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NEWTON'S INEQUALITIES FOR FAMILIES OF COMPLEX NUMBERS

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Abstract
Contents
Home Page
Go Back
Close
Quit

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Abstract

We prove an extension of Newton's inequalities for self-adjoint families of complex numbers in the half plane $\operatorname{Re} z > 0$. The connection of our results with some inequalities on eigenvalues of nonnegative matrices is also discussed.

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Contents

1. Introduction

The well known inequalities of Newton represent quadratic relations among the elementary symmetric functions of n real variables. One of the various consequences of these inequalities is the arithmetic mean-geometric mean (AM-GM) inequality for real nonnegative numbers. The classical book [2] contains different proofs and a detailed study of these results. In the more recent literature, reference [5] offers new families of Newton-type inequalities and an extended treatment of various related issues.

This paper presents an extension of Newton's inequalities involving elementary symmetric functions of complex variables. In particular, we consider $n-$tuples of complex numbers which are symmetric with respect to the real axis and obtain a complex variant of Newton's inequalities and the AM-GM inequality. Families of complex numbers which satisfy the inequalities of Newton in their usual form are also studied and some relations with inequalities on matrix eigenvalues are pointed out.

Let $\mathcal{X}$ be an n-tuple of real numbers $x_1, \dots, x_n$. The i-th elementary symmetric function of $x_1, \dots, x_n$ will be denoted by $e_i(\mathcal{X})$, $i = 0, \dots, n$, i.e.

$$e_0(\mathcal{X}) = 1,\ e_i(\mathcal{X}) = \sum_{1 \le \nu_1 < \dots < \nu_i \le n} x_{\nu_1} x_{\nu_2} \dots x_{\nu_i}, \quad i = 1, \dots, n.$$

By $E_i(\mathcal{X})$ we shall denote the arithmetic mean of the products in $e_i(\mathcal{X})$, i.e.

$$E_i(\mathcal{X}) = \frac{e_i(\mathcal{X})}{\binom{n}{i}}, \quad i = 0, \dots, n.$$

Newton's inequalities are stated in the following theorem [2, Ch. IV].

Theorem 1.1. *If $\mathcal{X}$ is an n-tuple of real numbers $x_1, \dots, x_n$, $x_i \neq 0$, $i = 1, \dots, n$ then*

$$E_i^2(\mathcal{X}) > E_{i-1}(\mathcal{X})E_{i+1}(\mathcal{X}), \quad i = 1, \dots, n-1 \tag{1.1}$$

unless all entries of $\mathcal{X}$ coincide.

The requirement that $x_i \neq 0$ actually is not a restriction. In general, for real x_i, $i = 1, \dots, n$

$$E_i^2(\mathcal{X}) \geq E_{i-1}(\mathcal{X})E_{i+1}(\mathcal{X}), \quad i = 1, \dots, n-1$$

and only characterizing all cases of equality is more complicated.

Inequalities (1.1) originate from the problem of finding a lower bound for the number of imaginary (nonreal) roots of an algebraic equation. Such a lower bound is given by the Newton's rule: *Given an equation with real coefficients*

$$a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0, \quad a_0 \neq 0$$

the number of its imaginary roots cannot be less than the number of sign changes that occur in the sequence

$$a_0^2, \left(\frac{a_1}{\binom{n}{1}}\right)^2 - \frac{a_2}{\binom{n}{2}} \cdot \frac{a_0}{\binom{n}{0}}, \dots, \left(\frac{a_{n-1}}{\binom{n}{n-1}}\right)^2 - \frac{a_n}{\binom{n}{n}} \cdot \frac{a_{n-2}}{\binom{n}{n-2}}, \ a_n^2.$$

According to this rule, if all roots are real, then all entries in the above sequence must be nonnegative which yields Newton's inequalities.

A chain of inequalities, due to Maclaurin, can be derived from (1.1), e.g. see [2] and [5].

Theorem 1.2. *If $\mathcal{X}$ is an n-tuple of positive numbers, then*

$$E_1(\mathcal{X}) > E_2^{1/2}(\mathcal{X}) > \cdots > E_n^{1/n}(\mathcal{X}) \tag{1.2}$$

unless all entries of $\mathcal{X}$ coincide.

