

STABILITY OF A GENERALIZED MIXED TYPE ADDITIVE, QUADRATIC, CUBIC AND QUARTIC FUNCTIONAL EQUATION

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K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
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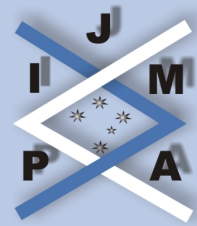
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Abstract:

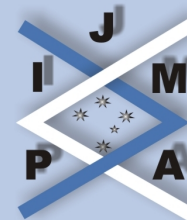
In this paper, we obtain the general solution and the generalized Hyers-Ulam-Rassias stability of the generalized mixed type of functional equation

$$\begin{aligned} & f(x + ay) + f(x - ay) \\ &= a^2 [f(x + y) + f(x - y)] + 2(1 - a^2) f(x) \\ & \quad + \frac{(a^4 - a^2)}{12} [f(2y) + f(-2y) - 4f(y) - 4f(-y)]. \end{aligned}$$

for fixed integers a with $a \neq 0, \pm 1$.

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1. Introduction

S.M. Ulam [31] is the pioneer of the stability problem in functional equations. In 1940, while he was delivering a talk before the Mathematics Club of the University of Wisconsin, he discussed a number of unsolved problems. Among them was the following question concerning the stability of homomorphisms:

"Let G be group and H be a metric group with metric $d(\cdot, \cdot)$. Given $\epsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \rightarrow H$ satisfies

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $a : G \rightarrow H$ with

$$d(f(x), a(x)) < \epsilon$$

for all $x \in G$."

In 1941, D.H. Hyers [12] gave the first affirmative partial answer to the question of Ulam for Banach spaces. He proved the following celebrated theorem.

Theorem 1.1 ([12]). *Let X, Y be Banach spaces and let $f : X \rightarrow Y$ be a mapping satisfying*

$$(1.1) \quad \|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

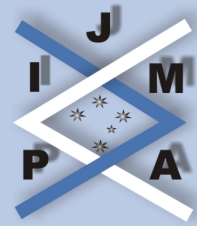
for all $x, y \in X$. Then the limit

$$(1.2) \quad a(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in X$ and $a : X \rightarrow Y$ is the unique additive mapping satisfying

$$(1.3) \quad \|f(x) - a(x)\| \leq \epsilon$$

for all $x \in X$.



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In 1950, Aoki [2] generalized the Hyers theorem for additive mappings. In 1978, Th.M. Rassias [26] provided a generalized version of the Hyers theorem which permitted the Cauchy difference to become unbounded. He proved the following:

Theorem 1.2 ([26]). *Let X be a normed vector space and Y be a Banach space. If a function $f : X \rightarrow Y$ satisfies the inequality*

$$(1.4) \quad \|f(x+y) - f(x) - f(y)\| \leq \theta (\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, where θ and p are constants with $\theta > 0$ and $p < 1$, then the limit

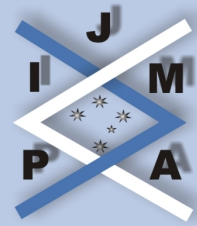
$$(1.5) \quad T(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in X$ and $T : X \rightarrow Y$ is the unique additive mapping which satisfies

$$(1.6) \quad \|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p} \|x\|^p$$

for all $x \in X$. If $p < 0$, then inequality (1.4) holds for $x, y \neq 0$ and (1.6) for $x \neq 0$. Also if for each $x \in X$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then T is linear.

It was shown by Z. Gajda [9], as well as Th.M. Rassias and P. Semrl [27] that one cannot prove a Th.M. Rassias type theorem when $p = 1$. The counter examples of Z. Gajda, as well as of Th.M. Rassias and P. Semrl [27] have stimulated several mathematicians to invent new definitions of approximately additive or approximately linear mappings; P. Gavruta [10] and S.M. Jung [17] among others have studied the Hyers-Ulam-Rassias stability of functional equations. The inequality (1.4) that was introduced by Th.M. Rassias [26] provided much influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as the Hyers-Ulam-Rassias stability of functions.



In 1982, J.M. Rassias [24] following the spirit of the approach of Th.M. Rassias [26] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

Theorem 1.3 ([24]). *Let X be a real normed linear space and Y be a real completed normed linear space. Assume that $f : X \rightarrow Y$ is an approximately additive mapping for which there exists constants $\theta > 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the inequality*

$$(1.7) \quad \|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$. Then the limit

$$(1.8) \quad L(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in X$ and $L : X \rightarrow Y$ is the unique additive mapping which satisfies

$$(1.9) \quad \|f(x) - L(x)\| \leq \frac{\theta}{|2 - 2^r|} \|x\|^r$$

for all $x \in X$. If, in addition $f : X \rightarrow Y$ is a mapping such that the transformation $t \rightarrow f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an \mathbb{R} -linear mapping.

However, the case $r = 1$ in inequality (1.9) is singular. A counter example has been given by P. Gavruta [11]. The above-mentioned stability involving a product of different powers of norms was called Ulam-Gavruta-Rassias stability by M.A. Sibaha et al., [30], as well as by K. Ravi and M. Arunkumar [28]. This stability result was also called the Hyers-Ulam-Rassias stability involving a product of different powers of norms by Park [23].

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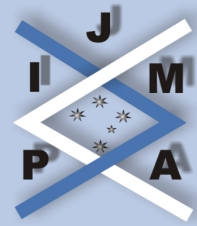
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In 1994, a generalization of Th.M. Rassias' theorem and J.M. Rassias' theorem was obtained by P. Gavruta [10], who replaced the factors $\theta(\|x\|^p + \|y\|^p)$ and $\theta(\|x\|^p\|y\|^p)$ by a general control function $\varphi(x, y)$. In the past few years several mathematicians have published various generalizations and applications of Hyers-Ulam-Rassias stability to a number of functional equations and mappings (see [4, 5, 13, 18, 19]). Very recently, J.M. Rassias [29] in the inequality (1.7) replaced the bound by a mixed one involving the product and sum of powers of norms, that is, $\theta\{\|x\|^p\|y\|^p + (\|x\|^{2p} + \|y\|^{2p})\}$.

The functional equation

$$(1.10) \quad f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is said to be a *quadratic functional equation* because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.10). A quadratic functional equation was used to characterize inner product spaces [1, 20]. It is well known that a function f is a solution of (1.10) if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all x (see [20]). The biadditive function B is given by

$$(1.11) \quad B(x, y) = \frac{1}{4}[f(x+y) + f(x-y)].$$

The functional equation

$$(1.12) \quad f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$$

is called a *cubic functional equation*, because the cubic function $f(x) = cx^3$ is a solution of the equation (1.12). The general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.12) was discussed by K.W. Jun and H.M. Kim [14]. They proved that a function f between real vector spaces

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X and Y is a solution of (1.12) if and only if there exists a unique function $C : X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$ and C is symmetric for each fixed one variable and is additive for fixed two variables.

The *quartic functional equation*

$$(1.13) \quad f(x + 2y) + f(x - 2y) - 6f(x) = 4[f(x + y) + f(x - y)] + 24f(y)$$

was introduced by J.M. Rassias [25]. Later S.H. Lee et al., [21] remodified J.M. Rassias's equation and obtained a new quartic functional equation of the form

$$(1.14) \quad f(2x + y) + f(2x - y) = 4[f(x + y) + f(x - y)] + 24f(x) - 6f(y)$$

and discussed its general solution. In fact S.H. Lee et al., [21] proved that a function f between vector spaces X and Y is a solution of (1.14) if and only if there exists a unique symmetric multi-additive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f(x) = Q(x, x, x, x)$ for all $x \in X$. It is easy to show that the function $f(x) = kx^4$ is the solution of (1.13) and (1.14).

A function

$$(1.15) \quad f(x) = Q(x_1, x_2, x_3, x_4)$$

is called symmetric multi additive if Q is additive with respect to each variable x_i , $i = 1, 2, 3, 4$ in (1.15).

A function f is defined as

$$f(x) = \frac{\beta(x) - \alpha(x)}{12}$$

where $\alpha(x) = f(2x) - 16f(x)$, $\beta(x) = f(2x) - 4f(x)$, further, f satisfies $f(2x) = 4f(x)$ and $f(2x) = 16f(x)$ is said to be a *quadratic - quartic function*.

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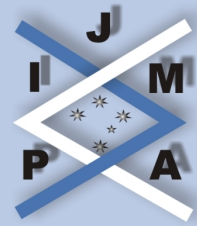
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K.W. Jun and H.M. Kim [16] introduced the following generalized *quadratic and additive type functional equation*

$$(1.16) \quad f\left(\sum_{i=1}^n x_i\right) + (n-2) \sum_{i=1}^n f(x_i) = \sum_{1 \leq i < j \leq n} f(x_i + x_j)$$

in the class of functions between real vector spaces. For $n = 3$, Pl. Kannapan proved that a function f satisfies the functional equation (1.16) if and only if there exists a symmetric bi-additive function A and an additive function B such that $f(x) = B(x, x) + A(x)$ for all x (see [20]). The Hyers-Ulam stability for the equation (1.16) when $n = 3$ was proved by S.M. Jung [18]. The Hyers-Ulam-Rassias stability for the equation (1.16) when $n = 4$ was also investigated by I.S. Chang et al., [3].

The general solution and the generalized Hyers-Ulam stability for the *quadratic and additive type functional equation*

$$(1.17) \quad f(x + ay) + af(x - y) = f(x - ay) + af(x + y)$$

for any positive integer a with $a \neq -1, 0, 1$ was discussed by K.W. Jun and H.M. Kim [15]. Recently A. Najati and M.B. Moghimi [22] investigated the generalized Hyers-Ulam-Rassias stability for a *quadratic and additive type functional equation* of the form

$$(1.18) \quad f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 2f(2x) - 4f(x)$$

Very recently, the authors [6, 7] investigated a mixed type functional equation of cubic and quartic type and obtained its general solution. The stability of generalized mixed type functional equations of the form

$$(1.19) \quad f(x + ky) + f(x - ky) = k^2 [f(x + y) + f(x - y)] + 2(1 - k^2) f(x)$$

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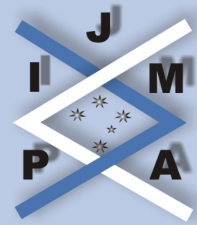
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for fixed integers k with $k \neq 0, \pm 1$ in quasi -Banach spaces was investigated by M. Eshaghi Gordji and H. Khodaie [8]. The mixed type functional equation (1.19) is additive, quadratic and cubic.

In this paper, the authors introduce a mixed type functional equation of the form

$$(1.20) \quad f(x + ay) + f(x - ay) \\ = a^2 [f(x + y) + f(x - y)] + 2(1 - a^2) f(x) \\ + \frac{a^4 - a^2}{12} [f(2y) + f(-2y) - 4f(y) - 4f(-y)]$$

which is additive, quadratic, cubic and quartic and obtain its general solution and generalized Hyers-Ulam-Rassias stability for fixed integers a with $a \neq 0, \pm 1$.

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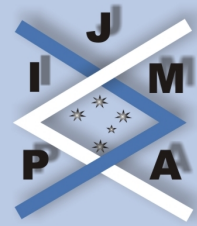
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2. General Solution

In this section, we present the general solution of the functional equation (1.20). Throughout this section let E_1 and E_2 be real vector spaces.

