(0) = 1$. For any two analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) =$



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$\sum_{n=0}^{\infty} b_n z^n$, the Hadamard product or convolution of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The function $f \in \mathcal{A}$ is called Φ -like if

$$(1.3) \quad \Re \left(\frac{z f'(z)}{\Phi(f(z))} \right) > 0 \quad (z \in \Delta).$$

The concept of Φ -like functions was introduced by Brickman [3] and he established that a function $f \in \mathcal{A}$ is univalent if and only if f is Φ -like for some Φ . For $\Phi(w) = w$, the function f is starlike. In a later investigation, Ruscheweyh [7] introduced and studied the following more general class of Φ -like functions.

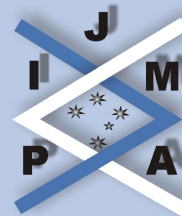
Definition 1.1. Let Φ be analytic in a domain containing $f(\Delta)$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(w) \neq 0$ for $w \in f(\Delta) \setminus \{0\}$. Let $q(z)$ be a fixed analytic function in Δ , $q(0) = 1$. The function $f \in \mathcal{A}$ is called Φ -like with respect to q if

$$(1.4) \quad \frac{z f'(z)}{\Phi(f(z))} \prec q(z) \quad (z \in \Delta).$$

When $\Phi(w) = w$, we denote the class of all Φ -like functions with respect to q by $S^*(q)$.

Using the definition of Φ -like functions, we introduce the following class of functions.

Definition 1.2. Let g be a fixed function in \mathcal{A} . Let Φ be analytic in a domain containing $f(\Delta)$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(w) \neq 0$ for $w \in f(\Delta) \setminus \{0\}$. Let $q(z)$ be a fixed analytic function in Δ , $q(0) = 1$. The function $f \in \mathcal{A}$ is called Φ -like with



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respect to $S_g^*(q)$ if

$$(1.5) \quad \frac{z(f * g)'(z)}{\Phi(f * g)(z)} \prec q(z) \quad (z \in \Delta).$$

We note that $S_{\frac{z}{1-z}}^*(q) := S^*(q)$.

In the present investigation, we obtain sufficient conditions for a normalized analytic function f to satisfy

$$q_1(z) \prec \frac{z(f * g)'(z)}{\Phi(f * g)(z)} \prec q_2(z).$$

We shall need the following definition and results to prove our main results. In this sequel, unless otherwise stated, α and γ are complex numbers.

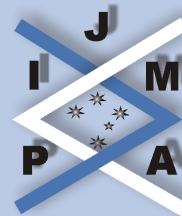
Definition 1.3 ([4, Definition 2, p. 817]). Let Q be the set of all functions f that are analytic and injective on $\bar{\Delta} - E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\Delta : \lim_{z \rightarrow \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

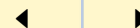
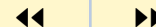
Lemma 1.1 ([4, Theorem 3.4h, p. 132]). *Let q be univalent in the open unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(\omega) \neq 0$ when $\omega \in q(\Delta)$. Set $\xi(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + \xi(z)$. Suppose that*

1. $\xi(z)$ is starlike univalent in Δ , and
2. $\Re \left\{ \frac{zh'(z)}{\xi(z)} \right\} > 0$ ($z \in \Delta$).



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If p is analytic in Δ with $p(\Delta) \subseteq D$ and

$$(1.6) \quad \theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ and q is the best dominant.

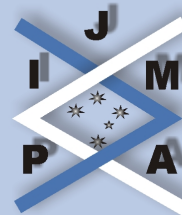
Lemma 1.2. [2, Corollary 3.1, p. 288] Let q be univalent in Δ , ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that

1. $\Re \left[\frac{\vartheta'(q(z))}{\varphi(q(z))} \right] > 0$ for $z \in \Delta$, and
2. $\xi(z) = zq'(z)\varphi(q(z))$ is starlike univalent function in Δ .

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\Delta) \subset D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ , and

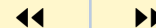
$$(1.7) \quad \vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)),$$

then $q(z) \prec p(z)$ and q is the best subdominant.