The above theorem implies the well known AM-GM inequality $E_1(\mathcal{X}) \geq E_n^{1/n}(\mathcal{X})$ for every $\mathcal{X}$ with nonnegative entries.

Newton did not give a proof of his rule and subsequently inequalities (1.1) and (1.2) were proved by Maclaurin. A proof of (1.1) based on a lemma of Maclaurin is given in Ch. IV of [2] and an inductive proof is presented in Ch. II of [2]. In the same reference it is also shown that the difference $E_i^2(\mathcal{X}) - E_{i-1}(\mathcal{X})E_{i+1}(\mathcal{X})$ can be represented as a sum of obviously nonnegative terms formed by the entries of $\mathcal{X}$ which again proves (1.1). Yet another equality which implies Newton's inequalities is the following.

Let $f(z) = \sum_{i=0}^{n} a_i z^{n-i}$ be a monic polynomial with $a_i \in \mathbb{C}$, $i = 1, \ldots, n$. For each $i = 1, \ldots, n-1$ such that $a_{i+1} \neq 0$, we have

$$\left(\frac{a_i}{\binom{n}{i}}\right)^2 - \frac{a_{i-1}}{\binom{n}{i-1}} \cdot \frac{a_{i+1}}{\binom{n}{i+1}} = \frac{1}{i(i+1)^2}\left(\prod_{k=1}^{i+1}\lambda_k\right)^2 \sum_{j<k}\left(\lambda_j^{-1} - \lambda_k^{-1}\right)^2, \tag{1.3}$$

where λ_k, $k = 1, \ldots, i+1$ are zeros of the $(n-i-1)$-st derivative $f^{(n-i-1)}(z)$ of $f(z)$. Indeed, let e_k, $k = 0, \ldots, i+1$ denote the elementary symmetric functions of $\lambda_1, \ldots, \lambda_{i+1}$. Since

$$f^{(n-i-1)}(z) = \sum_{k=0}^{i+1} \frac{(n-k)!}{(i+1-k)!} a_k z^{i+1-k},$$

we have

$$e_k = (-1)^k \frac{(i+1)!(n-k)!}{n!(i+1-k)!} a_k, \;\; k = 0, \dots, i+1$$

and hence

$$\left(\frac{a_i}{\binom{n}{i}} \right)^2 - \frac{a_{i-1}}{\binom{n}{i-1}} \cdot \frac{a_{i+1}}{\binom{n}{i+1}} = \frac{e_{i+1}^2}{i(i+1)^2} \left(i \left(\frac{e_i}{e_{i+1}} \right)^2 - 2(i+1) \frac{e_{i-1}}{e_{i+1}} \right) \tag{1.4}$$

which gives equality (1.3).

Now, if all zeros of $f(z)$ are real, then by the Rolle theorem all zeros of each derivative of $f(z)$ are also real and thus Newton's inequalities follow from (1.3).

2. Complex Newton's Inequalities

In what follows, we shall consider n-tuples of complex numbers $z_1, \dots, z_n$ denoted by $\mathcal{Z}$. As in the real case, $e_i(\mathcal{Z})$ will be the i-th elementary symmetric function of $\mathcal{Z}$ and $E_i(\mathcal{Z}) = e_i(\mathcal{Z}) \big/ \binom{n}{i}$, $i = 0, \dots, n$. In the next theorem, it is assumed that $\mathcal{Z}$ satisfies the following two conditions.

(C1) $\operatorname{Re} z_i \geq 0,\ i = 1, \dots, n$ where $\operatorname{Re} z_i = 0$ only if $z_i = 0$;

(C2) $\mathcal{Z}$ is self-conjugate, i.e. the non-real entries of $\mathcal{Z}$ appear in complex conjugate pairs.

Note that $\mathcal{Z}$ satisfies (C2) if and only if all elementary symmetric functions of $\mathcal{Z}$ are real. Conditions (C1) and (C2) together imply that $e_i(\mathcal{Z}) \geq 0$, $i = 0, \dots, n$.

