STABILITY OF A GENERALIZED MIXED TYPE ADDITIVE, QUADRATIC, CUBIC AND QUARTIC FUNCTIONAL EQUATION

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Abstract: In this paper, we obtain the general solution and the generalized Hyers-Ulam-Rassias stability of the generalized mixed type of functional equation

$$f(x+ay)+f(x-ay) = a^2\left[f(x+y)+f(x-y)\right]+2\left(1-a^2\right)f(x) + \frac{\left(a^4-a^2\right)}{12}\left[f(2y)+f(-2y)-4f(y)-4f(-y)\right].$$

for fixed integers a with $a \neq 0, \pm 1$.

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Contents

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1. Introduction

S.M. Ulam [31] is the pioneer of the stability problem in functional equations. In 1940, while he was delivering a talk before the Mathematics Club of the University of Wisconsin, he discussed a number of unsolved problems. Among them was the following question concerning the stability of homomorphisms:

"Let G be group and H be a metric group with metric $d(\cdot,\cdot)$. Given $\epsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \to H$ satisfies

$$d\left(f(xy), f(x)f(y)\right) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $a : G \to H$ with

$$d\left(f(x), a(x)\right) < \varepsilon$$

for all $x \in G$."

In 1941, D.H. Hyers [12] gave the first affirmative partial answer to the question of Ulam for Banach spaces. He proved the following celebrated theorem.

Theorem 1.1 ([12]). *Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \tag{1.1}$$

for all $x, y \in X$. Then the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all $x \in X$ and $a : X \to Y$ is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \leq \varepsilon \tag{1.3}$$

for all $x \in X$.

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In 1950, Aoki [2] generalized the Hyers theorem for additive mappings. In 1978, Th.M. Rassias [26] provided a generalized version of the Hyers theorem which permitted the Cauchy difference to become unbounded. He proved the following:

Theorem 1.2 ([26]). *Let X be a normed vector space and Y be a Banach space. If a function $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta\left(||x||^p + ||y||^p\right) \tag{1.4}$$

for all $x, y \in X$, where θ and p are constants with $\theta > 0$ and $p < 1$, then the limit

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.5}$$

exists for all $x \in X$ and $T : X \to Y$ is the unique additive mapping which satisfies

$$\|f(x) - T(x)\| \le \frac{2\theta}{2 - 2^p}||x||^p \tag{1.6}$$

for all $x \in X$. If $p < 0$, then inequality (1.4) holds for $x, y \neq 0$ and (1.6) for $x \neq 0$. Also if for each $x \in X$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then T is linear.

It was shown by Z. Gajda [9], as well as Th.M. Rassias and P. Semrl [27] that one cannot prove a Th.M. Rassias type theorem when $p = 1$. The counter examples of Z. Gajda, as well as of Th.M. Rassias and P. Semrl [27] have stimulated several mathematicians to invent new definitions of approximately additive or approximately linear mappings; P. Gavruta [10] and S.M. Jung [17] among others have studied the Hyers-Ulam-Rassias stability of functional equations. The inequality (1.4) that was introduced by Th.M. Rassias [26] provided much influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as the Hyers-Ulam-Rassias stability of functions.

journal of inequalities in pure and applied mathematics
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In 1982, J.M. Rassias [24] following the spirit of the approach of Th.M. Rassias [26] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

Theorem 1.3 ([24]). *Let X be a real normed linear space and Y be a real completed normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exists constants $\theta > 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q \tag{1.7}$$

for all $x, y \in X$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.8}$$

exists for all $x \in X$ and $L : X \to Y$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2 - 2^r|} \|x\|^r \tag{1.9}$$

for all $x \in X$. If, in addition $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}-$ linear mapping.

However, the case $r = 1$ in inequality (1.9) is singular. A counter example has been given by P. Gavruta [11]. The above-mentioned stability involving a product of different powers of norms was called Ulam-Gavruta-Rassias stability by M.A. Sibaha et al., [30], as well as by K. Ravi and M. Arunkumar [28]. This stability result was also called the Hyers-Ulam-Rassias stability involving a product of different powers of norms by Park [23].

journal of inequalities in pure and applied mathematics
issn: 1443-5756

In 1994, a generalization of Th.M. Rassias' theorem and J.M. Rassias' theorem was obtained by P. Gavruta [10], who replaced the factors $\theta\left(||x||^p + ||y||^p\right)$ and $\theta\left(||x||^p||y||^p\right)$ by a general control function $\varphi(x, y)$. In the past few years several mathematicians have published various generalizations and applications of Hyers-Ulam- Rassias stability to a number of functional equations and mappings (see [4, 5, 13, 18, 19]). Very recently, J.M. Rassias [29] in the inequality (1.7) replaced the bound by a mixed one involving the product and sum of powers of norms, that is, $\theta\{||x||^p||y||^p + (||x||^{2p} + ||y||^{2p})\}$.

The functional equation

$$f\left(x+y\right)+f\left(x-y\right)=2f\left(x\right)+2f\left(y\right) \tag{1.10}$$

is said to be a *quadratic functional equation* because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.10). A quadratic functional equation was used to characterize inner product spaces [1, 20]. It is well known that a function f is a solution of (1.10) if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all x (see [20]). The biadditive function B is given by

$$B\left(x,y\right)=\frac{1}{4}\left[f\left(x+y\right)+f\left(x-y\right)\right]. \tag{1.11}$$

The functional equation

$$f\left(2x+y\right)+f\left(2x-y\right)=2f\left(x+y\right)+2f\left(x-y\right)+12f\left(x\right) \tag{1.12}$$

is called a *cubic functional equation*, because the cubic function $f(x) = cx^3$ is a solution of the equation (1.12). The general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.12) was discussed by K.W. Jun and H.M. Kim [14]. They proved that a function f between real vector spaces

journal of inequalities in pure and applied mathematics
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X and Y is a solution of (1.12) if and only if there exists a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$ and C is symmetric for each fixed one variable and is additive for fixed two variables.

The *quartic functional equation*

$$f(x + 2y) + f(x - 2y) - 6f(x) = 4[f(x + y) + f(x - y)] + 24f(y) \tag{1.13}$$

was introduced by J.M. Rassias [25]. Later S.H. Lee et al., [21] remodified J.M. Rassias's equation and obtained a new quartic functional equation of the form

$$f(2x + y) + f(2x - y) = 4[f(x + y) + f(x - y)] + 24f(x) - 6f(y) \tag{1.14}$$

and discussed its general solution. In fact S.H. Lee et al., [21] proved that a function f between vector spaces X and Y is a solution of (1.14) if and only if there exists a unique symmetric multi - additive function $Q : X \times X \times X \times X \to Y$ such that $f(x) = Q(x, x, x, x)$ for all $x \in X$. It is easy to show that the function $f(x) = kx^4$ is the solution of (1.13) and (1.14).

A function

$$f(x) = Q(x_1, x_2, x_3, x_4) \tag{1.15}$$

is called symmetric multi additive if Q is additive with respect to each variable x_i, $i = 1, 2, 3, 4$ in (1.15).