Theorem 2.1. *Let $f : E_1 \rightarrow E_2$ be a function satisfying (1.20) for all $x, y \in E_1$. If f is even then f is quadratic - quartic.*

Proof. Let f be an even function, i.e., $f(-x) = f(x)$. Then equation (1.20) becomes

$$(2.1) \quad \begin{aligned} f(x+ay) + f(x-ay) \\ = a^2 [f(x+y) + f(x-y)] + 2(1-a^2)f(x) \\ \quad + \frac{a^4-a^2}{6} [f(2y) - 4f(y)] \end{aligned}$$

for all $x, y \in E_1$. Interchanging x and y in (2.1) and using the evenness of f , we get

$$(2.2) \quad \begin{aligned} f(ax+y) + f(ax-y) \\ = a^2 [f(x+y) + f(x-y)] + 2(1-a^2)f(y) \\ \quad + \frac{a^4-a^2}{6} [f(2x) - 4f(x)] \end{aligned}$$

for all $x, y \in E_1$. Setting (x, y) as $(0, 0)$ in (2.2), we obtain $f(0) = 0$. Replacing y by $x+y$ in (2.2) and using the evenness of f , we have

$$(2.3) \quad \begin{aligned} f((a+1)x+y) + f((a-1)x-y) \\ = a^2 [f(2x+y) + f(y)] + 2(1-a^2)f(x+y) \\ \quad + \frac{a^4-a^2}{6} [f(2x) - 4f(x)] \end{aligned}$$

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for all $x, y \in E_1$. Replacing y by $x - y$ in (2.2), we obtain

$$(2.4) \quad \begin{aligned} f((a+1)x - y) + f((a-1)x + y) \\ = a^2 [f(2x - y) + f(y)] + 2(1 - a^2) f(x - y) \\ + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)] \end{aligned}$$

for all $x, y \in E_1$. Adding (2.3) and (2.4), we get

$$(2.5) \quad \begin{aligned} f((a+1)x + y) + f((a-1)x - y) + f((a+1)x - y) \\ + f((a-1)x + y) \\ = a^2 [f(2x + y) + f(2x - y) + 2f(y)] + 2(1 - a^2) [f(x + y) + f(x - y)] \\ + \frac{a^4 - a^2}{6} [2f(2x) - 8f(x)] \end{aligned}$$

for all $x, y \in E_1$. Replacing y by $ax + y$ in (2.2), we obtain

$$(2.6) \quad \begin{aligned} f(2ax + y) + f(y) = a^2 [f((a+1)x + y) + f((1-a)x - y)] \\ + 2(1 - a^2) f(ax + y) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)] \end{aligned}$$

for all $x, y \in E_1$. Replacing y by $ax - y$ in (2.3), we get

$$(2.7) \quad \begin{aligned} f(2ax - y) + f(y) = a^2 [f((a+1)x - y) + f((1-a)x + y)] \\ + 2(1 - a^2) f(ax - y) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)] \end{aligned}$$

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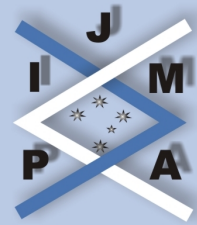
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for all $x, y \in E_1$. Adding (2.6) and (2.7), we obtain

$$(2.8) \quad f(2ax + y) + f(2ax - y) + 2f(y) \\
= a^2 [f((a+1)x + y) + f((a+1)x - y) + f((a-1)x + y) + f((a-1)x - y)] \\
+ 2(1 - a^2) [f(ax + y) + f(ax - y)] + \frac{a^4 - a^2}{3} [f(2x) - 4f(x)]$$

for all $x, y \in E_1$. Using (2.5) in (2.8), we arrive at

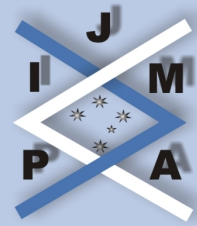
$$(2.9) \quad f(2ax + y) + f(2ax - y) + 2f(y) \\
= a^4 [f(2x + y) + f(2x - y)] + 2a^4 f(y) + 2a^2 (1 - a^2) [f(x + y) + f(x - y)] \\
+ \frac{a^2(a^4 - a^2)}{3} [f(2x) - 4f(x)] + 2(1 - a^2) [f(ax + y) + f(ax - y)] \\
+ \frac{a^4 - a^2}{3} [f(2x) - f(x)]$$

for all $x, y \in E_1$. Replacing x by $2x$ in (2.2), we get

$$(2.10) \quad f(2ax + y) + f(2ax - y) \\
= a^2 [f(2x + y) + f(2x - y)] + 2(1 - a^2) f(y) \\
+ \frac{a^4 - a^2}{6} [f(4x) - 4f(2x)]$$

for all $x, y \in E_1$. Using (2.10) in (2.9), we obtain

$$(2.11) \quad a^2 [f(2x + y) + f(2x - y)] + 2(1 - a^2) f(y) \\
+ \frac{a^4 - a^2}{6} [f(4x) - 4f(2x)] + 2f(y)$$



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$$\begin{aligned}
 &= a^4 [f(2x + y) + f(2x - y)] + 2a^2 (1 - a^2) [f(x + y) + f(x - y)] \\
 &\quad + \frac{a^2 (a^4 - a^2)}{3} [f(2x) - 4f(x)] + 2(1 - a^2) [f(ax + y) + f(ax - y)] \\
 &\quad\quad\quad + \frac{a^4 - a^2}{3} [f(2x) - 4f(x)] + 2a^4 f(y)
 \end{aligned}$$

for all $x, y \in E_1$. Using (2.2) in (2.11), we get

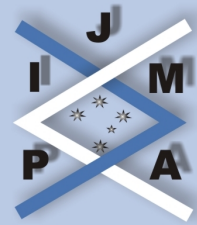
$$\begin{aligned}
 (2.12) \quad &a^2 [f(2x + y) + f(2x - y)] \\
 &\quad + 2(1 - a^2) f(y) + \frac{a^4 - a^2}{6} [f(4x) - 4f(2x)] + 2f(y) \\
 &= a^4 [f(2x + y) + f(2x - y)] + 2a^2 (1 - a^2) [f(x + y) + f(x - y)] \\
 &\quad + \frac{a^2 (a^4 - a^2)}{3} [f(2x) - 4f(x)] + \frac{a^4 - a^2}{3} [f(2x) - 4f(x)] \\
 &\quad + 2a^4 f(y) + 2(1 - a^2) \left[a^2 (f(x + y) + f(x - y)) \right. \\
 &\quad\quad\quad \left. + 2(1 - a^2) f(y) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)] \right]
 \end{aligned}$$

for all $x, y \in E_1$. Letting $y = 0$ in (2.2), we obtain

$$(2.13) \quad 2f(ax) = 2a^2 f(x) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)]$$

for all $x, y \in E_1$. Replacing y by x in (2.2), we get

$$\begin{aligned}
 (2.14) \quad &f((a + 1)x) + f((a - 1)x) \\
 &= a^2 f(2x) + 2(1 - a^2) f(x) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)]
 \end{aligned}$$



for all $x \in E_1$. Replacing y by ax in (2.2), we obtain

$$(2.15) \quad f(2ax) = a^2 [f((1+a)x) + f((1-a)x)] \\ + 2(1-a^2)f(ax) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)]$$

for all $x \in E_1$. Letting $y = 0$ in (2.10), we get

$$(2.16) \quad f(2ax) = a^2 f(2x) + \frac{a^4 - a^2}{12} [f(4x) - 4f(2x)]$$

for all $x \in E_1$. From (2.15) and (2.16), we arrive at

$$(2.17) \quad a^2 f(2x) + \frac{a^4 - a^2}{12} [f(4x) - 4f(2x)] = a^2 [f((1+a)x) + f((1-a)x)] \\ + 2(1-a^2)f(ax) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)]$$

for all $x \in E_1$. Using (2.13) and (2.14) in (2.17), we obtain

$$(2.18) \quad a^2 f(2x) + \frac{a^4 - a^2}{12} [f(4x) - 4f(2x)] \\ = a^2 \left[a^2 f(2x) + 2(1-a^2)f(x) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)] \right] \\ + (1-a^2) \left[2a^2 f(x) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)] \right] \\ + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)]$$

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for all $x \in E_1$. Comparing (2.12) and (2.18), we arrive at

$$(2.19) \quad f(2x + y) + f(2x - y) \\ = 4[f(x + y) + f(x - y)] - 8f(x) + 2f(2x) - 6f(y)$$

for all $x, y \in E_1$. Replacing y by $2y$ in (2.19), we get

$$(2.20) \quad f(2x + 2y) + f(2x - 2y) \\ = 4[f(x + 2y) + f(x - 2y)] - 8f(x) + 2f(2x) - 6f(2y)$$

for all $x, y \in E_1$. Interchanging x and y in (2.19) and using the evenness of f , we obtain

$$(2.21) \quad f(x + 2y) + f(x - 2y) \\ = 4[f(x + y) + f(x - y)] - 8f(y) + 2f(2y) - 6f(x)$$

for all $x, y \in E_1$. Using (2.21) in (2.20), we get

$$(2.22) \quad f(2x + 2y) + f(2x - 2y) \\ = 16[f(x + y) + f(x - y)] + 2f(2y) - 32f(y) + 2f(2x) - 32f(x)$$

for all $x, y \in E_1$. Rearranging (2.22), we have

$$(2.23) \quad \{f(2x + 2y) - 16f(x + y)\} + \{f(2x - 2y) - 16f(x - y)\} \\ = 2\{f(2x) - 16f(x)\} + 2\{f(2y) - 16f(y)\}$$

for all $x, y \in E_1$. Let $\alpha : E_1 \rightarrow E_2$ defined by

$$(2.24) \quad \alpha(x) = f(2x) - 16f(x), \quad \forall x \in E_1.$$

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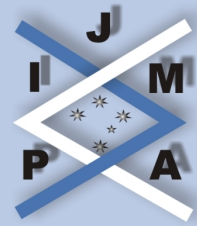


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Applying (2.24) in (2.23), we arrive at

$$(2.25) \quad \alpha(x+y) + \alpha(x-y) = 2\alpha(x) + 2\alpha(y) \quad \forall x \in E_1.$$

Hence $\alpha : E_1 \rightarrow E_2$ is quadratic mapping.

Since α is quadratic, we have $\alpha(2x) = 4\alpha(x)$ for all $x \in E_1$. Then

$$(2.26) \quad f(4x) = 20f(2x) - 64f(x)$$

for all $x \in E_1$. Replacing (x, y) by $(2x, 2y)$ in (2.19), we get

$$(2.27) \quad f(2(2x+y)) + f(2(2x-y)) \\ = 4[f(2(x+y)) + f(2(x-y))] - 8f(2x) + 2f(4x) - 6f(2y)$$

for all $x, y \in E_1$. Using (2.26) in (2.27), we obtain

$$(2.28) \quad f(2(2x+y)) + f(2(2x-y)) \\ = 4[f(2(x+y)) + f(2(x-y))] + 32\{f(2x) - 4f(x)\} - 6f(2y)$$

for all $x, y \in E_1$. Multiplying (2.19) by 4, we arrive at

$$(2.29) \quad 4f(2x+y) + 4f(2x-y) \\ = 16[f(x+y) + f(x-y)] + 8\{f(2x) - 4f(x)\} - 24f(y)$$

for all $x, y \in E_1$. Subtracting (2.29) from (2.28), we get

$$(2.30) \quad \{f(2(2x+y)) - 4f(2x+y)\} + \{f(2(2x-y)) - 4f(2x-y)\} \\ = 4\{f(2(x+y)) - 4f(x+y)\} + 4\{f(2(x-y)) - 4f(x-y)\} \\ + 24\{f(2x) - 4f(x)\} - 6\{f(2y) - 4f(y)\}$$

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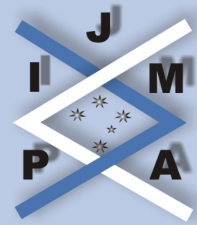
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for all $x, y \in E_1$. Let $\beta : E_1 \rightarrow E_2$ be defined by

$$(2.31) \quad \beta(x) = f(2x) - 4f(x), \forall x \in E_1.$$

Applying (2.30) in (2.31), we arrive at

$$(2.32) \quad \beta(2x + y) + \beta(2x - y) = 4[\beta(x + y) + \beta(x - y)] + 24\beta(x) - 6\beta(y)$$

for all $x, y \in E_1$. Hence $\beta : E_1 \rightarrow E_2$ is quartic mapping.