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2. Main Results

By making use of Lemma 1.1, we prove the following result.

Theorem 2.1. Let $q(z) \neq 0$ be analytic and univalent in Δ with $q(0) = 1$ such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in Δ . Let $q(z)$ satisfy

$$(2.1) \quad \Re \left[1 + \frac{\alpha q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} \right] > 0.$$

Let

$$(2.2) \quad \Psi(\alpha, \gamma, g; z) := \alpha \left\{ \frac{z(f * g)'(z)}{\Phi(f * g)(z)} \right\} \\ + \gamma \left\{ 1 + \frac{z(f * g)''(z)}{(f * g)'(z)} - \frac{z(\Phi(f * g)(z))'}{\Phi(f * g)(z)} \right\}.$$

If q satisfies

$$(2.3) \quad \Psi(\alpha, \gamma, g; z) \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)},$$

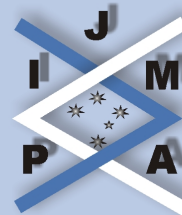
then

$$\frac{z(f * g)'(z)}{\Phi(f * g)(z)} \prec q(z)$$

and q is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$(2.4) \quad p(z) := \frac{z(f * g)'(z)}{\Phi(f * g)(z)}.$$



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Then the function $p(z)$ is analytic in Δ with $p(0) = 1$. By a straightforward computation

$$\frac{zp'(z)}{p(z)} = \left\{ 1 + \frac{z(f * g)''(z)}{(f * g)'(z)} - \frac{z[\Phi(f * g)(z)]'}{\Phi(f * g)(z)} \right\}$$

which, in light of hypothesis (2.3) of Theorem 2.1, yields the following subordination

$$(2.5) \quad \alpha p(z) + \frac{\gamma zp'(z)}{p(z)} \prec \alpha q(z) + \frac{\gamma zq'(z)}{q(z)}.$$

By setting

$$\theta(\omega) := \alpha\omega \quad \text{and} \quad \phi(\omega) := \frac{\gamma}{\omega},$$

it can be easily observed that $\theta(\omega)$ and $\phi(\omega)$ are analytic in $\mathbb{C} \setminus \{0\}$ and that

$$\phi(\omega) \neq 0 \quad (\omega \in \mathbb{C} \setminus \{0\}).$$

Also, by letting

$$(2.6) \quad \xi(z) = zq'(z)\phi(q(z)) = \frac{\gamma}{q(z)}zq'(z).$$

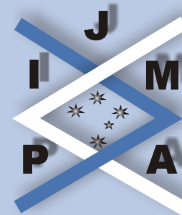
and

$$(2.7) \quad h(z) = \theta\{q(z)\} + \xi(z) = \alpha q(z) + \frac{\gamma}{q(z)}zq'(z),$$

we find that $\xi(z)$ is starlike univalent in Δ and that

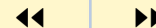
$$\Re \left[1 + \frac{\alpha q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} \right] > 0$$

by the hypothesis (2.1). The assertion of Theorem 2.1 now follows by an application of Lemma 1.1. ■



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When $\Phi(\omega) = \omega$ in Theorem 2.1 we get:

Corollary 2.2. Let $q(z) \neq 0$ be univalent in Δ with $q(0) = 1$. If q satisfies

$$(\alpha - \gamma) \frac{z(f * g)'(z)}{(f * g)(z)} + \gamma \left\{ 1 + \frac{z(f * g)''(z)}{(f * g)'(z)} \right\} \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)},$$

then

$$\frac{z(f * g)'(z)}{(f * g)(z)} \prec q(z)$$

and q is the best dominant.

For $g(z) = \frac{z}{1-z}$ and $\Phi(\omega) = \omega$, we get the following corollary.