A function f is defined as

$$f(x) = \frac{\beta(x) - \alpha(x)}{12}$$

where $\alpha(x) = f(2x) - 16f(x), \beta(x) = f(2x) - 4f(x)$, further, f satisfies $f(2x) = 4f(x)$ and $f(2x) = 16f(x)$ is said to be a *quadratic - quartic function*.

journal of inequalities in pure and applied mathematics
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K.W. Jun and H.M. Kim [16] introduced the following generalized *quadratic and additive type functional equation*

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f(x_i + x_j) \tag{1.16}$$

in the class of functions between real vector spaces. For $n = 3$, Pl. Kannappan proved that a function f satisfies the functional equation (1.16) if and only if there exists a symmetric bi-additive function A and an additive function B such that $f(x) = B(x,x) + A(x)$ for all x (see [20]). The Hyers-Ulam stability for the equation (1.16) when $n = 3$ was proved by S.M. Jung [18]. The Hyers-Ulam-Rassias stability for the equation (1.16) when $n = 4$ was also investigated by I.S. Chang et al., [3].

The general solution and the generalized Hyers-Ulam stability for the *quadratic and additive type functional equation*

$$f(x+ay) + af(x-y) = f(x-ay) + af(x+y) \tag{1.17}$$

for any positive integer a with $a \neq -1, 0, 1$ was discussed by K.W. Jun and H.M. Kim [15]. Recently A. Najati and M.B. Moghimi [22] investigated the generalized Hyers-Ulam-Rassias stability for a *quadratic and additive type functional equation* of the form

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 2f(2x) - 4f(x) \tag{1.18}$$

Very recently, the authors [6, 7] investigated a mixed type functional equation of cubic and quartic type and obtained its general solution. The stability of generalized mixed type functional equations of the form

$$f(x+ky) + f(x-ky) = k^2\left[f(x+y) + f(x-y)\right] + 2\left(1-k^2\right)f(x) \tag{1.19}$$

journal of inequalities in pure and applied mathematics
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for fixed integers k with $k \neq 0, \pm 1$ in quasi -Banach spaces was investigated by M. Eshaghi Gordji and H. Khodaie [8]. The mixed type functional equation (1.19) is additive, quadratic and cubic.

In this paper, the authors introduce a mixed type functional equation of the form

$$\begin{aligned} f(x+ay)+f(x-ay) &= a^2\left[f(x+y)+f(x-y)\right]+2\left(1-a^2\right)f(x) \\ &\quad +\frac{a^4-a^2}{12}\left[f(2y)+f(-2y)-4f(y)-4f(-y)\right] \end{aligned} \tag{1.20}$$

which is additive, quadratic, cubic and quartic and obtain its general solution and generalized Hyers-Ulam-Rassias stability for fixed integers a with $a \neq 0, \pm 1$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. General Solution

In this section, we present the general solution of the functional equation (1.20). Throughout this section let E_1 and E_2 be real vector spaces.

Theorem 2.1. *Let $f : E_1 \to E_2$ be a function satisfying (1.20) for all $x, y \in E_1$. If f is even then f is quadratic - quartic.*

Proof. Let f be an even function, i.e., $f(-x) = f(x)$. Then equation (1.20) becomes

$$
\begin{aligned}
(2.1)\quad f(x+ay)+f(x-ay) &= a^2\left[f(x+y)+f(x-y)\right]+2\left(1-a^2\right)f(x)\\
&\quad +\frac{a^4-a^2}{6}\left[f(2y)-4f(y)\right]
\end{aligned}
$$

for all $x, y \in E_1$. Interchanging x and y in (2.1) and using the evenness of f, we get

$$
\begin{aligned}
(2.2)\quad f(ax+y)+f(ax-y) &= a^2\left[f(x+y)+f(x-y)\right]+2\left(1-a^2\right)f(y)\\
&\quad +\frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right]
\end{aligned}
$$

for all $x, y \in E_1$. Setting (x, y) as $(0, 0)$ in (2.2), we obtain $f(0) = 0$. Replacing y by $x + y$ in (2.2) and using the evenness of f, we have

$$
\begin{aligned}
(2.3)\quad f((a+1)x+y)+f((a-1)x-y) &= a^2\left[f(2x+y)+f(y)\right]+2\left(1-a^2\right)f(x+y)\\
&\quad +\frac{a^4-a^2}{6}\left[f(2x)-4f(x)\right]
\end{aligned}
$$

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
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for all $x, y \in E_1$. Replacing y by $x - y$ in (2.2), we obtain

$$(2.4)\quad f\left(\left(a+1\right)x-y\right)+f\left(\left(a-1\right)x+y\right) = a^2\left[f\left(2x-y\right)+f\left(y\right)\right]+2\left(1-a^2\right)f\left(x-y\right) + \frac{a^4-a^2}{6}\left[f\left(2x\right)-4f\left(x\right)\right]$$

for all $x, y \in E_1$. Adding (2.3) and (2.4), we get

$$(2.5)\quad f\left(\left(a+1\right)x+y\right)+f\left(\left(a-1\right)x-y\right)+f\left(\left(a+1\right)x-y\right) + f\left(\left(a-1\right)x+y\right) = a^2\left[f\left(2x+y\right)+f\left(2x-y\right)+2f\left(y\right)\right]+2\left(1-a^2\right)\left[f\left(x+y\right)+f\left(x-y\right)\right] + \frac{a^4-a^2}{6}\left[2f\left(2x\right)-8f\left(x\right)\right]$$

for all $x, y \in E_1$. Replacing y by $ax + y$ in (2.2), we obtain

$$(2.6)\quad f\left(2ax+y\right)+f\left(y\right) = a^2\left[f\left(\left(a+1\right)x+y\right)+f\left(\left(1-a\right)x-y\right)\right] + 2\left(1-a^2\right)f\left(ax+y\right)+\frac{a^4-a^2}{6}\left[f\left(2x\right)-4f\left(x\right)\right]$$

for all $x, y \in E_1$. Replacing y by $ax - y$ in (2.3), we get

$$(2.7)\quad f\left(2ax-y\right)+f\left(y\right) = a^2\left[f\left(\left(a+1\right)x-y\right)+f\left(\left(1-a\right)x+y\right)\right] + 2\left(1-a^2\right)f\left(ax-y\right)+\frac{a^4-a^2}{6}\left[f\left(2x\right)-4f\left(x\right)\right]$$

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x, y \in E_1$. Adding (2.6) and (2.7), we obtain

$$f(2ax+y)+f(2ax-y)+2f(y) = a^2\left[f((a+1)x+y)+f((a+1)x-y)+f((a-1)x+y)+f((a-1)x-y)\right] + 2\left(1-a^2\right)\left[f(ax+y)+f(ax-y)\right]+\frac{a^4-a^2}{3}\left[f(2x)-4f(x)\right] \tag{2.8}$$

for all $x, y \in E_1$. Using (2.5) in (2.8), we arrive at

$$f(2ax+y)+f(2ax-y)+2f(y) = a^4\left[f(2x+y)+f(2x-y)\right]+2a^4f(y)+2a^2\left(1-a^2\right)\left[f(x+y)+f(x-y)\right] + \frac{a^2\left(a^4-a^2\right)}{3}\left[f(2x)-4f(x)\right]+2\left(1-a^2\right)\left[f(ax+y)+f(ax-y)\right] + \frac{a^4-a^2}{3}\left[f(2x)-f(x)\right] \tag{2.9}$$

for all $x, y \in E_1$. Replacing x by $2x$ in (2.2), we get

$$f(2ax+y)+f(2ax-y) = a^2\left[f(2x+y)+f(2x-y)\right]+2\left(1-a^2\right)f(y) + \frac{a^4-a^2}{6}\left[f(4x)-4f(2x)\right] \tag{2.10}$$

for all $x, y \in E_1$. Using (2.10) in (2.9), we obtain

$$a^2\left[f(2x+y)+f(2x-y)\right]+2\left(1-a^2\right)f(y) + \frac{a^4-a^2}{6}\left[f(4x)-4f(2x)\right]+2f(y) \tag{2.11}$$