On the other hand, we have

$$(2.33) \quad f(x) = \frac{\beta(x) - \alpha(x)}{12} \quad \forall x \in E_1.$$

This means that f is quadratic-quartic function. This completes the proof of the theorem. \square

Theorem 2.2. *Let $f : E_1 \rightarrow E_2$ be a function satisfying (1.20) for all $x, y \in E_1$. If f is odd then f is additive - cubic.*

Proof. Let f be an odd function (i.e., $f(-x) = -f(x)$). Then equation (1.20) becomes

$$(2.34) \quad f(x + ay) + f(x - ay) = a^2[f(x + y) + f(x - y)] + 2(1 - a^2)f(x)$$

for all $x, y \in E_1$. By Lemma 2.2 of [13], f is additive-cubic. \square

Theorem 2.3. *Let $f : E_1 \rightarrow E_2$ be a function satisfying (1.20) for all $x, y \in E_1$ if and only if there exists functions $A : E_1 \rightarrow E_2$, $B : E_1 \times E_1 \rightarrow E_2$, $C : E_1 \times E_1 \times E_1 \rightarrow E_2$ and $D : E_1 \times E_1 \times E_1 \times E_1 \rightarrow E_2$ such that*

$$(2.35) \quad f(x) = A(x) + B(x, x) + C(x, x, x) + D(x, x, x, x)$$

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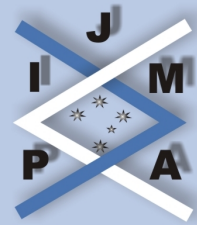
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for all $x \in E_1$, where A is additive, B is symmetric bi-additive, C is symmetric for each fixed one variable and is additive for fixed two variables and D is symmetric multi-additive.

Proof. Let $f : E_1 \rightarrow E_2$ be a function satisfying (1.20). We decompose f into even and odd parts by setting

$$f_e(x) = \frac{1}{2} \{f(x) + f(-x)\}, \quad f_o(x) = \frac{1}{2} \{f(x) - f(-x)\}$$

for all $x \in E_1$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in E_1$. It is easy to show that the functions f_e and f_o satisfy (1.20). Hence by Theorem 2.1 and 2.2, we see that the function f_e is quadratic-quartic and f_o is additive-cubic, respectively. Thus there exist a symmetric bi-additive function $B : E_1 \times E_1 \rightarrow E_2$ and a symmetric multi-additive function $D : E_1 \times E_1 \times E_1 \times E_1 \rightarrow E_2$ such that $f_e(x) = B(x, x) + D(x, x, x, x)$ for all $x \in E_1$, and the function $A : E_1 \rightarrow E_2$ is additive and $C : E_1 \times E_1 \times E_1 \rightarrow E_2$ such that $f_o(x) = A(x) + C(x, x, x)$, where C is symmetric for each fixed one variable and is additive for fixed two variables. Hence we get (2.35) for all $x \in E_1$.

Conversely let $f(x) = A(x) + B(x, x) + C(x, x, x) + D(x, x, x, x)$ for all $x \in E_1$, where A is additive, B is symmetric bi-additive, C is symmetric for each fixed one variable and is additive for fixed two variables and D is symmetric multi-additive. Then it is easy to show that f satisfies (1.20). \square

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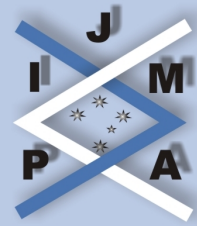
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3. Stability of the Functional Equation (1.20)

In this section, we investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.20). Throughout this section, let E_1 be a real normed space and E_2 be a Banach space. Define a difference operator $Df : E_1 \times E_1 \rightarrow E_2$ by

$$\begin{aligned} Df(x, y) &= f(x + ay) + f(x - ay) - a^2[f(x + y) + f(x - y)] - 2(1 - a^2)f(x) \\ &\quad - \frac{a^4 - a^2}{12}[f(2y) + f(-2y) - 4f(y) - 4f(-y)] \end{aligned}$$

for all $x, y \in E_1$.

Theorem 3.1. Let $\phi_b : E_1 \times E_1 \rightarrow [0, \infty)$ be a function such that

$$(3.1) \quad \sum_{n=0}^{\infty} \frac{\phi_b(2^n x, 2^n y)}{4^n} \quad \text{converges and} \quad \lim_{n \rightarrow \infty} \frac{\phi_b(2^n x, 2^n y)}{4^n} = 0$$

for all $x, y \in E_1$ and let $f : E_1 \rightarrow E_2$ be an even function which satisfies the inequality

$$(3.2) \quad \|Df(x, y)\| \leq \phi_b(x, y)$$

for all $x, y \in E_1$. Then there exists a unique quadratic function $B : E_1 \rightarrow E_2$ such that

$$(3.3) \quad \|f(2x) - 16f(x) - B(x)\| \leq \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(2^k x)}{4^k}$$

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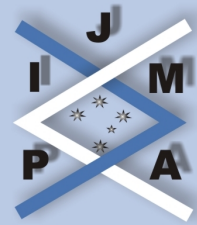
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for all $x \in E_1$, where the mapping $B(x)$ and $\Phi_b(2^k x)$ are defined by

$$(3.4) \quad B(x) = \lim_{n \rightarrow \infty} \frac{1}{4^n} \{f(2^{n+1}x) - 16f(2^n x)\}$$

$$(3.5) \quad \Phi_b(2^k x) = \frac{1}{a^4 - a^2} \left[12(1 - a^2) \phi_b(0, 2^k x) + 12a^2 \phi_b(2^k x, 2^k x) \right. \\ \left. + 6\phi_b(0, 2^{k+1}x) + 12\phi_b(2^k ax, 2^k x) \right]$$

for all $x \in E_1$.

Proof. Using the evenness of f , from (3.2) we get

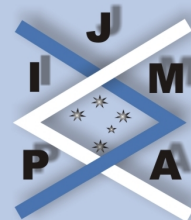
$$(3.6) \quad \left\| f(x + ay) + f(x - ay) - a^2[f(x + y) + f(x - y)] - 2(1 - a^2)f(x) \right. \\ \left. - \frac{(a^4 - a^2)}{12}[2f(2y) - 8f(y)] \right\| \leq \phi_b(x, y)$$

for all $x, y \in E_1$. Interchanging x and y in (3.6), we obtain

$$(3.7) \quad \left\| f(ax + y) + f(ax - y) - a^2[f(x + y) + f(x - y)] - 2(1 - a^2)f(y) \right. \\ \left. - \frac{(a^4 - a^2)}{12}[2f(2x) - 8f(x)] \right\| \leq \phi_b(y, x)$$

for all $x, y \in E_1$. Letting $y = 0$ in (3.7), we get

$$(3.8) \quad \left\| 2f(ax) - 2a^2f(x) - \frac{(a^4 - a^2)}{12}[2f(2x) - 8f(x)] \right\| \leq \phi_b(0, x)$$



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for all $x \in E_1$. Putting $y = x$ in (3.7), we obtain

$$(3.9) \quad \left\| f((a+1)x) + f((a-1)x) - a^2 f(2x) - 2(1-a^2)f(x) - \frac{(a^4-a^2)}{12}[2f(2x) - 8f(x)] \right\| \leq \phi_b(x, x)$$

for all $x \in E_1$. Replacing x by $2x$ in (3.8), we get

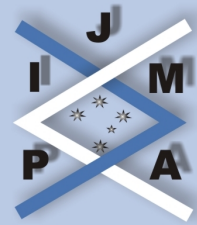
$$(3.10) \quad \left\| 2f(2ax) - 2a^2 f(2x) - \frac{(a^4-a^2)}{12}[2f(4x) - 8f(2x)] \right\| \leq \phi_b(0, 2x)$$

for all $x \in E_1$. Setting y by ax in (3.7), we obtain

$$(3.11) \quad \left\| f(2ax) - a^2 [f((1+a)x) + f((1-a)x)] - 2(1-a^2)f(ax) - \frac{(a^4-a^2)}{12}[2f(2x) - 8f(x)] \right\| \leq \phi_b(ax, x)$$

for all $x \in E_1$. Multiplying (3.8), (3.9), (3.10) and (3.11) by $12(1-a^2)$, $12a^2$, 6 and 12 respectively, we have

$$\begin{aligned} & (a^4 - a^2) \|f(4x) - 20f(2x) + 64f(x)\| \\ &= \left\| \left\{ 24(1-a^2)f(ax) - 24a^2(1-a^2)f(x) - \frac{12(1-a^2)(a^4-a^2)}{12}[2f(2x) - 8f(x)] \right\} \right. \\ & \quad + \left\{ 12a^2 f((a+1)x) + 12a^2 f((a-1)x) - 12a^4 f(2x) \right. \\ & \quad \left. \left. - 24a^2(1-a^2)f(x) - \frac{12a^2(a^4-a^2)}{12}[2f(2x) - 8f(x)] \right\} \right\| \end{aligned}$$



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$$\begin{aligned}
 & + \left\{ -12f(2ax) + 12a^2f(2x) + \frac{6(a^4 - a^2)}{12}[2f(4x) - 8f(2x)] \right\} \\
 & + \left\{ 12f(2ax) - 12a^2[f((1+a)x) + f((1-a)x)] \right. \\
 & \left. - 24(1-a^2)f(ax) - \frac{12(a^4 - a^2)}{12}[2f(2x) - 8f(x)] \right\} \Big\|
 \end{aligned}$$

$$\leq 12(1-a^2)\phi_b(0, x) + 12a^2\phi_b(x, x) + 6\phi_b(0, 2x) + 12\phi_b(ax, x)$$

for all $x \in E_1$. Hence from the above inequality, we get

$$\begin{aligned}
 (3.12) \quad & \|f(4x) - 20f(2x) + 64f(x)\| \\
 & \leq \frac{1}{(a^4 - a^2)} [12(1-a^2)\phi_b(0, x) + 12a^2\phi_b(x, x) + 6\phi_b(0, 2x) + 12\phi_b(ax, x)]
 \end{aligned}$$

for all $x \in E_1$. From (3.12), we arrive at

$$(3.13) \quad \|f(4x) - 20f(2x) + 64f(x)\| \leq \Phi_b(x),$$

where

$$\Phi_b(x) = \frac{1}{a^4 - a^2} [12(1-a^2)\phi_b(0, x) + 12a^2\phi_b(x, x) + 6\phi_b(0, 2x) + 12\phi_b(ax, x)]$$

for all $x \in E_1$. It is easy to see from (3.13) that

$$(3.14) \quad \|f(4x) - 16f(2x) - 4\{f(2x) - 16f(x)\}\| \leq \Phi_b(x)$$

for all $x \in E_1$. Using (2.24) in (3.14), we obtain

$$(3.15) \quad \|\alpha(2x) - 4\alpha(x)\| \leq \Phi_b(x)$$



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for all $x \in E_1$. From (3.15), we have

$$(3.16) \quad \left\| \frac{\alpha(2x)}{4} - \alpha(x) \right\| \leq \frac{\Phi_b(x)}{4}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 4 in (3.16), we obtain

$$(3.17) \quad \left\| \frac{\alpha(2^2x)}{4^2} - \frac{\alpha(2x)}{4} \right\| \leq \frac{\Phi_b(2x)}{4^2}$$