Corollary 2.3. Let $q(z) \neq 0$ be univalent in Δ with $q(0) = 1$. If q satisfies

$$(\alpha - \gamma) \frac{z f'(z)}{f(z)} + \gamma \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)},$$

then

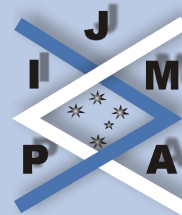
$$\frac{z f'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

For the choice $\alpha = \gamma = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) in Corollary 2.3, we have the following result of Ravichandran and Jayamala [6].

Corollary 2.4. If $f \in \mathcal{A}$ and

$$1 + \frac{z f''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz} + \frac{(A - B)z}{(1 + Az)(1 + Bz)},$$



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then

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Theorem 2.5. Let $\gamma \neq 0$. Let $q(z) \neq 0$ be convex univalent in Δ with $q(0) = 1$ such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in Δ . Suppose that $q(z)$ satisfies

$$(2.8) \quad \Re \left[\frac{\alpha q(z)}{\gamma} \right] > 0.$$

If $f \in \mathcal{A}$, $\frac{z(f * g)'(z)}{\Phi(f * g)(z)} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$, $\Psi(\alpha, \gamma, g; z)$ as defined by (2.2) is univalent in Δ and

$$(2.9) \quad \alpha q(z) + \frac{\gamma z q'(z)}{q(z)} \prec \Psi(\alpha, \gamma, g; z),$$

then

$$q(z) \prec \frac{z(f * g)'(z)}{\Phi(f * g)(z)}$$

and q is the best subdominant.

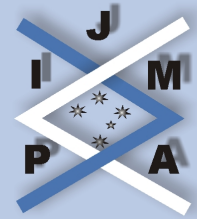
Proof. By setting

$$\vartheta(w) := \alpha w \quad \text{and} \quad \varphi(w) := \frac{\gamma}{w},$$

it is easily observed that $\vartheta(w)$ is analytic in \mathbb{C} , $\varphi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that

$$\varphi(w) \neq 0, \quad (w \in \mathbb{C} \setminus \{0\}).$$

The assertion of Theorem 2.5 follows by an application of Lemma 1.2. ■



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For $\Phi(\omega) = \omega$ in Theorem 2.5, we get

Corollary 2.6. Let $q(z) \neq 0$ be convex univalent in Δ with $q(0) = 1$. If $f \in \mathcal{A}$ and

$$\alpha q(z) + \frac{\gamma z q'(z)}{q(z)} \prec (\alpha - \gamma) \left\{ \frac{z(f * g)'(z)}{(f * g)(z)} \right\} + \gamma \left\{ 1 + \frac{z(f * g)''(z)}{(f * g)'(z)} \right\},$$

then

$$q(z) \prec \frac{z(f * g)'(z)}{(f * g)(z)}$$

and q is the best subordinate.

Combining Theorem 2.1 and Theorem 2.5 we get the following sandwich theorem.

Theorem 2.7. Let q_1 be convex univalent and q_2 be univalent in Δ satisfying (2.8) and (2.1) respectively such that $q_1(0) = 1$, $q_2(0) = 1$, $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ are starlike univalent in Δ with

$$q_1(z) \neq 0 \quad \text{and} \quad q_2(z) \neq 0.$$

Let $f \in \mathcal{A}$, $\frac{z(f * g)'(z)}{\Phi(f * g)(z)} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$, and $\Psi(\alpha, \gamma, g; z)$ as defined by (2.2) be univalent in Δ . Further, if

$$\alpha q_1(z) + \frac{\gamma z q_1'(z)}{q_1(z)} \prec \Psi(\alpha, \gamma, g; z) \prec \alpha q_2(z) + \frac{\gamma z q_2'(z)}{q_2(z)},$$

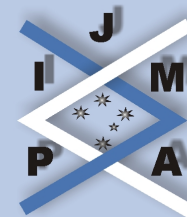
then

$$q_1(z) \prec \frac{z(f * g)'(z)}{\Phi(f * g)(z)} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinate and best dominant.

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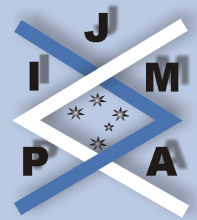
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journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756