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$= a^4 \left[f\left(2x+y\right)+f\left(2x-y\right)\right]+2a^2\left(1-a^2\right)\left[f\left(x+y\right)+f\left(x-y\right)\right] + \frac{a^2\left(a^4-a^2\right)}{3}\left[f\left(2x\right)-4f\left(x\right)\right]+2\left(1-a^2\right)\left[f\left(ax+y\right)+f\left(ax-y\right)\right] + \frac{a^4-a^2}{3}\left[f\left(2x\right)-4f\left(x\right)\right]+2a^4f\left(y\right)$$

for all $x, y \in E_1$. Using (2.2) in (2.11), we get

$$(2.12)\quad a^2\left[f\left(2x+y\right)+f\left(2x-y\right)\right] + 2\left(1-a^2\right)f\left(y\right)+\frac{a^4-a^2}{6}\left[f\left(4x\right)-4f\left(2x\right)\right]+2f\left(y\right) = a^4\left[f\left(2x+y\right)+f\left(2x-y\right)\right]+2a^2\left(1-a^2\right)\left[f\left(x+y\right)+f\left(x-y\right)\right] + \frac{a^2\left(a^4-a^2\right)}{3}\left[f\left(2x\right)-4f\left(x\right)\right]+\frac{a^4-a^2}{3}\left[f\left(2x\right)-4f\left(x\right)\right] + 2a^4f\left(y\right)+2\left(1-a^2\right)\left[a^2\left(f\left(x+y\right)+f\left(x-y\right)\right) + 2\left(1-a^2\right)f\left(y\right)+\frac{a^4-a^2}{6}\left[f\left(2x\right)-4f\left(x\right)\right]\right]$$

for all $x, y \in E_1$. Letting $y = 0$ in (2.2), we obtain

$$(2.13)\qquad 2f\left(ax\right) = 2a^2f\left(x\right)+\frac{a^4-a^2}{6}\left[f\left(2x\right)-4f\left(x\right)\right]$$

for all $x, y \in E_1$. Replacing y by x in (2.2), we get

$$(2.14)\quad f\left(\left(a+1\right)x\right)+f\left(\left(a-1\right)x\right) = a^2f\left(2x\right)+2\left(1-a^2\right)f\left(x\right)+\frac{a^4-a^2}{6}\left[f\left(2x\right)-4f\left(x\right)\right]$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. Replacing y by ax in (2.2), we obtain

$$f(2ax) = a^2 [f((1+a)x) + f((1-a)x)] + 2(1-a^2) f(ax) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)] \tag{2.15}$$

for all $x \in E_1$. Letting $y = 0$ in (2.10), we get

$$f(2ax) = a^2 f(2x) + \frac{a^4 - a^2}{12} [f(4x) - 4f(2x)] \tag{2.16}$$

for all $x \in E_1$. From (2.15) and (2.16), we arrive at

$$a^2 f(2x) + \frac{a^4 - a^2}{12} [f(4x) - 4f(2x)] = a^2 [f((1+a)x) + f((1-a)x)] + 2(1-a^2) f(ax) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)] \tag{2.17}$$

for all $x \in E_1$. Using (2.13) and (2.14) in (2.17), we obtain

$$\begin{aligned} a^2 f(2x) + \frac{a^4 - a^2}{12} [f(4x) - 4f(2x)] &= a^2 \left[a^2 f(2x) + 2(1-a^2) f(x) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)] \right] \\ &\quad + (1-a^2) \left[2a^2 f(x) + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)] \right] \\ &\quad + \frac{a^4 - a^2}{6} [f(2x) - 4f(x)] \end{aligned} \tag{2.18}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. Comparing (2.12) and (2.18), we arrive at

$$f(2x+y)+f(2x-y) = 4[f(x+y)+f(x-y)] - 8f(x) + 2f(2x) - 6f(y) \tag{2.19}$$

for all $x, y \in E_1$. Replacing y by $2y$ in (2.19), we get

$$f(2x+2y)+f(2x-2y) = 4[f(x+2y)+f(x-2y)] - 8f(x) + 2f(2x) - 6f(2y) \tag{2.20}$$

for all $x, y \in E_1$. Interchanging x and y in (2.19) and using the evenness of f, we obtain

$$f(x+2y)+f(x-2y) = 4[f(x+y)+f(x-y)] - 8f(y) + 2f(2y) - 6f(x) \tag{2.21}$$

for all $x, y \in E_1$. Using (2.21) in (2.20), we get

$$f(2x+2y)+f(2x-2y) = 16[f(x+y)+f(x-y)] + 2f(2y) - 32f(y) + 2f(2x) - 32f(x) \tag{2.22}$$

for all $x, y \in E_1$. Rearranging (2.22), we have

$$\{f(2x+2y) - 16f(x+y)\} + \{f(2x-2y) - 16f(x-y)\} = 2\{f(2x) - 16f(x)\} + 2\{f(2y) - 16f(y)\} \tag{2.23}$$

for all $x, y \in E_1$. Let $\alpha : E_1 \to E_2$ defined by

$$\alpha(x) = f(2x) - 16f(x), \qquad \forall x \in E_1. \tag{2.24}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Applying (2.24) in (2.23), we arrive at

$$\alpha(x+y)+\alpha(x-y)=2\alpha(x)+2\alpha(y) \qquad \forall x\in E_1. \tag{2.25}$$

Hence $\alpha : E_1 \to E_2$ is quadratic mapping.