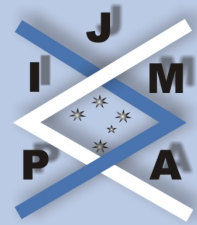
for all $x \in E_1$. From (3.16) and (3.17), we arrive at

$$(3.18) \quad \left\| \frac{\alpha(2^2x)}{4^2} - \alpha(x) \right\| \leq \left\| \frac{\alpha(2^2x)}{4^2} - \frac{\alpha(2x)}{4} \right\| + \left\| \frac{\alpha(2x)}{4} - \alpha(x) \right\| \\ \leq \frac{1}{4} \left[\Phi_b(x) + \frac{\Phi_b(2x)}{4} \right]$$

for all $x \in E_1$. In general for any positive integer n , we get

$$(3.19) \quad \left\| \frac{\alpha(2^n x)}{4^n} - \alpha(x) \right\| \leq \frac{1}{4} \sum_{k=0}^{n-1} \frac{\Phi_b(2^k x)}{4^k} \\ \leq \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(2^k x)}{4^k}$$

for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{ \frac{\alpha(2^n x)}{4^n} \right\}$, replace



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x by $2^m x$ and divide by 4^m in (3.19). For any $m, n > 0$, we have

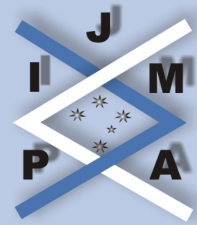
$$\begin{aligned} \left\| \frac{\alpha(2^{n+m}x)}{4^{n+m}} - \frac{\alpha(2^m x)}{4^m} \right\| &= \frac{1}{4^m} \left\| \frac{\alpha(2^n 2^m x)}{4^n} - \alpha(2^m x) \right\| \\ &\leq \frac{1}{4} \sum_{k=0}^{n-1} \frac{\Phi_b(2^{k+m}x)}{4^{k+m}} \\ &\leq \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(2^{k+m}x)}{4^{k+m}} \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty \end{aligned}$$

for all $x \in E_1$. Hence the sequence $\left\{ \frac{\alpha(2^n x)}{4^n} \right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a quadratic mapping $B : E_1 \rightarrow E_2$ such that

$$B(x) = \lim_{n \rightarrow \infty} \frac{\alpha(2^n x)}{4^n} \quad \forall x \in E_1.$$

Letting $n \rightarrow \infty$ in (3.19) and using (2.24), we see that (3.3) holds for all $x \in E_1$. To prove that B satisfies (1.20), replace (x, y) by $(2^n x, 2^n y)$ and divide by 4^n in (3.2). We obtain

$$\begin{aligned} \frac{1}{4^n} \left\| f(2^n(x+ay)) + f(2^n(x-ay)) - a^2[f(2^n(x+y)) + f(2^n(x-y))] \right. \\ \left. - 2(1-a^2)f(2^n x) - \frac{(a^4-a^2)}{12}[f(2^n(2y)) + f(2^n(-2y))] \right. \\ \left. - \frac{(a^4-a^2)}{12}[-4f(2^n y) - 4f(2^n(-y))] \right\| \leq \frac{\phi(2^n x, 2^n y)}{4^n} \end{aligned}$$



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for all $x, y \in E_1$. Letting $n \rightarrow \infty$ in the above inequality, we see that

$$\left\| B(x+ay) + B(x-ay) - a^2[B(x+y) + B(x-y)] - 2(1-a^2)B(x) - \frac{(a^4-a^2)}{12}[B(2y) + B(-2y) - 4B(y) - 4Bf(-y)] \right\| \leq 0,$$

which gives

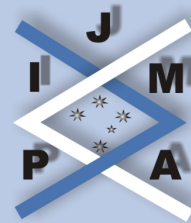
$$\begin{aligned} B(x+ay) + B(x-ay) &= a^2[B(x+y) + B(x-y)] + 2(1-a^2)B(x) \\ &\quad + \frac{(a^4-a^2)}{12}[B(2y) + B(-2y) - 4B(y) - 4Bf(-y)] \end{aligned}$$

for all $x, y \in E_1$. Hence B satisfies (1.20). To prove that B is unique, let B' be another quadratic function satisfying (1.20) and (3.3). We have

$$\begin{aligned} \|B(x) - B'(x)\| &= \frac{1}{4^n} \|B(2^n x) - B'(2^n x)\| \\ &\leq \frac{1}{4^n} \{ \|B(2^n x) - \alpha(2^n x)\| + \|\alpha(2^n x) - B'(2^n x)\| \} \\ &\leq \frac{1}{4^n} \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_b(2^k x)}{4^k} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

for all $x \in E_1$. Hence B is unique. This completes the proof of the theorem. \square

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.20).



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Corollary 3.2. Let ε, p be nonnegative real numbers. Suppose that an even function $f : E_1 \rightarrow E_2$ satisfies the inequality

$$(3.20) \quad \|Df(x, y)\| \leq \begin{cases} \varepsilon (\|x\|^p + \|y\|^p), & 0 \leq p < 2; \\ \varepsilon, & 0 \leq p < 1; \\ \varepsilon \|x\|^p \|y\|^p, & 0 \leq p < 1; \\ \varepsilon (\|x\|^p \|y\|^p + \{\|x\|^{2p} + \|y\|^{2p}\}), & 0 \leq p < 1; \end{cases}$$

for all $x, y \in E_1$. Then there exists a unique quadratic function $B : E_1 \rightarrow E_2$ such that

$$(3.21) \quad \|f(2x) - 16f(x) - B(x)\| \leq \begin{cases} \frac{\lambda_1 \|x\|^p}{4-2^p}, \\ 10\lambda_2, \\ \frac{\lambda_3 \|x\|^{2p}}{4-2^{2p}}, \\ \frac{\lambda_4 \|x\|^{2p}}{4-2^{2p}}, \end{cases}$$

where

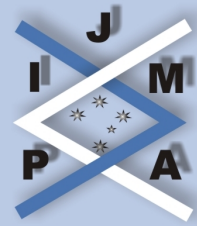
$$\lambda_1 = \frac{\varepsilon \{24 + 12a^2 + 12(a^p) + 6(2^p)\}}{a^4 - a^2}, \quad \lambda_2 = \frac{\varepsilon}{a^4 - a^2},$$

$$\lambda_3 = \frac{12\varepsilon \{a^2 + a^p\}}{a^4 - a^2} \quad \text{and} \quad \lambda_4 = \frac{\varepsilon \{24 + 24a^2 + 12(a^p) + 12(a^{2p}) + 6(2^{2p})\}}{a^4 - a^2}$$

for all $x \in E_1$.

Theorem 3.3. Let $\phi_d : E_1 \times E_1 \rightarrow [0, \infty)$ be a function such that

$$(3.22) \quad \sum_{n=0}^{\infty} \frac{\phi_d(2^n x, 2^n y)}{16^n} \text{ converges and } \lim_{n \rightarrow \infty} \frac{\phi_d(2^n x, 2^n y)}{16^n} = 0$$



for all $x, y \in E_1$ and let $f : E_1 \rightarrow E_2$ be an even function which satisfies the inequality

$$(3.23) \quad \|Df(x, y)\| \leq \phi_d(x, y)$$

for all $x, y \in E_1$. Then there exists a unique quartic function $D : E_1 \rightarrow E_2$ such that

$$(3.24) \quad \|f(2x) - 4f(x) - D(x)\| \leq \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d(2^k x)}{16^k}$$

for all $x \in E_1$, where the mapping $D(x)$ and $\Phi_d(2^k x)$ are defined by

$$(3.25) \quad D(x) = \lim_{n \rightarrow \infty} \frac{1}{16^n} \{f(2^{n+1}x) - 4f(2^n x)\},$$

$$(3.26) \quad \Phi_d(2^k x) = \frac{1}{a^4 - a^2} \left[12(1 - a^2) \phi_d(0, 2^k x) + 12a^2 \phi_d(2^k x, 2^k x) \right. \\ \left. + 6\phi_d(0, 2^{k+1}x) + 12\phi_d(2^k ax, 2^k x) \right]$$

for all $x \in E_1$.

Proof. Along similar lines to those in the proof of Theorem 3.1, we have

$$(3.27) \quad \|f(4x) - 20f(2x) + 64f(x)\| \leq \Phi_d(x),$$

where

$$\Phi_d(x) = \frac{1}{a^4 - a^2} \left[12(1 - a^2) \phi_d(0, x) + 12a^2 \phi_d(x, x) + 6\phi_d(0, 2x) + 12\phi_d(ax, x) \right]$$

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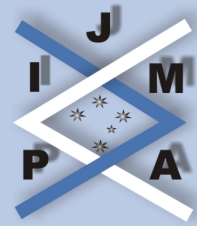
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for all $x \in E_1$. It is easy to see from (3.27) that

$$(3.28) \quad \|f(4x) - 4f(2x) - 16\{f(2x) - 4f(x)\}\| \leq \Phi_d(x)$$

for all $x \in E_1$. Using (2.31) in (3.28), we obtain

$$(3.29) \quad \|\beta(2x) - 16\beta(x)\| \leq \Phi_d(x)$$

for all $x \in E_1$. From (3.29), we have

$$(3.30) \quad \left\| \frac{\beta(2x)}{16} - \beta(x) \right\| \leq \frac{\Phi_d(x)}{16}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 16 in (3.30), we obtain

$$(3.31) \quad \left\| \frac{\beta(2^2x)}{16^2} - \frac{\beta(2x)}{16} \right\| \leq \frac{\Phi_d(2x)}{16^2}$$

for all $x \in E_1$. From (3.30) and (3.31), we arrive at

$$(3.32) \quad \begin{aligned} \left\| \frac{\beta(2^2x)}{16^2} - \beta(x) \right\| &\leq \left\| \frac{\beta(2^2x)}{16^2} - \frac{\beta(2x)}{16} \right\| + \left\| \frac{\beta(2x)}{16} - \beta(x) \right\| \\ &\leq \frac{1}{16} \left[\Phi_d(x) + \frac{\Phi_d(2x)}{16} \right] \end{aligned}$$

for all $x \in E_1$. In general for any positive integer n , we get

$$(3.33) \quad \begin{aligned} \left\| \frac{\beta(2^n x)}{16^n} - \beta(x) \right\| &\leq \frac{1}{16} \sum_{k=0}^{n-1} \frac{\Phi_d(2^k x)}{16^k} \\ &\leq \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d(2^k x)}{16^k} \end{aligned}$$

Stability of Generalized Mixed Type

K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009

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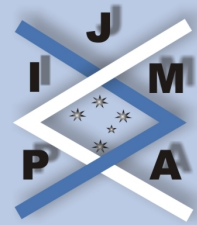
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for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{ \frac{\beta(2^n x)}{16^n} \right\}$, replace x by $2^m x$ and divide by 16^m in (3.33). For any $m, n > 0$, we then have

$$\begin{aligned} \left\| \frac{\beta(2^{n+m}x)}{16^{n+m}} - \frac{\beta(2^m x)}{16^m} \right\| &= \frac{1}{16^m} \left\| \frac{\beta(2^n 2^m x)}{16^n} - \beta(2^m x) \right\| \\ &\leq \frac{1}{16} \sum_{k=0}^{n-1} \frac{\Phi_d(2^{k+m}x)}{16^{k+m}} \\ &\leq \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d(2^{k+m}x)}{16^{k+m}} \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty \end{aligned}$$

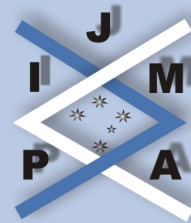
for all $x \in E_1$. Hence the sequence $\left\{ \frac{\beta(2^n x)}{16^n} \right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a quartic mapping $D : E_1 \rightarrow E_2$ such that

$$D(x) = \lim_{n \rightarrow \infty} \frac{\beta(2^n x)}{16^n} \quad \forall x \in E_1.$$

Letting $n \rightarrow \infty$ in (3.33) and using (2.31) we see that (3.24) holds for all $x \in E_1$. The proof that D satisfies (1.20) and is unique is similar to that for Theorem 3.1. \square

The following corollary is an immediate consequence of Theorem 3.3 concerning the stability of (1.20).