Theorem 2.1. *Let $\mathcal{Z}$ be an n-tuple of complex numbers $z_1, \dots, z_n$ satisfying conditions (C1) and (C2) and let $-\varphi \leq \arg z_i \leq \varphi$, $i = 1, \dots, n$ where $0 \leq \varphi < \pi/2$. Then*

$$c^2 E_i^2(\mathcal{Z}) \geq E_{i-1}(\mathcal{Z}) E_{i+1}(\mathcal{Z}), \quad i = 1, \dots, n-1 \tag{2.1}$$

and

$$c^{n-1} E_1(\mathcal{Z}) \geq c^{n-2} E_2^{1/2}(\mathcal{Z}) \geq \cdots \geq c E_{n-1}^{1/(n-1)}(\mathcal{Z}) \geq E_n^{1/n}(\mathcal{Z}) \tag{2.2}$$

where $c = (1 + \tan^2 \varphi)^{1/2}$.

Title Page
Contents
◀◀ ▶▶
◀ ▶
Go Back
Close
Quit
Page 7 of 17

Proof. Let W_φ be defined by

$$W_\varphi = \{z \in \mathbb{C} : -\varphi \le \arg z \le \varphi\}$$

and consider the polynomial

$$f(z) = \prod_{i=1}^{n}(z - z_i) = \sum_{i=0}^{n} a_i z^{n-i} \tag{2.3}$$

with coefficients

$$a_i = (-1)^i \binom{n}{i} E_i(\mathcal{Z}), \quad i = 0, \dots, n. \tag{2.4}$$

If for some $i = 1, \dots, n-1$, $E_{i+1}(\mathcal{Z}) = 0$ then the corresponding inequality in (2.1) is obviously satisfied. For each $i = 1, \dots, n-1$ such that $E_{i+1}(\mathcal{Z}) \ne 0$ let $\lambda_1, \dots, \lambda_{i+1}$ denote the zeros of $f^{(n-i-1)}(z)$. As in (1.4), it is easily seen that

$$\begin{aligned} &c^2 E_i^2(\mathcal{Z}) - E_{i-1}(\mathcal{Z}) E_{i+1}(\mathcal{Z}) \\ &= \frac{1}{i(i+1)^2} \left(\prod_{k=1}^{i+1} \lambda_k \right)^2 \left(i(1+\tan^2\varphi) \left(\sum_{k=1}^{i+1} \lambda_k^{-1} \right)^2 - 2(i+1) \sum_{j<k} \lambda_j^{-1} \lambda_k^{-1} \right). \end{aligned} \tag{2.5}$$

Let $\alpha_k = \operatorname{Re} \lambda_k^{-1}$ and $\beta_k = \operatorname{Im} \lambda_k^{-1}$, $k = 1, \dots, i+1$. Since the zeros of $f(z)$ lie in the convex area W_φ, by the Gauss-Lucas theorem, λ_k, and hence λ_k^{-1}, $k = 1, \dots, i+1$ also lie in W_φ which implies that

$$\alpha_k \ge \frac{|\beta_k|}{\tan\varphi}, \quad k = 1, \dots, i+1. \tag{2.6}$$

Using (2.6) and the inequality $\operatorname{Re}\lambda_j^{-1}\lambda_k^{-1} \le \alpha_j\alpha_k + |\beta_j|\,|\beta_k|$ in (2.5), it is obtained

$$c^2E_i^2(\mathcal{Z}) - E_{i-1}(\mathcal{Z})E_{i+1}(\mathcal{Z}) \ge \frac{1}{i(i+1)^2}\left(\prod_{k=1}^{i+1}\lambda_k\right)^2 \sum_{j<k}\left((\alpha_j-\alpha_k)^2 + (|\beta_j|-|\beta_k|)^2\right),$$

which proves (2.1).