Since α is quadratic, we have $\alpha(2x) = 4\alpha(x)$ for all $x \in E_1$. Then

$$f(4x) = 20f(2x) - 64f(x) \tag{2.26}$$

for all $x \in E_1$. Replacing (x, y) by $(2x, 2y)$ in (2.19), we get

$$\begin{aligned} f(2(2x+y)) + f(2(2x-y)) \\ = 4[f(2(x+y)) + f(2(x-y))] - 8f(2x) + 2f(4x) - 6f(2y) \end{aligned} \tag{2.27}$$

for all $x, y \in E_1$. Using (2.26) in (2.27), we obtain

$$\begin{aligned} f(2(2x+y)) + f(2(2x-y)) \\ = 4[f(2(x+y)) + f(2(x-y))] + 32\{f(2x) - 4f(x)\} - 6f(2y) \end{aligned} \tag{2.28}$$

for all $x, y \in E_1$. Multiplying (2.19) by 4, we arrive at

$$\begin{aligned} 4f(2x+y) + 4f(2x-y) \\ = 16[f(x+y) + f(x-y)] + 8\{f(2x) - 4f(x)\} - 24f(y) \end{aligned} \tag{2.29}$$

for all $x, y \in E_1$. Subtracting (2.29) from (2.28), we get

$$\begin{aligned} &\{f(2(2x+y)) - 4f(2x+y)\} + \{f(2(2x-y)) - 4f(2x-y)\} \\ &= 4\{f(2(x+y)) - 4f(x+y)\} + 4\{f(2(x-y)) - 4f(x-y)\} \\ &\quad + 24\{f(2x) - 4f(x)\} - 6\{f(2y) - 4f(y)\} \end{aligned} \tag{2.30}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x, y \in E_1$. Let $\beta : E_1 \to E_2$ be defined by

$$\beta(x) = f(2x) - 4f(x), \forall x \in E_1. \tag{2.31}$$

Applying (2.30) in (2.31), we arrive at

$$\beta(2x+y) + \beta(2x-y) = 4[\beta(x+y) + \beta(x-y)] + 24\beta(x) - 6\beta(y) \tag{2.32}$$

for all $x, y \in E_1$. Hence $\beta : E_1 \to E_2$ is quartic mapping.

On the other hand, we have

$$f(x) = \frac{\beta(x) - \alpha(x)}{12} \qquad \forall x \in E_1. \tag{2.33}$$

This means that f is quadratic-quartic function. This completes the proof of the theorem. □

Theorem 2.2. *Let $f : E_1 \to E_2$ be a function satisfying (1.20) for all $x, y \in E_1$. If f is odd then f is additive - cubic.*

Proof. Let f be an odd function (i.e., $f(-x) = -f(x)$). Then equation (1.20) becomes

$$f(x+ay) + f(x-ay) = a^2[f(x+y) + f(x-y)] + 2(1-a^2) f(x) \tag{2.34}$$

for all $x, y \in E_1$. By Lemma 2.2 of [13], f is additive-cubic. □

Theorem 2.3. *Let $f : E_1 \to E_2$ be a function satisfying (1.20) for all $x, y \in E_1$ if and only if there exists functions $A : E_1 \to E_2$, $B : E_1 \times E_1 \to E_2$, $C : E_1 \times E_1 \times E_1 \to E_2$ and $D : E_1 \times E_1 \times E_1 \times E_1 \to E_2$ such that*

$$f(x) = A(x) + B(x,x) + C(x,x,x) + D(x,x,x,x) \tag{2.35}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$, where A is additive, B is symmetric bi-additive, C is symmetric for each fixed one variable and is additive for fixed two variables and D is symmetric multi-additive.

Proof. Let $f : E_1 \to E_2$ be a function satisfying (1.20). We decompose f into even and odd parts by setting

$$f_e(x) = \frac{1}{2}\{f(x) + f(-x)\}, \qquad f_o(x) = \frac{1}{2}\{f(x) - f(-x)\}$$

for all $x \in E_1$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in E_1$. It is easy to show that the functions f_e and f_o satisfy (1.20). Hence by Theorem 2.1 and 2.2, we see that the function f_e is quadratic-quartic and f_o is additive-cubic, respectively. Thus there exist a symmetric bi-additive function $B : E_1 \times E_1 \to E_2$ and a symmetric multi-additive function $D : E_1 \times E_1 \times E_1 \times E_1 \to E_2$ such that $f_e(x) = B(x,x) + D(x,x,x,x)$ for all $x \in E_1$, and the function $A : E_1 \to E_2$ is additive and $C : E_1 \times E_1 \times E_1 \to E_2$ such that $f_o(x) = A(x) + C(x,x,x)$, where C is symmetric for each fixed one variable and is additive for fixed two variables. Hence we get (2.35) for all $x \in E_1$.

Conversely let $f(x) = A(x) + B(x,x) + C(x,x,x) + D(x,x,x,x)$ for all $x \in E_1$, where A is additive, B is symmetric bi-additive, C is symmetric for each fixed one variable and is additive for fixed two variables and D is symmetric multi-additive. Then it is easy to show that f satisfies (1.20). □

vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. Stability of the Functional Equation (1.20)

In this section, we investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.20). Throughout this section, let E_1 be a real normed space and E_2 be a Banach space. Define a difference operator $Df : E_1 \times E_1 \to E_2$ by

$$\begin{aligned} &Df(x,y) \\ &\quad = f(x+ay) + f(x-ay) - a^2 [f(x+y) + f(x-y)] - 2(1-a^2) f(x) \\ &\qquad - \frac{a^4 - a^2}{12} [f(2y) + f(-2y) - 4f(y) - 4f(-y)] \end{aligned}$$

for all $x, y \in E_1$.

Theorem 3.1. *Let $\phi_b : E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$(3.1) \qquad \sum_{n=0}^{\infty} \frac{\phi_b(2^n x, 2^n y)}{4^n} \quad \textit{converges and} \quad \lim_{n\to\infty} \frac{\phi_b(2^n x, 2^n y)}{4^n} = 0$$

for all $x, y \in E_1$ and let $f : E_1 \to E_2$ be an even function which satisfies the inequality

$$(3.2) \qquad \|Df(x,y)\| \le \phi_b(x,y)$$

for all $x, y \in E_1$. Then there exists a unique quadratic function $B : E_1 \to E_2$ such that

$$(3.3) \qquad \|f(2x) - 16f(x) - B(x)\| \le \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b(2^k x)}{4^k}$$

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009
journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$, where the mapping $B(x)$ and $\Phi_b(2^k x)$ are defined by

$$B(x) = \lim_{n\to\infty} \frac{1}{4^n} \left\{ f\left(2^{n+1}x\right) - 16 f\left(2^n x\right) \right\} \tag{3.4}$$

$$\Phi_b\left(2^k x\right) = \frac{1}{a^4 - a^2} \Big[12\left(1-a^2\right)\phi_b\left(0, 2^k x\right) + 12a^2\phi_b\left(2^k x, 2^k x\right) + 6\phi_b\left(0, 2^{k+1}x\right) + 12\phi_b\left(2^k a x, 2^k x\right) \Big] \tag{3.5}$$

for all $x \in E_1$.