Corollary 3.4. *Let ε, p be nonnegative real numbers. Suppose that an even function*



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$f : E_1 \rightarrow E_2$ satisfies the inequality

$$(3.34) \quad \|Df(x, y)\| \leq \begin{cases} \varepsilon (\|x\|^p + \|y\|^p), & 0 \leq p < 4; \\ \varepsilon, & \\ \varepsilon \|x\|^p \|y\|^p, & 0 \leq p < 2; \\ \varepsilon (\|x\|^p \|y\|^p + \{\|x\|^{2p} + \|y\|^{2p}\}), & 0 \leq p < 2 \end{cases}$$

for all $x, y \in E_1$. Then there exists a unique quartic function $D : E_1 \rightarrow E_2$ such that

$$(3.35) \quad \|f(2x) - 4f(x) - D(x)\| \leq \begin{cases} \frac{\lambda_1 \|x\|^p}{16-2^p}, \\ 2\lambda_2, \\ \frac{\lambda_3 \|x\|^{2p}}{16-2^{2p}}, \\ \frac{\lambda_4 \|x\|^{2p}}{16-2^{2p}}, \end{cases}$$

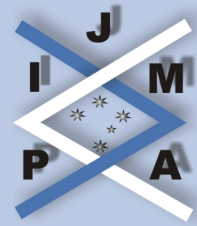
for all $x \in E_1$, where λ_i ($i = 1, 2, 3, 4$) are given in Corollary 3.2.

Theorem 3.5. Let $\phi : E_1 \times E_1 \rightarrow [0, \infty)$ be a function such that

$$(3.36) \quad \sum_{n=0}^{\infty} \frac{\phi_b(2^n x, 2^n y)}{4^n}, \quad \sum_{n=0}^{\infty} \frac{\phi_d(2^n x, 2^n y)}{16^n} \quad \text{converges}$$

and

$$(3.37) \quad \lim_{n \rightarrow \infty} \frac{\phi_b(2^n x, 2^n y)}{4^n} = 0 = \lim_{n \rightarrow \infty} \frac{\phi_d(2^n x, 2^n y)}{16^n}$$



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for all $x, y \in E_1$. Suppose that an even function $f : E_1 \rightarrow E_2$ satisfies the inequalities (3.2) and (3.23) for all $x, y \in E_1$. Then there exists a unique quadratic function $B : E_1 \rightarrow E_2$ and a unique quartic function $D : E_1 \rightarrow E_2$ such that

$$(3.38) \quad \|f(x) - B(x) - D(x)\| \leq \frac{1}{12} \left\{ \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(2^k x)}{4^k} + \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d(2^k x)}{16^k} \right\}$$

for all $x \in E_1$, where $\Phi_b(2^k x)$ and $\Phi_d(2^k x)$ are defined in (3.5) and (3.26), respectively for all $x \in E_1$.

Proof. By Theorems 3.1 and 3.3, there exists a unique quadratic function $B_1 : E_1 \rightarrow E_2$ and a unique quartic function $D_1 : E_1 \rightarrow E_2$ such that

$$(3.39) \quad \|f(2x) - 16f(x) - B_1(x)\| \leq \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(2^k x)}{4^k}$$

and

$$(3.40) \quad \|f(2x) - 4f(x) - D_1(x)\| \leq \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d(2^k x)}{16^k}$$

for all $x \in E_1$. Now from (3.39) and (3.40), one can see that

$$\begin{aligned} & \left\| f(x) + \frac{1}{12} B_1(x) - \frac{1}{12} D_1(x) \right\| \\ &= \left\| \left\{ -\frac{f(2x)}{12} + \frac{16f(x)}{12} + \frac{B_1(x)}{12} \right\} + \left\{ \frac{f(2x)}{12} - \frac{4f(x)}{12} - \frac{D_1(x)}{12} \right\} \right\| \\ &\leq \frac{1}{12} \{ \|f(2x) - 16f(x) - B_1(x)\| + \|f(2x) - 4f(x) - D_1(x)\| \} \end{aligned}$$

$$\leq \frac{1}{12} \left\{ \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(2^k x)}{4^k} + \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d(2^k x)}{16^k} \right\}$$

for all $x \in E_1$. Thus we obtain (3.38) by defining $B(x) = \frac{-1}{12}B_1(x)$ and $D(x) = \frac{1}{12}D_1(x)$, where $\Phi_b(2^k x)$ and $\Phi_d(2^k x)$ are defined in (3.5) and (3.26), respectively for all $X \in E_1$. \square

The following corollary is the immediate consequence of Theorem 3.5 concerning the stability of (1.20).

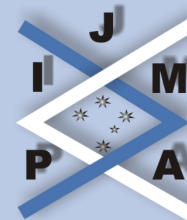
Corollary 3.6. *Let ϵ, p be nonnegative real numbers. Suppose an even function $f : E_1 \rightarrow E_2$ satisfies the inequality*

$$(3.41) \quad \|Df(x, y)\| \leq \begin{cases} \epsilon (\|x\|^p + \|y\|^p), & 0 \leq p < 2; \\ \epsilon, & \\ \epsilon \|x\|^p \|y\|^p, & 0 \leq p < 1; \\ \epsilon (\|x\|^p \|y\|^p + \{\|x\|^{2p} + \|y\|^{2p}\}), & 0 \leq p < 1 \end{cases}$$

for all $x, y \in E_1$. Then there exists a unique quadratic function $B : E_1 \rightarrow E_2$ and a unique quartic function $D : E_1 \rightarrow E_2$ such that

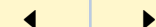
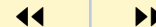
$$(3.42) \quad \|f(x) - B(x) - D(x)\| \leq \begin{cases} \frac{\lambda_1 \|x\|^p}{12} \left\{ \frac{1}{4-2^p} + \frac{1}{16-2^{2p}} \right\}, \\ \lambda_2 \\ \frac{\lambda_3 \|x\|^{2p}}{12} \left\{ \frac{1}{4-2^{2p}} + \frac{1}{16-2^{4p}} \right\}, \\ \frac{\lambda_4 \|x\|^p}{12} \left\{ \frac{1}{4-2^{2p}} + \frac{1}{16-2^{2p}} \right\}, \end{cases}$$

for all $x \in E_1$, where λ_i ($i = 1, 2, 3, 4$) are given in Corollary 3.2.



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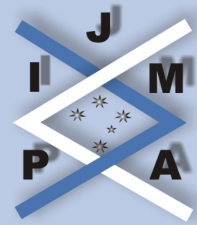
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Theorem 3.7. Let $\phi_a : E_1 \times E_1 \rightarrow [0, \infty)$ be a function such that

$$(3.43) \quad \sum_{n=0}^{\infty} \frac{\phi_a(2^n x, 2^n y)}{2^n} \text{ converges and } \lim_{n \rightarrow \infty} \frac{\phi_a(2^n x, 2^n y)}{2^n} = 0$$

for all $x, y \in E_1$ and let $f : E_1 \rightarrow E_2$ be an odd function with $f(0) = 0$ which satisfies the inequality

$$(3.44) \quad \|Df(x, y)\| \leq \phi_a(x, y)$$

for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \rightarrow E_2$ such that

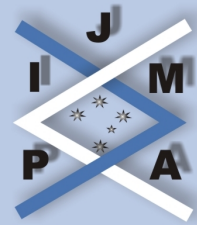
$$(3.45) \quad \|f(2x) - 8f(x) - A(x)\| \leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a(2^k x)}{2^k}$$

for all $x \in E_1$, where the mapping $A(x)$ and $\Phi_a(2^k x)$ are defined by

$$(3.46) \quad A(x) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \{f(2^{n+1}x) - 8f(2^n x)\}$$

$$(3.47) \quad \begin{aligned} \Phi_a(2^k x) = & \frac{1}{a^4 - a^2} [(5 - 4a^2) \phi_a(2^k x, 2^k x) + a^2 \phi_a(2^{k+1} x, 2^{k+1} x) \\ & + 2a^2 \phi_a(2^{k+1} x, 2^k x) + (4 - 2a^2) \phi_a(2^k x, 2^{k+1} x) + \phi_a(2^k x, 2^k 3x) \\ & + 2\phi_a(2^k(1+a)x, 2^k x) + 2\phi_a(2^k(1-a)x, 2^k x) \\ & + \phi_a(2^k(1+2a)x, 2^k x) + \phi_a(2^k(1-2a)x, 2^k x)] \end{aligned}$$

for all $x \in E_1$.



Proof. Using the oddness of f and from (3.44), we get

$$(3.48) \quad \left\| f(x+ay) + f(x-ay) - a^2[f(x+y) + f(x-y)] - 2(1-a^2)f(x) \right\| \leq \phi_a(x, y)$$

for all $x \in E_1$. Replacing y by x in (3.48), we obtain

$$(3.49) \quad \left\| f((1+a)x) + f((1-a)x) - a^2f(2x) - 2(1-a^2)f(x) \right\| \leq \phi_a(x, x)$$

for all $x \in E_1$. Replacing x by $2x$ in (3.49), we get

$$(3.50) \quad \left\| f(2(1+a)x) + f(2(1-a)x) - a^2f(4x) - 2(1-a^2)f(2x) \right\| \leq \phi_a(2x, 2x)$$

for all $x \in E_1$. Again replacing (x, y) by $(2x, x)$ in (3.48), we obtain

$$(3.51) \quad \left\| f((2+a)x) + f((2-a)x) - a^2f(3x) - a^2f(x) - 2(1-a^2)f(2x) \right\| \leq \phi_a(2x, x)$$

for all $x \in E_1$. Replacing y by $2x$ in (3.48), we get

$$(3.52) \quad \left\| f((1+2a)x) + f((1-2a)x) - a^2f(3x) + a^2f(x) - 2(1-a^2)f(x) \right\| \leq \phi_a(x, 2x)$$

for all $x \in E_1$. Replacing y by $3x$ in (3.48), we obtain

$$(3.53) \quad \left\| f((1+3a)x) + f((1-3a)x) - a^2f(4x) + a^2f(2x) - 2(1-a^2)f(x) \right\| \leq \phi_a(x, 3x)$$