Inequalities (2.2) can be obtained from (2.1) similarly as in the real case. From (2.1) we have

$$c^2E_1^2c^4E_2^4\cdots c^{2i}E_i^{2i} \ge E_0E_2(E_1E_3)^2\cdots(E_{i-1}E_{i+1})^i$$

which gives $c^{i(i+1)}E_i^{i+1} \ge E_{i+1}^i$, or equivalently

$$cE_1 \ge E_2^{1/2},\ cE_2^{1/2} \ge E_3^{1/3}, \dots, cE_{n-1}^{1/(n-1)} \ge E_n^{1/n}.$$

Multiplying each inequality $cE_i^{1/i} \ge E_{i+1}^{1/(i+1)}$ by c^{n-i-1} for $i = 1,\dots,n-2$, we obtain (2.2). □

Inequalities (2.2) yield a complex version of the AM-GM inequality, i.e.

$$c^{n-1}E_1(\mathcal{Z}) \ge E_n^{1/n}(\mathcal{Z}) \tag{2.7}$$

for every $\mathcal{Z}$ satisfying conditions (C1) and (C2). It is easily seen that a case of equality occurs in (2.1), (2.2) and (2.7) if $n = 2$ and $\mathcal{Z}$ consists of a pair of complex conjugate numbers $z_1 = \alpha + i\beta$ and $z_2 = \alpha - i\beta$ with $\tan\varphi = \beta/\alpha$.

Another simple observation is that under the conditions of Theorem 2.1, inequalities (2.1) also hold for $-\mathcal{Z}$ given by $-z_1, \dots, -z_n$. This follows immediately since $E_i(-\mathcal{Z}) = (-1)^i E_i(\mathcal{Z})$, $i = 0, \dots, n$.

The next theorem indicates that if $\mathcal{Z}$ satisfies an additional condition then one can find n-tuples of complex numbers satisfying a complete analog of Newton's inequalities.

Theorem 2.2. *Let $\mathcal{Z}$ be an n-tuple of complex numbers $z_1, \dots, z_n$ satisfying condition (C2) and let*

$$E_1^2(\mathcal{Z}) - E_2(\mathcal{Z}) > 0. \tag{2.8}$$

Then there is a real $r \geq 0$ such that the shifted n-tuple $\mathcal{Z}_\alpha$

$$z_1 - \alpha,\ z_2 - \alpha, \dots, z_n - \alpha \tag{2.9}$$

satisfies

$$E_i^2(\mathcal{Z}_\alpha) > E_{i-1}(\mathcal{Z}_\alpha) E_{i+1}(\mathcal{Z}_\alpha), \quad i = 1, \dots, n-1 \tag{2.10}$$

for all real α with $|\alpha| \geq r$.

Proof. The complex numbers (2.9) are zeros of the polynomial

$$f(z+\alpha) = \frac{f^{(n)}(\alpha)}{n!} z^n + \frac{f^{(n-1)}(\alpha)}{(n-1)!} z^{n-1} + \dots + f(\alpha),$$

where $f(z)$ is given by (2.3) and (2.4). Thus

$$E_i(\mathcal{Z}_\alpha) = \frac{(-1)^i}{\binom{n}{i}} \cdot \frac{f^{(n-i)}(\alpha)}{(n-i)!}, \quad i = 0, \dots, n.$$

By writing $f^{(n-i)}(\alpha)$ in the form

$$f^{(n-i)}(\alpha) = (n-i)! \sum_{k=0}^{i} \binom{n-k}{n-i} a_k \alpha^{i-k}, \quad i = 0, \dots, n$$

and taking into account (2.4), it is obtained

$$E_i(\mathcal{Z}_\alpha) = (-1)^i \sum_{k=0}^{i} (-1)^k \binom{i}{k} E_k(\mathcal{Z}) \alpha^{i-k}, \quad i = 0, \dots, n \tag{2.11}$$

Now, using (2.11) one can easily find that

$$\begin{aligned} E_i^2(\mathcal{Z}_\alpha) - E_{i-1}(\mathcal{Z}_\alpha)E_{i+1}(\mathcal{Z}_\alpha) &= 0 \cdot \alpha^{2i} + 0 \cdot \alpha^{2i-1} + \left(E_1^2(\mathcal{Z}) - E_2(\mathcal{Z})\right) \alpha^{2i-2} + \\ &\qquad \cdots + E_i^2(\mathcal{Z}) - E_{i-1}(\mathcal{Z})E_{i+1}(\mathcal{Z}). \end{aligned} \tag{2.12}$$