Proof. Using the evenness of f, from (3.2) we get

$$\left\| f(x+ay) + f(x-ay) - a^2\left[f(x+y) + f(x-y)\right] - 2\left(1-a^2\right)f(x) - \frac{\left(a^4-a^2\right)}{12}\left[2f(2y) - 8f(y)\right] \right\| \le \phi_b(x,y) \tag{3.6}$$

for all $x, y \in E_1$. Interchanging x and y in (3.6), we obtain

$$\left\| f(ax+y) + f(ax-y) - a^2\left[f(x+y) + f(x-y)\right] - 2\left(1-a^2\right)f(y) - \frac{\left(a^4-a^2\right)}{12}\left[2f(2x) - 8f(x)\right] \right\| \le \phi_b(y,x) \tag{3.7}$$

for all $x, y \in E_1$. Letting $y = 0$ in (3.7), we get

$$\left\| 2f(ax) - 2a^2 f(x) - \frac{\left(a^4-a^2\right)}{12}\left[2f(2x) - 8f(x)\right] \right\| \le \phi_b(0,x) \tag{3.8}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. Putting $y = x$ in (3.7), we obtain

$$\begin{aligned} &\left\| f\left((a+1)x\right) + f\left((a-1)x\right) - a^2 f(2x) - 2\left(1-a^2\right) f(x) \right. \\ &\qquad \left. - \frac{\left(a^4-a^2\right)}{12}\left[2f(2x) - 8f(x)\right] \right\| \leq \phi_b(x,x) \end{aligned} \tag{3.9}$$

for all $x \in E_1$. Replacing x by $2x$ in (3.8), we get

$$\left\| 2f(2ax) - 2a^2 f(2x) - \frac{\left(a^4-a^2\right)}{12}\left[2f(4x) - 8f(2x)\right] \right\| \leq \phi_b(0,2x) \tag{3.10}$$

for all $x \in E_1$. Setting y by ax in (3.7), we obtain

$$\begin{aligned} &\left\| f(2ax) - a^2\left[f\left((1+a)x\right) + f\left((1-a)x\right)\right] - 2\left(1-a^2\right) f(ax) \right. \\ &\qquad \left. - \frac{\left(a^4-a^2\right)}{12}\left[2f(2x) - 8f(x)\right] \right\| \leq \phi_b(ax,x) \end{aligned} \tag{3.11}$$

for all $x \in E_1$. Multiplying (3.8), (3.9), (3.10) and (3.11) by $12(1-a^2)$, $12a^2$, 6 and 12 respectively, we have

$$\begin{aligned} &\left(a^4-a^2\right)\left\| f(4x) - 20f(2x) + 64f(x) \right\| \\ &= \left\| \left\{ 24\left(1-a^2\right) f(ax) - 24a^2\left(1-a^2\right) f(x) \right. \right. \\ &\qquad \left. - \frac{12\left(1-a^2\right)\left(a^4-a^2\right)}{12}\left[2f(2x) - 8f(x)\right] \right\} \\ &\qquad + \left\{ 12a^2 f\left((a+1)x\right) + 12a^2 f\left((a-1)x\right) - 12a^4 f(2x) \right. \\ &\qquad \left. - 24a^2\left(1-a^2\right) f(x) - \frac{12a^2\left(a^4-a^2\right)}{12}\left[2f(2x) - 8f(x)\right] \right\} \end{aligned}$$

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$+\left\{-12f(2ax)+12a^2f(2x)+\frac{6(a^4-a^2)}{12}[2f(4x)-8f(2x)]\right\}$$
$$+\Big\{12f(2ax)-12a^2[f((1+a)x)+f((1-a)x)]$$
$$-24(1-a^2)f(ax)-\frac{12(a^4-a^2)}{12}[2f(2x)-8f(x)]\Big\}\Big\|$$
$$\leq 12(1-a^2)\phi_b(0,x)+12a^2\phi_b(x,x)+6\phi_b(0,2x)+12\phi_b(ax,x)$$

for all $x \in E_1$. Hence from the above inequality, we get

$$\|f(4x)-20f(2x)+64f(x)\| \leq \frac{1}{(a^4-a^2)}\left[12(1-a^2)\phi_b(0,x)+12a^2\phi_b(x,x)+6\phi_b(0,2x)+12\phi_b(ax,x)\right] \tag{3.12}$$

for all $x \in E_1$. From (3.12), we arrive at

$$\|f(4x)-20f(2x)+64f(x)\| \leq \Phi_b(x), \tag{3.13}$$

where

$$\Phi_b(x)=\frac{1}{a^4-a^2}\left[12(1-a^2)\phi_b(0,x)+12a^2\phi_b(x,x)+6\phi_b(0,2x)+12\phi_b(ax,x)\right]$$

for all $x \in E_1$. It is easy to see from (3.13) that

$$\|f(4x)-16f(2x)-4\{f(2x)-16f(x)\}\| \leq \Phi_b(x) \tag{3.14}$$

for all $x \in E_1$. Using (2.24) in (3.14), we obtain

$$\|\alpha(2x)-4\alpha(x)\| \leq \Phi_b(x) \tag{3.15}$$

vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. From (3.15), we have

$$\left\| \frac{\alpha(2x)}{4} - \alpha(x) \right\| \le \frac{\Phi_b(x)}{4} \tag{3.16}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 4 in (3.16), we obtain

$$\left\| \frac{\alpha(2^2x)}{4^2} - \frac{\alpha(2x)}{4} \right\| \le \frac{\Phi_b(2x)}{4^2} \tag{3.17}$$

for all $x \in E_1$. From (3.16) and (3.17), we arrive at

$$\begin{aligned} \left\| \frac{\alpha(2^2x)}{4^2} - \alpha(x) \right\| &\le \left\| \frac{\alpha(2^2x)}{4^2} - \frac{\alpha(2x)}{4} \right\| + \left\| \frac{\alpha(2x)}{4} - \alpha(x) \right\| \\ &\le \frac{1}{4}\left[\Phi_b(x) + \frac{\Phi_b(2x)}{4} \right] \end{aligned} \tag{3.18}$$

for all $x \in E_1$. In general for any positive integer n, we get

$$\begin{aligned} \left\| \frac{\alpha(2^nx)}{4^n} - \alpha(x) \right\| &\le \frac{1}{4}\sum_{k=0}^{n-1} \frac{\Phi_b(2^kx)}{4^k} \\ &\le \frac{1}{4}\sum_{k=0}^{\infty} \frac{\Phi_b(2^kx)}{4^k} \end{aligned} \tag{3.19}$$

for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{ \frac{\alpha(2^nx)}{4^n} \right\}$, replace

journal of inequalities in pure and applied mathematics
issn: 1443-5756

x by $2^m x$ and divide by 4^m in (3.19). For any $m, n > 0$, we have

$$\left\| \frac{\alpha\left(2^{n+m}x\right)}{4^{n+m}} - \frac{\alpha\left(2^m x\right)}{4^m} \right\| = \frac{1}{4^m} \left\| \frac{\alpha\left(2^n 2^m x\right)}{4^n} - \alpha\left(2^m x\right) \right\|$$
$$\leq \frac{1}{4} \sum_{k=0}^{n-1} \frac{\Phi_b\left(2^{k+m}x\right)}{4^{k+m}}$$
$$\leq \frac{1}{4} \sum_{k=0}^{\infty} \frac{\Phi_b\left(2^{k+m}x\right)}{4^{k+m}}$$
$$\to 0 \qquad \text{as} \qquad m \to \infty$$

for all $x \in E_1$. Hence the sequence $\left\{\frac{\alpha(2^n x)}{4^n}\right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a quadratic mapping $B : E_1 \to E_2$ such that

$$B(x) = \lim_{n\to\infty} \frac{\alpha\left(2^n x\right)}{4^n} \qquad \forall x \in E_1.$$