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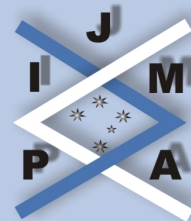
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for all $x \in E_1$. Replacing (x, y) by $((1 + a)x, x)$ in (3.48), we get

$$(3.54) \quad \left\| f((1 + 2a)x) + f(x) - a^2 f((2 + a)x) \right. \\ \left. - a^2 f(ax) - 2(1 - a^2) f((1 + a)x) \right\| \leq \phi_a((1 + a)x, x)$$

for all $x \in E_1$. Again replacing (x, y) by $((1 - a)x, x)$ in (3.48), we obtain

$$(3.55) \quad \left\| f((1 - 2a)x) + f(x) - a^2 f((2 - a)x) \right. \\ \left. + a^2 f(ax) - 2(1 - a^2) f((1 - a)x) \right\| \leq \phi_a((1 - a)x, x)$$

for all $x \in E_1$. Adding (3.54) and (3.55), we arrive at

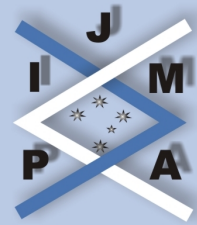
$$(3.56) \quad \left\| f((1 + 2a)x) + f((1 - 2a)x) + 2f(x) - a^2 f((2 + a)x) \right. \\ \left. - a^2 f((2 - a)x) - 2(1 - a^2) f((1 + a)x) - 2(1 - a^2) f((1 - a)x) \right\| \\ \leq \phi_a((1 + a)x, x) + \phi_a((1 - a)x, x)$$

for all $x \in E_1$. Replacing (x, y) by $((1 + 2a)x, x)$ in (3.48), we get

$$(3.57) \quad \left\| f((1 + 3a)x) + f((1 + a)x) - a^2 f(2(1 + a)x) - a^2 f(2ax) \right. \\ \left. - 2(1 - a^2) f((1 + 2a)x) \right\| \leq \phi_a((1 + 2a)x, x)$$

for all $x \in E_1$. Again replacing (x, y) by $((1 - 2a)x, x)$ in (3.48), we obtain

$$(3.58) \quad \left\| f((1 - 3a)x) + f((1 - a)x) - a^2 f(2(1 - a)x) + a^2 f(2ax) \right. \\ \left. - 2(1 - a^2) f((1 - 2a)x) \right\| \leq \phi_a((1 - 2a)x, x)$$



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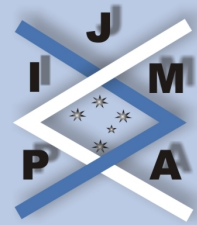
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for all $x \in E_1$. Adding (3.57) and (3.58), we arrive at

$$\begin{aligned}
 (3.59) \quad & \left\| f((1+3a)x) + f((1-3a)x) + f((1+a)x) \right. \\
 & + f((1-a)x) - a^2 f(2(1+a)x) - a^2 f(2(1-a)x) \\
 & \left. - 2(1-a^2)f((1+2a)x) - 2(1-a^2)f((1-2a)x) \right\| \\
 & \leq \phi_a((1+2a)x, x) + \phi_a((1-2a)x, x)
 \end{aligned}$$

for all $x \in E_1$. Now multiplying (3.49) by $2(1-a^2)$, (3.51) by a^2 and adding (3.52) and (3.56), we have

$$\begin{aligned}
 & (a^4 - a^2) \|f(3x) - 4f(2x) + 5f(x)\| \\
 = & \left\| \left\{ 2(1-a^2)f((1+a)x) + 2(1-a^2)f((1-a)x) - 2a^2(1-a^2)f(2x) \right. \right. \\
 & - 4(1-a^2)^2 f(x) \left. \right\} + \left\{ a^2 f((2+a)x) + a^2 f((2-a)x) - a^4 f(3x) \right. \\
 & - a^4 f(x) - 2a^2(1-a^2)f(2x) \left. \right\} + \left\{ -f((1+2a)x) \right. \\
 & - f((1-2a)x) + a^2 f(3x) - a^2 f(x) + 2(1-a^2)f(x) \left. \right\} \\
 & + \left\{ f((1+2a)x) + f((1-2a)x) + 2f(x) - a^2 f((2+a)x) \right. \\
 & - a^2 f((2-a)x) - 2(1-a^2)f((1+a)x) - 2(1-a^2)f((1-a)x) \left. \right\} \left\| \right. \\
 \leq & 2(1-a^2)\phi_a(x, x) + a^2\phi_a(2x, x) + \phi_a(x, 2x) \\
 & + \phi_a((1+a)x, x) + \phi_a((1-a)x, x)
 \end{aligned}$$



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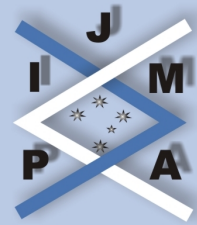
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for all $x \in E_1$. Hence from the above inequality, we get

$$(3.60) \quad \|f(3x) - 4f(2x) + 5f(x)\| \\
\leq \frac{1}{(a^4 - a^2)} [2(1 - a^2)\phi_a(x, x) + a^2\phi_a(2x, x) \\
+ \phi_a(x, 2x) + \phi_a((1 + a)x, x) + \phi_a((1 - a)x, x)]$$

for all $x \in E_1$. Now multiplying (3.50) by a^2 , (3.52) by $2(1 - a^2)$ and adding (3.49), (3.53) and (3.59), we have

$$(a^4 - a^2) \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\
= \|\{-f((1 + a)x) - f((1 - a)x) + a^2f(2x) + 2(1 - a^2)f(x)\} \\
+ \{a^2f(2(1 + a)x) + a^2f(2(1 - a)x) - a^4f(4x) - 2a^2(1 - a^2)f(2x)\} \\
+ \{2(1 - a^2)f((1 + 2a)x) + 2(1 - a^2)f((1 - 2a)x) - 2a^2(1 - a^2)f(3x) \\
+ 2a^2(1 - a^2)f(x) - 4(1 - a^2)^2f(x)\} + \{-f((1 + 3a)x) \\
- f((1 - 3a)x) + a^2f(4x) - a^2f(2x) + 2(1 - a^2)f(x)\} + \{f((1 + 3a)x) \\
+ f((1 - 3a)x) + f((1 + a)x) + f((1 - a)x) - a^2f(2(1 + a)x) \\
- a^2f(2(1 - a)x) - 2(1 - a^2)f((1 + 2a)x) - 2(1 - a^2)f((1 - 2a)x)\}\| \\
\leq \phi_a(x, x) + a^2\phi_a(2x, 2x) + 2(1 - a^2)\phi_a(x, 2x) \\
+ \phi_a(x, 3x) + \phi_a((1 + 2a)x, x) + \phi_a((1 - 2a)x, x)$$



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for all $x \in E_1$. Hence from the above inequality, we get

$$(3.61) \quad \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\
\leq \frac{1}{(a^4 - a^2)} [\phi_a(x, x) + a^2\phi_a(2x, 2x) + 2(1 - a^2)\phi_a(x, 2x) + \phi_a(x, 3x) \\
+ \phi_a((1 + 2a)x, x) + \phi_a((1 - 2a)x, x)]$$

for all $x \in E_1$. From (3.60) and (3.61), we arrive at

$$(3.62) \quad \|f(4x) - 10f(2x) + 16f(x)\| \\
= \|2f(3x) - 8f(2x) + 10f(x) + f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\
\leq 2\|f(3x) - 4f(2x) + 5f(x)\| \\
+ \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\
\leq \frac{1}{(a^4 - a^2)} [(5 - 4a^2)\phi_a(x, x) + a^2\phi_a(2x, 2x) + 2a^2\phi_a(2x, x) \\
+ (4 - 2a^2)\phi_a(x, 2x) + \phi_a(x, 3x) + 2\phi_a((1 + a)x, x) \\
+ 2\phi_a((1 - a)x, x) + \phi_a((1 + 2a)x, x) + \phi_a((1 - 2a)x, x)]$$

for all $x \in E_1$. From (3.62), we have

$$(3.63) \quad \|f(4x) - 10f(2x) + 16f(x)\| \leq \Phi_a(x),$$

where

$$\Phi_a(x) = \frac{1}{(a^4 - a^2)} [(5 - 4a^2)\phi_a(x, x) + a^2\phi_a(2x, 2x) + 2a^2\phi_a(2x, x) \\
+ (4 - 2a^2)\phi_a(x, 2x) + \phi_a(x, 3x) + 2\phi_a((1 + a)x, x) \\
+ 2\phi_a((1 - a)x, x) + \phi_a((1 + 2a)x, x) + \phi_a((1 - 2a)x, x)]$$



for all $x \in E$. It is easy to see from (3.63)

$$(3.64) \quad \|f(4x) - 8f(2x) - 2\{f(2x) - 8f(x)\}\| \leq \Phi_a(x)$$

for all $x \in E_1$. Define a mapping $\gamma : E_1 \rightarrow E_2$ by

$$(3.65) \quad \gamma(x) = f(2x) - 8f(x)$$

for all $x \in E_1$. Using (3.65) in (3.64), we obtain

$$(3.66) \quad \|\gamma(2x) - 2\gamma(x)\| \leq \Phi_a(x)$$

for all $x \in E_1$. From (3.66), we have

$$(3.67) \quad \left\| \frac{\gamma(2x)}{2} - \gamma(x) \right\| \leq \frac{\Phi_a(x)}{2}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 2 in (3.67), we obtain

$$(3.68) \quad \left\| \frac{\gamma(2^2x)}{2^2} - \frac{\gamma(2x)}{2} \right\| \leq \frac{\Phi_a(2x)}{2^2}$$

for all $x \in E_1$. From (3.67) and (3.68), we arrive at

$$(3.69) \quad \begin{aligned} \left\| \frac{\gamma(2^2x)}{2^2} - \gamma(x) \right\| &\leq \left\| \frac{\gamma(2^2x)}{2^2} - \frac{\gamma(2x)}{2} \right\| + \left\| \frac{\gamma(2x)}{2} - \gamma(x) \right\| \\ &\leq \frac{1}{2} \left[\Phi_a(x) + \frac{\Phi_a(2x)}{2} \right] \end{aligned}$$

for all $x \in E_1$. In general for any positive integer n , we get

$$(3.70) \quad \left\| \frac{\gamma(2^n x)}{2^n} - \gamma(x) \right\| \leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\Phi_a(2^k x)}{2^k} \leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a(2^k x)}{2^k}$$

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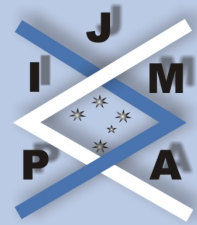
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for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{ \frac{\gamma(2^n x)}{2^n} \right\}$, replace x by $2^m x$ and divide by 2^m in (3.70). Then for any $m, n > 0$, we have

$$\begin{aligned} \left\| \frac{\gamma(2^{n+m}x)}{2^{n+m}} - \frac{\gamma(2^m x)}{2^m} \right\| &= \frac{1}{2^m} \left\| \frac{\gamma(2^n 2^m x)}{2^n} - \gamma(2^m x) \right\| \\ &\leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\Phi_a(2^{k+m}x)}{2^{k+m}} \\ &\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a(2^{k+m}x)}{2^{k+m}} \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty \end{aligned}$$

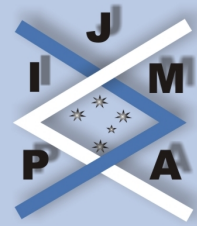
for all $x \in E_1$. Hence the sequence $\left\{ \frac{\gamma(2^n x)}{2^n} \right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a additive mapping $A : E_1 \rightarrow E_2$ such that

$$A(x) = \lim_{n \rightarrow \infty} \frac{\gamma(2^n x)}{2^n} \quad \forall x \in E_1.$$

Letting $n \rightarrow \infty$ in (3.70) and using (3.65) we see that (3.45) holds for all $x \in E_1$. The proof that A satisfies (1.20) and is unique is similar to that of Theorem 3.1. \square

The following corollary is the immediate consequence of Theorem 3.7 concerning the stability of (1.20).