From (2.8) and (2.12), it is seen that for each $i = 1, \dots, n-1$ there is $r_i \geq 0$ such that the right-hand side of (2.12) is greater than zero for all $|\alpha| \geq r_i$. Hence, inequalities (2.10) are satisfied for all $|\alpha| \geq r$, where $r = \max\{r_i : i = 1, \dots, n-1\}$. □

If α in the above proposition is chosen such that $\mathrm{Re}(z_i - \alpha) > 0$, $i = 1, \dots, n$ then all the elementary symmetric functions of $\mathcal{Z}_\alpha$ are positive and inequalities (2.10) yield

$$E_1(\mathcal{Z}_\alpha) > E_2^{1/2}(\mathcal{Z}_\alpha) > \cdots > E_n^{1/n}(\mathcal{Z}_\alpha). \tag{2.13}$$

In this case, the AM-GM inequality for $\mathcal{Z}_\alpha$ follows from (2.13).

3. Newton's Inequalities on Matrix Eigenvalues

In a recent work [3] the inequalities of Newton are studied in relation with the eigenvalues of a special class of matrices, namely M-matrices. An $n \times n$ real matrix A is an M-matrix iff [1]

$$A = \alpha I - P, \tag{3.1}$$

where P is a matrix with nonnegative entries and $\alpha > \rho(P)$, where $\rho(P)$ is the spectral radius (Perron root) of P. Let $\mathcal{Z}$ and $\mathcal{Z}_\alpha$ denote the $n-$tuples $z_1, \dots, z_n$ and $\alpha - z_1, \dots, \alpha - z_n$ of the eigenvalues of P and A, respectively. In terms of this notation, it is proved in [3] that

$$E_i^2(\mathcal{Z}_\alpha) \geq E_{i-1}(\mathcal{Z}_\alpha) E_{i+1}(\mathcal{Z}_\alpha), \quad i = 1, \dots, n-1 \tag{3.2}$$

for all $\alpha > \rho(P)$, i.e. the eigenvalues of A satisfy Newton's inequalities. The proof is based on inequalities involving principal minors of A and nonnegativity of a quadratic form. As a consequence of (3.2) and the property of M-matrices that $E_i(\mathcal{Z}_\alpha) > 0$, $i = 1, \dots, n$, the eigenvalues of A satisfy the AM-GM inequality, a fact which can be directly seen from

$$det\, A \leq \prod_{i=1}^{n} a_{ii} \leq \left(\frac{1}{n} \sum_{i=1}^{n} a_{ii} \right)^n,$$

where $a_{ii} > 0$, $i = 1, \dots, n$ are the diagonal entries of A, the first inequality is the Hadamard inequality for M-matrices and the second inequality is the usual AM-GM inequality.

In view of Theorem 2.2 above, it is easily seen that one can find other matrix classes described in the form (3.1) and satisfying Newton's inequalities. In particular, if $\mathcal{Z}$ denotes the $n-$tuple of the eigenvalues of a real matrix $B = [b_{ij}]$, $i,j = 1,\dots,n$ then the left hand side of (2.8) can be written as

$$E_1^2(\mathcal{Z}) - E_2(\mathcal{Z}) = \frac{1}{n^2}\left(\sum_{i=1}^{n} b_{ii}\right)^2 - \frac{2}{n(n-1)}\sum_{i<j}(b_{ii}b_{jj} - b_{ij}b_{ji}). \tag{3.3}$$

By the first inequality of Newton applied to $b_{11},\dots,b_{nn}$, it follows from (3.3) that condition (2.8) is satisfied if

$$b_{ij}b_{ji} \geq 0, \quad 1 \leq i < j \leq n \tag{3.4}$$

with at least one strict inequality. According to Theorem 2.2, in this case there is $r \geq 0$ such that the eigenvalues of $A = \alpha I - B$ satisfy (2.10) for $|\alpha| \geq r$. It should be noted that matrices satisfying (3.4) include the class of weakly sign symmetric matrices.

Next, we consider the inequalities of Loewy, London and Johnson [1] (LLJ inequalities) on the eigenvalues of nonnegative matrices and point out a close relation with Newton's inequalities.