Letting $n \to \infty$ in (3.19) and using (2.24), we see that (3.3) holds for all $x \in E_1$. To prove that B satisfies (1.20), replace (x, y) by $(2^n x, 2^n y)$ and divide by 4^n in (3.2). We obtain

$$\frac{1}{4^n} \Big\| f\left(2^n(x+ay)\right) + f\left(2^n(x-ay)\right) - a^2\left[f\left(2^n(x+y)\right) + f\left(2^n(x-y)\right)\right]$$
$$- 2\left(1-a^2\right) f\left(2^n x\right) - \frac{\left(a^4-a^2\right)}{12}\left[f\left(2^n(2y)\right) + f\left(2^n(-2y)\right)\right]$$
$$- \frac{\left(a^4-a^2\right)}{12}\left[-4f\left(2^n y\right) - 4f\left(2^n(-y)\right)\right] \Big\| \leq \frac{\phi\left(2^n x, 2^n y\right)}{4^n}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x, y \in E_1$. Letting $n \to \infty$ in the above inequality, we see that

$$\Big\| B(x+ay) + B(x-ay) - a^2 [B(x+y) + B(x-y)] - 2\left(1-a^2\right) B(x) - \frac{(a^4-a^2)}{12}[B(2y) + B(-2y) - 4B(y) - 4Bf(-y)] \Big\| \le 0,$$

which gives

$$\begin{aligned} B(x+ay) + B(x-ay) &= a^2 [B(x+y) + B(x-y)] + 2\left(1-a^2\right) B(x) \\ &\quad + \frac{(a^4-a^2)}{12} [B(2y) + B(-2y) - 4B(y) - 4Bf(-y)] \end{aligned}$$

for all $x, y \in E_1$. Hence B satisfies (1.20). To prove that B is unique, let B' be another quadratic function satisfying (1.20) and (3.3). We have

$$\begin{aligned} \|B(x) - B'(x)\| &= \frac{1}{4^n} \|B(2^n x) - B'(2^n x)\| \\ &\le \frac{1}{4^n} \{\|B(2^n x) - \alpha(2^n x)\| + \|\alpha(2^n x) - B'(2^n x)\|\} \\ &\le \frac{1}{4^n} \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_b\left(2^k x\right)}{4^k} \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

for all $x \in E_1$. Hence B is unique. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.20).

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Corollary 3.2. *Let ε, p be nonnegative real numbers. Suppose that an even function $f : E_1 \to E_2$ satisfies the inequality*

$$(3.20)\qquad \|Df(x,y)\| \le \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \le p < 2; \\ \varepsilon, & 0 \le p < 1; \\ \varepsilon \|x\|^p \|y\|^p, & 0 \le p < 1; \\ \varepsilon\left(\|x\|^p \|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & \end{cases}$$

for all $x, y \in E_1$. Then there exists a unique quadratic function $B : E_1 \to E_2$ such that

$$(3.21)\qquad \|f(2x) - 16f(x) - B(x)\| \le \begin{cases} \frac{\lambda_1 \|x\|^p}{4-2^p}, \\ 10\lambda_2, \\ \frac{\lambda_3 \|x\|^{2p}}{4-2^{2p}}, \\ \frac{\lambda_4 \|x\|^{2p}}{4-2^{2p}}, \end{cases}$$

where

$$\lambda_1 = \frac{\varepsilon\left\{24 + 12a^2 + 12\left(a^p\right) + 6\left(2^p\right)\right\}}{a^4 - a^2}, \qquad \lambda_2 = \frac{\varepsilon}{a^4 - a^2},$$

$$\lambda_3 = \frac{12\varepsilon\left\{a^2 + a^p\right\}}{a^4 - a^2} \quad \text{and} \quad \lambda_4 = \frac{\varepsilon\left\{24 + 24a^2 + 12\left(a^p\right) + 12\left(a^{2p}\right) + 6\left(2^{2p}\right)\right\}}{a^4 - a^2}$$

for all $x \in E_1$.

Theorem 3.3. *Let $\phi_d : E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$(3.22)\qquad \sum_{n=0}^{\infty} \frac{\phi_d\left(2^n x, 2^n y\right)}{16^n} \quad \text{converges and} \quad \lim_{n\to\infty} \frac{\phi_d\left(2^n x, 2^n y\right)}{16^n} = 0$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x, y \in E_1$ and let $f : E_1 \to E_2$ be an even function which satisfies the inequality

$$\|Df(x,y)\| \le \phi_d(x,y) \tag{3.23}$$

for all $x, y \in E_1$. Then there exists a unique quartic function $D : E_1 \to E_2$ such that

$$\|f(2x) - 4f(x) - D(x)\| \le \frac{1}{16}\sum_{k=0}^{\infty} \frac{\Phi_d(2^k x)}{16^k} \tag{3.24}$$

for all $x \in E_1$, where the mapping $D(x)$ and $\Phi_d(2^k x)$ are defined by

$$D(x) = \lim_{n\to\infty} \frac{1}{16^n}\left\{f\left(2^{n+1}x\right) - 4f\left(2^n x\right)\right\}, \tag{3.25}$$

$$\Phi_d\left(2^k x\right) = \frac{1}{a^4 - a^2}\Big[12\left(1-a^2\right)\phi_d\left(0, 2^k x\right) + 12a^2\phi_d\left(2^k x, 2^k x\right) + 6\phi_d\left(0, 2^{k+1}x\right) + 12\phi_d\left(2^k a x, 2^k x\right)\Big] \tag{3.26}$$

for all $x \in E_1$.

Proof. Along similar lines to those in the proof of Theorem 3.1, we have

$$\|f(4x) - 20f(2x) + 64f(x)\| \le \Phi_d(x), \tag{3.27}$$

where

$$\Phi_d(x) = \frac{1}{a^4 - a^2}\left[12\left(1-a^2\right)\phi_d(0,x) + 12a^2\phi_d(x,x) + 6\phi_d(0,2x) + 12\phi_d(ax,x)\right]$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. It is easy to see from (3.27) that

$$\|f(4x) - 4f(2x) - 16\{f(2x) - 4f(x)\}\| \le \Phi_d(x) \tag{3.28}$$

for all $x \in E_1$. Using (2.31) in (3.28), we obtain

$$\|\beta(2x) - 16\beta(x)\| \le \Phi_d(x) \tag{3.29}$$

for all $x \in E_1$. From (3.29), we have

$$\left\|\frac{\beta(2x)}{16} - \beta(x)\right\| \le \frac{\Phi_d(x)}{16} \tag{3.30}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 16 in (3.30), we obtain

$$\left\|\frac{\beta(2^2x)}{16^2} - \frac{\beta(2x)}{16}\right\| \le \frac{\Phi_d(2x)}{16^2} \tag{3.31}$$

for all $x \in E_1$. From (3.30) and (3.31), we arrive at

$$\begin{aligned}\left\|\frac{\beta(2^2x)}{16^2} - \beta(x)\right\| &\le \left\|\frac{\beta(2^2x)}{16^2} - \frac{\beta(2x)}{16}\right\| + \left\|\frac{\beta(2x)}{16} - \beta(x)\right\| \\ &\le \frac{1}{16}\left[\Phi_d(x) + \frac{\Phi_d(2x)}{16}\right]\end{aligned} \tag{3.32}$$