Corollary 3.8. *Let ε, p be nonnegative real numbers. Suppose that an odd function*



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$f : E_1 \rightarrow E_2$ with $f(0) = 0$ satisfies the inequality

$$(3.71) \quad \|Df(x, y)\| \leq \begin{cases} \varepsilon (\|x\|^p + \|y\|^p), & 0 \leq p < 2; \\ \varepsilon, & \\ \varepsilon \|x\|^p \|y\|^p, & 0 \leq p < \frac{1}{2}; \\ \varepsilon (\|x\|^p \|y\|^p + \{\|x\|^{2p} + \|y\|^{2p}\}), & 0 \leq p < \frac{1}{2} \end{cases}$$

for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \rightarrow E_2$ such that

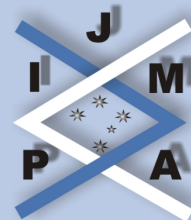
$$(3.72) \quad \|f(2x) - 8f(x) - A(x)\| \leq \begin{cases} \frac{\lambda_5 \|x\|^p}{2-2^p}, \\ \lambda_6, \\ \frac{\lambda_7 \|x\|^{2p}}{2-2^{2p}}, \\ \frac{\lambda_8 \|x\|^{2p}}{2-2^{2p}}, \end{cases}$$

where

$$\lambda_5 = \frac{\varepsilon}{a^4 - a^2} \{ 21 - 8a^2 + 2^p (2a^2 + 4) + 3^p + 2(1+a)^p + 2(1-a)^p + (1+2a)^p + (1-2a)^p \},$$

$$\lambda_6 = \frac{\varepsilon (16 - 3a^2)}{a^4 - a^2},$$

$$\lambda_7 = \frac{\varepsilon}{a^4 - a^2} \{ 5 - 4a^2 + 2^{2p} a^2 + 4(2^p) + 3^p + 2(1+a)^p + 2(1-a)^p + (1+2a)^p + (1-2a)^p \}$$



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and

$$\lambda_8 = \frac{\varepsilon}{a^4 - a^2} \{26 - 12a^2 + 2^{2p} (3a^2 + 4) + 3^{2p} \\ + 2(1+a)^{2p} + 2(1-a)^{2p} + (1+2a)^{2p} + (1-2a)^{2p} + 4(2^p) \\ + 3^p + 2(1+a)^p + 2(1-a)^p + (1+2a)^p + (1-2a)^p\}$$

for all $x \in E_1$.

Theorem 3.9. Let $\phi_c : E_1 \times E_1 \rightarrow [0, \infty)$ be a function such that

$$(3.73) \quad \sum_{n=0}^{\infty} \frac{\phi_c(2^n x, 2^n y)}{8^n} \text{ converges and } \lim_{n \rightarrow \infty} \frac{\phi_c(2^n x, 2^n y)}{8^n} = 0$$

for all $x, y \in E_1$ and let $f : E_1 \rightarrow E_2$ be an odd function with $f(0) = 0$ that satisfies the inequality

$$(3.74) \quad \|Df(x, y)\| \leq \phi_c(x, y)$$

for all $x, y \in E_1$. Then there exists a unique cubic function $C : E_1 \rightarrow E_2$ such that

$$(3.75) \quad \|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c(2^k x)}{8^k}$$

for all $x \in E_1$, where the mapping $C(x)$ and $\Phi_c(2^k x)$ are defined by

$$(3.76) \quad C(x) = \lim_{n \rightarrow \infty} \frac{1}{8^n} \{f(2^{n+1}x) - 2f(2^n x)\}$$



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$$(3.77) \quad \Phi_c(2^k x) = \frac{1}{a^4 - a^2} [(5 - 4a^2) \phi_c(2^k x, 2^k x) + a^2 \phi_c(2^{k+1} x, 2^{k+1} x) \\ + 2a^2 \phi_c(2^{k+1} x, 2^k x) + (4 - 2a^2) \phi_c(2^k x, 2^{k+1} x) + \phi_c(2^k x, 2^k 3x) \\ + 2\phi_c(2^k(1+a)x, 2^k x) + 2\phi_c(2^k(1-a)x, 2^k x) \\ + \phi_c(2^k(1+2a)x, 2^k x) + \phi_c(2^k(1-2a)x, 2^k x)]$$

for all $x \in E_1$.

Proof. Following along similar lines to those in the proof of Theorem 3.7, we have

$$(3.78) \quad \|f(4x) - 10f(2x) + 16f(x)\| \leq \Phi_c(x),$$

where

$$\Phi_c(x) = \frac{1}{(a^4 - a^2)} [(5 - 4a^2) \phi_c(x, x) + a^2 \phi_c(2x, 2x) + 2a^2 \phi_c(2x, x) \\ + (4 - 2a^2) \phi_c(x, 2x) + \phi_c(x, 3x) + 2\phi_c((1+a)x, x) \\ + 2\phi_c((1-a)x, x) + \phi_c((1+2a)x, x) + \phi_c((1-2a)x, x)]$$

for all $x \in E_1$. It is easy to see from (3.78) that

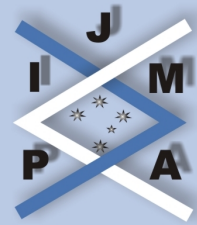
$$(3.79) \quad \|f(4x) - 2f(2x) - 8\{f(2x) - 2f(x)\}\| \leq \Phi_c(x)$$

for all $x \in E_1$. Define a mapping $\delta : E_1 \rightarrow E_2$ by

$$(3.80) \quad \delta(x) = f(2x) - 2f(x)$$

for all $x \in E_1$. Using (3.80) in (3.79), we obtain

$$(3.81) \quad \|\delta(2x) - 8\delta(x)\| \leq \Phi_c(x)$$



for all $x \in E_1$. From (3.81), we have

$$(3.82) \quad \left\| \frac{\delta(2x)}{8} - \delta(x) \right\| \leq \frac{\Phi_c(x)}{8}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 8 in (3.82), we obtain

$$(3.83) \quad \left\| \frac{\delta(2^2x)}{8^2} - \frac{\delta(2x)}{8} \right\| \leq \frac{\Phi_c(2x)}{8^2}$$

for all $x \in E_1$. From (3.82) and (3.83), we arrive at

$$(3.84) \quad \left\| \frac{\delta(2^2x)}{8^2} - \delta(x) \right\| \leq \left\| \frac{\delta(2^2x)}{8^2} - \frac{\delta(2x)}{8} \right\| + \left\| \frac{\delta(2x)}{8} - \delta(x) \right\| \\ \leq \frac{1}{8} \left[\Phi_c(x) + \frac{\Phi_c(2x)}{8} \right]$$

for all $x \in E_1$. In general for any positive integer n , we get

$$(3.85) \quad \left\| \frac{\delta(2^n x)}{8^n} - \delta(x) \right\| \leq \frac{1}{8} \sum_{k=0}^{n-1} \frac{\Phi_c(2^k x)}{8^k} \\ \leq \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c(2^k x)}{8^k}$$

for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{ \frac{\delta(2^n x)}{8^n} \right\}$, replace

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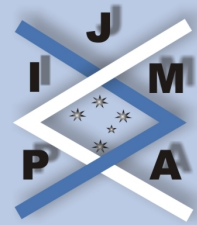
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x by $2^m x$ and divide by 8^m in (3.85). Then for any $m, n > 0$, we have

$$\begin{aligned} \left\| \frac{\delta(2^{n+m}x)}{8^{n+m}} - \frac{\delta(2^m x)}{8^m} \right\| &= \frac{1}{8^m} \left\| \frac{\delta(2^n 2^m x)}{8^n} - \delta(2^m x) \right\| \\ &\leq \frac{1}{8} \sum_{k=0}^{n-1} \frac{\Phi_c(2^{k+m}x)}{8^{k+m}} \\ &\leq \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c(2^{k+m}x)}{8^{k+m}} \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty \end{aligned}$$

for all $x \in E_1$. Hence the sequence $\left\{ \frac{\delta(2^n x)}{8^n} \right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a cubic mapping $C : E_1 \rightarrow E_2$ such that

$$C(x) = \lim_{n \rightarrow \infty} \frac{\delta(2^n x)}{8^n} \quad \forall x \in E_1.$$

Letting $n \rightarrow \infty$ in (3.84) and using (3.80) we see that (3.75) holds for all $x \in E_1$. The rest of the proof, which proves that C satisfies (1.20) and is unique, is similar to that of Theorem 3.1. \square

The following corollary is an immediate consequence of Theorem 3.9 concerning the stability of (1.20).

Corollary 3.10. *Let ε, p be nonnegative real numbers. Suppose that an odd function*

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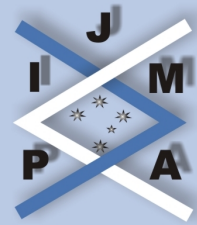
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$f : E_1 \rightarrow E_2$ with $f(0) = 0$ satisfies the inequality

$$(3.86) \quad \|Df(x, y)\| \leq \begin{cases} \varepsilon (\|x\|^p + \|y\|^p), & 0 \leq p < 3; \\ \varepsilon, & \\ \varepsilon \|x\|^p \|y\|^p, & 0 \leq p < \frac{3}{2}; \\ \varepsilon (\|x\|^p \|y\|^p + \{\|x\|^{2p} + \|y\|^{2p}\}), & 0 \leq p < \frac{3}{2} \end{cases}$$

for all $x, y \in E_1$. Then there exists a unique cubic function $C : E_1 \rightarrow E_2$ such that

$$(3.87) \quad \|f(2x) - 8f(x) - A(x)\| \leq \begin{cases} \frac{\lambda_5 \|x\|^p}{8-2^p}, \\ \frac{\lambda_6}{7}, \\ \frac{\lambda_7 \|x\|^{2p}}{8-2^{2p}}, \\ \frac{\lambda_8 \|x\|^{2p}}{8-2^{2p}}, \end{cases}$$

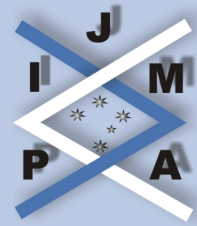
for all $x \in E_1$, where λ_i ($i = 5, 6, 7, 8$) are given in Corollary 3.8.

Theorem 3.11. Let $\phi : E_1 \times E_1 \rightarrow [0, \infty)$ be a function such that

$$(3.88) \quad \sum_{n=0}^{\infty} \frac{\phi_a(2^n x, 2^n y)}{2^n}, \quad \sum_{n=0}^{\infty} \frac{\phi_c(2^n x, 2^n y)}{8^n} \quad \text{converges}$$

and

$$(3.89) \quad \lim_{n \rightarrow \infty} \frac{\phi_a(2^n x, 2^n y)}{2^n} = 0 = \lim_{n \rightarrow \infty} \frac{\phi_c(2^n x, 2^n y)}{8^n}$$



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for all $x, y \in E_1$. Suppose that an odd function $f : E_1 \rightarrow E_2$ with $f(0) = 0$ satisfies the inequalities (3.44) and (3.74) for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \rightarrow E_2$ and a unique cubic function $C : E_1 \rightarrow E_2$ such that

$$(3.90) \quad \|f(x) - A(x) - C(x)\| \leq \frac{1}{6} \left\{ \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a(2^k x)}{2^k} + \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c(2^k x)}{8^k} \right\}$$

for all $x \in E_1$, where $\Phi_a(2^k x)$ and $\Phi_c(2^k x)$ are defined by (3.47) and (3.77), respectively for all $x \in E_1$.