Let $A \geq 0$ denote an entry-wise nonnegative matrix $A = [a_{ij}]$, $i,j = 1,\dots,n$, $\operatorname{tr} A$ be the trace of A, i.e. $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$ and let S_k denote the $k-$th power sum of the eigenvalues $z_1,\dots,z_n$ of A :

$$S_k = \sum_{i=1}^{n} z_i^k, \; k = 1,2,\dots.$$

Due to the nonnegativity of A, we have

$$tr(A^k) \geq \sum_{i=1}^{n} a_{ii}^k \tag{3.5}$$

and since $S_k = \mathrm{tr}(A^k)$, it follows that $S_k \geq 0$ for each $k = 1, 2, \ldots$. The LLJ inequalities actually show something more, i.e.

$$n^{m-1} S_{km} \geq (S_k)^m, \quad k, m = 1, 2, \ldots \tag{3.6}$$

or equivalently,

$$n^{m-1} \,\mathrm{tr}\left((A^k)^m\right) \geq \left(\mathrm{tr}(A^k)\right)^m, \quad k, m = 1, 2, \ldots. \tag{3.7}$$

Equalities hold in (3.6) and (3.7) if A is a scalar matrix $A = \alpha I$. Obviously, in order to prove (3.7) it suffices to show that

$$n^{m-1} \,\mathrm{tr}(A^m) \geq (\mathrm{tr}\, A)^m, \quad m = 1, 2, \ldots \tag{3.8}$$

for every $A \geq 0$. The key to the proof of (3.8) are inequalities

$$n^{m-1} \sum_{i=1}^{n} x_i^m - \left(\sum_{i=1}^{n} x_i\right)^m \geq 0, \quad m = 1, 2, \ldots \tag{3.9}$$

which hold for nonnegative $x_1, \ldots, x_n$ and can be deduced from Hölder's inequalities, e.g. see [1], [4]. Since $A \geq 0$, (3.9) together with (3.5) imply (3.8).

From the point of view of Newton's inequalities, it can be easily seen that the case $m = 2$ in (3.9) follows from

$$\begin{aligned} E_1^2(\mathcal{X}) - E_2(\mathcal{X}) &= \frac{1}{n^2(n-1)} \left((n-1)\, e_1^2(\mathcal{X}) - 2n\, e_2(\mathcal{X})\right) \\ &= \frac{1}{n^2(n-1)} \left(n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i \right)^2 \right) \\ &= \frac{1}{n^2(n-1)} \sum_{i<j} (x_i - x_j)^2 \geq 0. \end{aligned}$$

Thus, (3.9) holds for $m = 1$ (trivially), $m = 2$ and the rest of the inequalities can be obtained by induction on m. Also, following this approach, the inequalities in (3.6) for $m = 2$ and $k = 1, 2, \dots$ can be obtained directly from

$$\begin{aligned} n \sum_{i=1}^{n} z_i^{2k} - \left(\sum_{i=1}^{n} z_i^k \right)^2 &= (n-1)\, e_1^2(\mathcal{Z}^k) - 2n\, e_2(\mathcal{Z}^k) \\ &= (n-1) \left(\sum_{i=1}^{n} a_{ii}^{[k]} \right)^2 - 2n \sum_{i<j} \left(a_{ii}^{[k]} a_{jj}^{[k]} - a_{ij}^{[k]} a_{ji}^{[k]} \right) \\ &\geq (n-1) \left(\sum_{i=1}^{n} a_{ii}^{[k]} \right)^2 - 2n \sum_{i<j} a_{ii}^{[k]} a_{jj}^{[k]} \\ &= \sum_{i<j} \left(a_{ii}^{[k]} - a_{jj}^{[k]} \right)^2 \geq 0 \end{aligned}$$

where $\mathcal{Z}^k$ is the $n-$tuple $z_1^k, \dots, z_n^k$ of the eigenvalues of A^k and $a_{ij}^{[k]}$ denotes the $(i,j)-$th element of A^k, $i,j = 1,\dots,n$, $k = 1,2,\dots$. Clearly, equalities hold if and only if A^k is a scalar matrix.

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