for all $x \in E_1$. In general for any positive integer n, we get

$$\begin{aligned}\left\|\frac{\beta(2^nx)}{16^n} - \beta(x)\right\| &\le \frac{1}{16}\sum_{k=0}^{n-1}\frac{\Phi_d(2^kx)}{16^k} \\ &\le \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d(2^kx)}{16^k}\end{aligned} \tag{3.33}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{\frac{\beta(2^n x)}{16^n}\right\}$, replace x by $2^m x$ and divide by 16^m in (3.33). For any $m, n > 0$, we then have

$$\begin{aligned}\left\|\frac{\beta\left(2^{n+m}x\right)}{16^{n+m}} - \frac{\beta\left(2^m x\right)}{16^m}\right\| &= \frac{1}{16^m}\left\|\frac{\beta\left(2^n 2^m x\right)}{16^n} - \beta\left(2^m x\right)\right\| \\ &\leq \frac{1}{16}\sum_{k=0}^{n-1}\frac{\Phi_d\left(2^{k+m}x\right)}{16^{k+m}} \\ &\leq \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d\left(2^{k+m}x\right)}{16^{k+m}} \\ &\to 0 \quad \text{as} \quad m \to \infty\end{aligned}$$

for all $x \in E_1$. Hence the sequence $\left\{\frac{\beta(2^n x)}{16^n}\right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a quartic mapping $D : E_1 \to E_2$ such that

$$D(x) = \lim_{n\to\infty}\frac{\beta\left(2^n x\right)}{16^n} \qquad \forall x \in E_1.$$

Letting $n \to \infty$ in (3.33) and using (2.31) we see that (3.24) holds for all $x \in E_1$. The proof that D satisfies (1.20) and is unique is similar to that for Theorem 3.1. $\square$

The following corollary is an immediate consequence of Theorem 3.3 concerning the stability of (1.20).

Corollary 3.4. *Let ε, p be nonnegative real numbers. Suppose that an even function*

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$f : E_1 \to E_2$ satisfies the inequality

$$\|Df(x,y)\| \le \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \le p < 4; \\ \varepsilon, & \\ \varepsilon \|x\|^p \|y\|^p, & 0 \le p < 2; \\ \varepsilon\left(\|x\|^p \|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & 0 \le p < 2 \end{cases} \tag{3.34}$$

for all $x, y \in E_1$. Then there exists a unique quartic function $D : E_1 \to E_2$ such that

$$\|f(2x) - 4f(x) - D(x)\| \le \begin{cases} \frac{\lambda_1 \|x\|^p}{16-2^p}, \\ 2\lambda_2, \\ \frac{\lambda_3 \|x\|^{2p}}{16-2^{2p}}, \\ \frac{\lambda_4 \|x\|^{2p}}{16-2^{2p}}, \end{cases} \tag{3.35}$$

for all $x \in E_1$, where λ_i $(i = 1, 2, 3, 4)$ are given in Corollary 3.2.

Theorem 3.5. *Let $\phi : E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi_b(2^n x, 2^n y)}{4^n}, \quad \sum_{n=0}^{\infty} \frac{\phi_d(2^n x, 2^n y)}{16^n} \quad \textit{converges} \tag{3.36}$$

and

$$\lim_{n\to\infty} \frac{\phi_b(2^n x, 2^n y)}{4^n} = 0 = \lim_{n\to\infty} \frac{\phi_d(2^n x, 2^n y)}{16^n} \tag{3.37}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x, y \in E_1$. Suppose that an even function $f : E_1 \to E_2$ satisfies the inequalities (3.2) and (3.23) for all $x, y \in E_1$. Then there exists a unique quadratic function $B : E_1 \to E_2$ and a unique quartic function $D : E_1 \to E_2$ such that

$$\|f(x) - B(x) - D(x)\| \leq \frac{1}{12}\left\{\frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b\left(2^k x\right)}{4^k} + \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d\left(2^k x\right)}{16^k}\right\} \tag{3.38}$$

for all $x \in E_1$, where $\Phi_b\left(2^k x\right)$ and $\Phi_d\left(2^k x\right)$ are defined in (3.5) and (3.26), respectively for all $x \in E_1$.

Proof. By Theorems 3.1 and 3.3, there exists a unique quadratic function $B_1 : E_1 \to E_2$ and a unique quartic function $D_1 : E_1 \to E_2$ such that

$$\|f(2x) - 16f(x) - B_1(x)\| \leq \frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b\left(2^k x\right)}{4^k} \tag{3.39}$$

and

$$\|f(2x) - 4f(x) - D_1(x)\| \leq \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d\left(2^k x\right)}{16^k} \tag{3.40}$$

for all $x \in E_1$. Now from (3.39) and (3.40), one can see that

$$\begin{aligned}
&\left\|f(x) + \frac{1}{12}B_1(x) - \frac{1}{12}D_1(x)\right\| \\
&\qquad = \left\|\left\{-\frac{f(2x)}{12} + \frac{16f(x)}{12} + \frac{B_1(x)}{12}\right\} + \left\{\frac{f(2x)}{12} - \frac{4f(x)}{12} - \frac{D_1(x)}{12}\right\}\right\| \\
&\qquad \leq \frac{1}{12}\left\{\|f(2x) - 16f(x) - B_1(x)\| + \|f(2x) - 4f(x) - D_1(x)\|\right\}
\end{aligned}$$

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$\leq \frac{1}{12}\left\{\frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b\left(2^k x\right)}{4^k}+\frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d\left(2^k x\right)}{16^k}\right\}$$

for all $x \in E_1$. Thus we obtain (3.38) by defining $B(x) = \frac{-1}{12}B_1(x)$ and $D(x) = \frac{1}{12}D_1(x)$, where $\Phi_b\left(2^k x\right)$ and $\Phi_d\left(2^k x\right)$ are defined in (3.5) and (3.26), respectively for all $X \in E_1$. □

The following corollary is the immediate consequence of Theorem 3.5 concerning the stability of (1.20).

Corollary 3.6. *Let ϵ, p be nonnegative real numbers. Suppose an even function $f : E_1 \longrightarrow E_2$ satisfies the inequality*

$$(3.41)\qquad \|Df(x,y)\| \leq \begin{cases} \varepsilon\left(\|x\|^p+\|y\|^p\right), & 0 \leq p < 2; \\ \varepsilon, & \\ \varepsilon\|x\|^p\|y\|^p, & 0 \leq p < 1; \\ \varepsilon\left(\|x\|^p\|y\|^p+\left\{\|x\|^{2p}+\|y\|^{2p}\right\}\right), & 0 \leq p < 1 \end{cases}$$

for all $x, y \in E_1$. Then there exists a unique quadratic function $B : E_1 \longrightarrow E_2$ and a unique quartic function $D : E_1 \longrightarrow E_2$ such that

$$(3.42)\qquad \|f(x) - B(x) - D(x)\| \leq \begin{cases} \frac{\lambda_1\|x\|^p}{12}\left\{\frac{1}{4-2^p}+\frac{1}{16-2^p}\right\}, \\ \lambda_2 \\ \frac{\lambda_3\|x\|^{2p}}{12}\left\{\frac{1}{4-2^{2p}}+\frac{1}{16-2^{2p}}\right\}, \\ \frac{\lambda_4\|x\|^{p}}{12}\left\{\frac{1}{4-2^{2p}}+\frac{1}{16-2^{2p}}\right\}, \end{cases}$$