Proof. By Theorems 3.7 and 3.9, there exists a unique additive function $A_1 : E_1 \rightarrow E_2$ and a unique cubic function $C_1 : E_1 \rightarrow E_2$ such that

$$(3.91) \quad \|f(2x) - 8f(x) - A_1(x)\| \leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a(2^k x)}{2^k}$$

and

$$(3.92) \quad \|f(2x) - 2f(x) - C_1(x)\| \leq \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c(2^k x)}{8^k}$$

for all $x \in E_1$. Now from (3.91) and (3.92), one can see that

$$\begin{aligned} & \left\| f(x) + \frac{1}{6} A_1(x) - \frac{1}{6} C_1(x) \right\| \\ &= \left\| \left\{ -\frac{f(2x)}{6} + \frac{8f(x)}{6} + \frac{A_1(x)}{6} \right\} + \left\{ \frac{f(2x)}{6} - \frac{2f(x)}{6} - \frac{C_1(x)}{6} \right\} \right\| \\ &\leq \frac{1}{6} \{ \|f(2x) - 8f(x) - A_1(x)\| + \|f(2x) - 2f(x) - C_1(x)\| \} \end{aligned}$$

$$\leq \frac{1}{6} \left\{ \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a(2^k x)}{2^k} + \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c(2^k x)}{8^k} \right\}$$

for all $x \in E_1$. Thus we obtain (3.90) by defining $A(x) = \frac{-1}{6}A_1(x)$ and $C(x) = \frac{1}{6}C_1(x)$, where $\Phi_a(2^k x)$ and $\Phi_c(2^k x)$ are defined in (3.47) and (3.77), respectively for all $x \in E_1$. \square

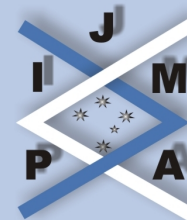
The following corollary is an immediate consequence of Theorem 3.11 concerning the stability of (1.20).

Corollary 3.12. *Let ε, p be nonnegative real numbers. Suppose that an odd function $f : E_1 \rightarrow E_2$ with $f(0) = 0$ satisfies the inequality*

$$(3.93) \quad \|Df(x, y)\| \leq \begin{cases} \varepsilon (\|x\|^p + \|y\|^p), & 0 \leq p < 1; \\ \varepsilon, & \\ \varepsilon \|x\|^p \|y\|^p, & 0 \leq p < \frac{1}{2}; \\ \varepsilon (\|x\|^p \|y\|^p + \{\|x\|^{2p} + \|y\|^{2p}\}), & 0 \leq p < \frac{1}{2} \end{cases}$$

for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \rightarrow E_2$ and a unique cubic function $C : E_1 \rightarrow E_2$ such that

$$(3.94) \quad \|f(x) - A(x) - C(x)\| \leq \begin{cases} \frac{\lambda_5 \|x\|^p}{6} \left\{ \frac{1}{2-2^p} + \frac{1}{8-2^p} \right\}, \\ \frac{4\lambda_6}{21}, \\ \frac{\lambda_7 \|x\|^{2p}}{6} \left\{ \frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}} \right\}, \\ \frac{\lambda_8 \|x\|^p}{6} \left\{ \frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}} \right\}, \end{cases}$$



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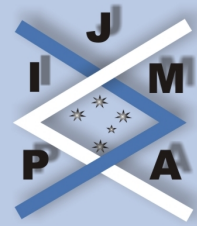
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for all $x \in E_1$, where λ_i ($i = 5, 6, 7, 8$) are given in Corollary 3.8.

Theorem 3.13. Let $\phi : E_1 \times E_1 \rightarrow [0, \infty)$ be a function that satisfies (3.36), (3.37), (3.88) and (3.89) for all $x, y \in E_1$. Suppose that a function $f : E_1 \rightarrow E_2$ with $f(0) = 0$ satisfies the inequalities (3.2), (3.23), (3.44) and (3.74) for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \rightarrow E_2$, a unique quadratic function $B : E_1 \rightarrow E_2$, a unique cubic function $C : E_1 \rightarrow E_2$ and a unique quartic function $D : E_1 \rightarrow E_2$ such that

$$(3.95) \quad \|f(x) - A(x) - B(x) - C(x) - D(x)\| \leq \frac{1}{2} \left\{ \tilde{\Phi}_a(x) + \tilde{\Phi}_b(x) + \tilde{\Phi}_c(x) + \tilde{\Phi}_d(x) \right\}$$

for all $x \in E_1$, where $\tilde{\Phi}_a(x)$, $\tilde{\Phi}_b(x)$, $\tilde{\Phi}_c(x)$ and $\tilde{\Phi}_d(x)$ are defined by

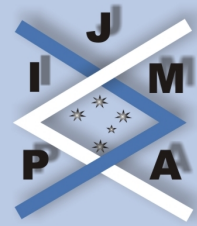
$$(3.96) \quad \tilde{\Phi}_a(x) = \frac{1}{6} \left\{ \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a(2^k x)}{2^k} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a(-2^k x)}{2^k} \right\},$$

$$(3.97) \quad \tilde{\Phi}_b(x) = \frac{1}{12} \left\{ \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(2^k x)}{4^k} + \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(-2^k x)}{4^k} \right\},$$

$$(3.98) \quad \tilde{\Phi}_c(x) = \frac{1}{6} \left\{ \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c(2^k x)}{8^k} + \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c(-2^k x)}{8^k} \right\},$$

$$(3.99) \quad \tilde{\Phi}_d(x) = \frac{1}{12} \left\{ \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d(2^k x)}{16^k} + \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d(-2^k x)}{16^k} \right\},$$

respectively for all $x \in E_1$.



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Proof. Let $f_e(x) = \frac{1}{2} \{f(x) + f(-x)\}$ for all $x \in E_1$. Then $f_e(0) = 0$, $f_e(x) = f_e(-x)$. Hence

$$\begin{aligned} \|Df_e(x, y)\| &= \frac{1}{2} \{\|Df(x, y) + Df(-x, -y)\|\} \\ &\leq \frac{1}{2} \{\|Df(x, y)\| + \|Df(-x, -y)\|\} \\ &\leq \frac{1}{2} \{\phi(x, y) + \phi(-x, -y)\} \end{aligned}$$

for all $x \in E_1$. Hence from Theorem 3.5, there exists a unique quadratic function $B : E_1 \rightarrow E_2$ and a unique quartic function $D : E_1 \rightarrow E_2$ such that

$$\begin{aligned} (3.100) \quad &\|f(x) - B(x) - D(x)\| \\ &\leq \frac{1}{2} \left\{ \frac{1}{12} \left[\frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(2^k x)}{4^k} + \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d(2^k x)}{16^k} \right] \right. \\ &\quad \left. + \frac{1}{12} \left[\frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(-2^k x)}{4^k} + \frac{1}{16} \sum_{k=0}^{\infty} \frac{\Phi_d(-2^k x)}{16^k} \right] \right\} \\ &\leq \frac{1}{2} \left\{ \tilde{\Phi}_b(x) + \tilde{\Phi}_d(x) \right\}, \end{aligned}$$

where $\tilde{\Phi}_b(x)$ and $\tilde{\Phi}_d(x)$ are given in (3.97) and (3.99) for all $x \in E_1$. Again $f_o(x) = \frac{1}{2} \{f(x) - f(-x)\}$ for all $x \in E_1$. Then $f_o(0) = 0$, $f_o(x) = -f_o(-x)$. Hence

$$\begin{aligned} \|Df_o(x, y)\| &= \frac{1}{2} \{\|Df(x, y) - Df(-x, -y)\|\} \\ &\leq \frac{1}{2} \{\|Df(x, y)\| + \|Df(-x, -y)\|\} \end{aligned}$$

$$\leq \frac{1}{2} \{ \phi(x, y) + \phi(-x, -y) \}$$

for all $x \in E_1$. Hence from Theorem 3.11, there exists a unique additive function $A : E_1 \rightarrow E_2$ and a unique cubic function $C : E_1 \rightarrow E_2$ such that

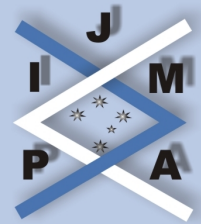
$$\begin{aligned} (3.101) \quad & \|f(x) - A(x) - C(x)\| \\ & \leq \frac{1}{2} \left\{ \frac{1}{6} \left[\frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a(2^k x)}{2^k} + \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c(2^k x)}{8^k} \right] \right. \\ & \quad \left. + \frac{1}{6} \left[\frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a(-2^k x)}{2^k} + \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c(-2^k x)}{8^k} \right] \right\} \\ & \leq \frac{1}{2} \{ \tilde{\Phi}_a(x) + \tilde{\Phi}_c(x) \}, \end{aligned}$$

where $\tilde{\Phi}_a(x)$ and $\tilde{\Phi}_c(x)$ are given in (3.96) and (3.98) for all $x \in E_1$. Since $f(x) = f_e(x) + f_o(x)$, then it follows from (3.100) and (3.101) that

$$\begin{aligned} & \|f(x) - A(x) - B(x) - C(x) - D(x)\| \\ & = \| \{f_e(x) - B(x) - D(x)\} + \{f_o(x) - C(x) - D(x)\} \| \\ & \leq \|f_e(x) - B(x) - D(x)\| + \|f_o(x) - C(x) - D(x)\| \\ & \leq \frac{1}{2} \{ \tilde{\Phi}_a(x) + \tilde{\Phi}_b(x) + \tilde{\Phi}_c(x) + \tilde{\Phi}_d(x) \} \end{aligned}$$

for all $x \in E_1$. Hence the proof of the theorem is complete. \square

The following corollary is an immediate consequence of Theorem 3.13 concerning the stability of (1.20).



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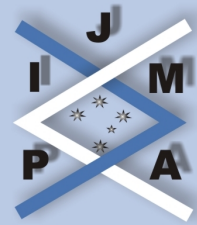
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Corollary 3.14. Let ε, p be nonnegative real numbers. Suppose a function $f : E_1 \rightarrow E_2$ with $f(0) = 0$ satisfies the inequality

$$(3.102) \quad \|D_f(x, y)\| \leq \begin{cases} \varepsilon (\|x\|^p + \|y\|^p), & 0 \leq p < 1; \\ \varepsilon, & \\ \varepsilon \|x\|^p \|y\|^p, & 0 \leq p < \frac{1}{2}; \\ \varepsilon (\|x\|^p \|y\|^p + \{\|x\|^{2p} + \|y\|^{2p}\}), & 0 \leq p < \frac{1}{2} \end{cases}$$

for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \rightarrow E_2$, a unique quadratic function $B : E_1 \rightarrow E_2$, a unique cubic function $C : E_1 \rightarrow E_2$ and a unique quartic function $D : E_1 \rightarrow E_2$ such that

$$(3.103) \quad \|f(x) - A(x) - B(x) - C(x) - D(x)\| \leq \begin{cases} \frac{1}{2} \left\{ \frac{\lambda_1}{6} \left\{ \frac{1}{4-2^p} + \frac{1}{16-2^{2p}} \right\} + \frac{\lambda_5}{3} \left\{ \frac{1}{2-2^p} + \frac{1}{8-2^p} \right\} \right\} \|x\|^p; \\ \frac{1}{2} \left\{ \lambda_2 + \frac{4\lambda_6}{21} \right\}; \\ \frac{1}{2} \left\{ \frac{\lambda_3}{6} \left\{ \frac{1}{4-2^{2p}} + \frac{1}{16-2^{2p}} \right\} + \frac{\lambda_7}{3} \left\{ \frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}} \right\} \right\} \|x\|^{2p}; \\ \frac{1}{2} \left\{ \frac{\lambda_4}{6} \left\{ \frac{1}{4-2^{2p}} + \frac{1}{16-2^{2p}} \right\} + \frac{\lambda_8}{3} \left\{ \frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}} \right\} \right\} \|x\|^{2p} \end{cases}$$

for all $x \in E_1$, where λ_i ($i = 1, \dots, 8$) are respectively, given in Corollaries 3.6 and 3.12.

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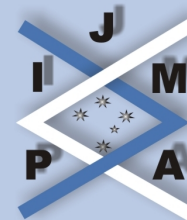
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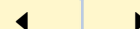
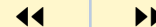
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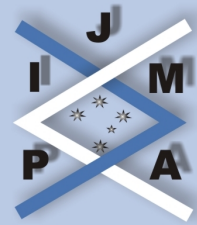
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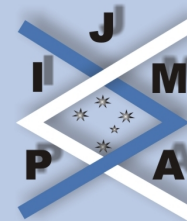
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