for all $x \in E_1$, where $\lambda_i\ (i = 1,2,3,4)$ are given in Corollary 3.2.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Theorem 3.7. *Let* $\phi_a : E_1 \times E_1 \to [0, \infty)$ *be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi_a(2^n x, 2^n y)}{2^n} \text{ converges and } \lim_{n\to\infty} \frac{\phi_a(2^n x, 2^n y)}{2^n} = 0 \tag{3.43}$$

for all $x, y \in E_1$ *and let* $f : E_1 \to E_2$ *be an odd function with* $f(0) = 0$ *which satisfies the inequality*

$$\|Df(x,y)\| \leq \phi_a(x,y) \tag{3.44}$$

for all $x, y \in E_1$. *Then there exists a unique additive function* $A : E_1 \to E_2$ *such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Phi_a(2^k x)}{2^k} \tag{3.45}$$

for all $x \in E_1$, *where the mapping* $A(x)$ *and* $\Phi_a(2^k x)$ *are defined by*

$$A(x) = \lim_{n\to\infty} \frac{1}{2^n} \left\{ f(2^{n+1}x) - 8f(2^n x) \right\} \tag{3.46}$$

$$\begin{aligned} \Phi_a(2^k x) = \frac{1}{a^4 - a^2} \Big[&(5 - 4a^2)\phi_a(2^k x, 2^k x) + a^2 \phi_a(2^{k+1}x, 2^{k+1}x) \\ &+ 2a^2 \phi_a(2^{k+1}x, 2^k x) + (4 - 2a^2)\phi_a(2^k x, 2^{k+1}x) + \phi_a(2^k x, 2^k 3x) \\ &+ 2\phi_a(2^k(1+a)x, 2^k x) + 2\phi_a(2^k(1-a)x, 2^k x) \\ &+ \phi_a(2^k(1+2a)x, 2^k x) + \phi_a(2^k(1-2a)x, 2^k x) \Big] \end{aligned} \tag{3.47}$$

for all $x \in E_1$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. Using the oddness of f and from (3.44), we get

$$\left\| f(x+ay) + f(x-ay) - a^2\left[f(x+y) + f(x-y)\right] - 2\left(1-a^2\right)f(x)\right\| \le \phi_a(x,y) \tag{3.48}$$

for all $x \in E_1$. Replacing y by x in (3.48), we obtain

$$\left\| f((1+a)x) + f((1-a)x) - a^2 f(2x) - 2\left(1-a^2\right)f(x)\right\| \le \phi_a(x,x) \tag{3.49}$$

for all $x \in E_1$. Replacing x by $2x$ in (3.49), we get

$$\left\| f(2(1+a)x) + f(2(1-a)x) - a^2 f(4x) - 2\left(1-a^2\right)f(2x)\right\| \le \phi_a(2x,2x) \tag{3.50}$$

for all $x \in E_1$. Again replacing (x,y) by $(2x,x)$ in (3.48), we obtain

$$\left\| f((2+a)x) + f((2-a)x) - a^2 f(3x) - a^2 f(x) - 2\left(1-a^2\right)f(2x)\right\| \le \phi_a(2x,x) \tag{3.51}$$

for all $x \in E_1$. Replacing y by $2x$ in (3.48), we get

$$\left\| f((1+2a)x) + f((1-2a)x) - a^2 f(3x) + a^2 f(x) - 2\left(1-a^2\right)f(x)\right\| \le \phi_a(x,2x) \tag{3.52}$$

for all $x \in E_1$. Replacing y by $3x$ in (3.48), we obtain

$$\left\| f((1+3a)x) + f((1-3a)x) - a^2 f(4x) + a^2 f(2x) - 2\left(1-a^2\right)f(x)\right\| \le \phi_a(x,3x) \tag{3.53}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. Replacing (x, y) by $((1+a)x, x)$ in (3.48), we get

$$\left\| f\left((1+2a)x\right) + f(x) - a^2 f\left((2+a)x\right) - a^2 f(ax) - 2\left(1-a^2\right) f\left((1+a)x\right) \right\| \le \phi_a\left((1+a)x, x\right) \tag{3.54}$$

for all $x \in E_1$. Again replacing (x, y) by $((1-a)x, x)$ in (3.48), we obtain

$$\left\| f\left((1-2a)x\right) + f(x) - a^2 f\left((2-a)x\right) + a^2 f(ax) - 2\left(1-a^2\right) f\left((1-a)x\right) \right\| \le \phi_a\left((1-a)x, x\right) \tag{3.55}$$

for all $x \in E_1$. Adding (3.54) and (3.55), we arrive at

$$\begin{aligned} &\Big\| f\left((1+2a)x\right) + f\left((1-2a)x\right) + 2f(x) - a^2 f\left((2+a)x\right) \\ &\quad - a^2 f\left((2-a)x\right) - 2\left(1-a^2\right) f\left((1+a)x\right) - 2\left(1-a^2\right) f\left((1-a)x\right) \Big\| \\ &\qquad \le \phi_a\left((1+a)x, x\right) + \phi_a\left((1-a)x, x\right) \end{aligned} \tag{3.56}$$

for all $x \in E_1$. Replacing (x, y) by $((1+2a)x, x)$ in (3.48), we get

$$\begin{aligned} &\Big\| f\left((1+3a)x\right) + f\left((1+a)x\right) - a^2 f\left(2(1+a)x\right) - a^2 f(2ax) \\ &\quad -2\left(1-a^2\right) f\left((1+2a)x\right) \Big\| \le \phi_a\left((1+2a)x, x\right) \end{aligned} \tag{3.57}$$

for all $x \in E_1$. Again replacing (x, y) by $((1-2a)x, x)$ in (3.48), we obtain

$$\begin{aligned} &\Big\| f\left((1-3a)x\right) + f\left((1-a)x\right) - a^2 f\left(2(1-a)x\right) + a^2 f(2ax) \\ &\quad -2\left(1-a^2\right) f\left((1-2a)x\right) \Big\| \le \phi_a\left((1-2a)x, x\right) \end{aligned} \tag{3.58}$$

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009
journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. Adding (3.57) and (3.58), we arrive at

$$(3.59)\quad \Big\| f\left(\left(1+3a\right)x\right) + f\left(\left(1-3a\right)x\right) + f\left(\left(1+a\right)x\right) + f\left(\left(1-a\right)x\right) - a^2 f\left(2\left(1+a\right)x\right) - a^2 f\left(2\left(1-a\right)x\right) - 2\left(1-a^2\right) f\left(\left(1+2a\right)x\right) - 2\left(1-a^2\right) f\left(\left(1-2a\right)x\right) \Big\| \leq \phi_a\left(\left(1+2a\right)x, x\right) + \phi_a\left(\left(1-2a\right)x, x\right)$$

for all $x \in E_1$. Now multiplying (3.49) by $2(1-a^2)$, (3.51) by a^2 and adding (3.52) and (3.56), we have

$$\begin{aligned}
&\left(a^4 - a^2\right)\left\| f\left(3x\right) - 4f\left(2x\right) + 5f\left(x\right)\right\| \\
&= \Big\| \left\{2\left(1-a^2\right) f\left(\left(1+a\right)x\right) + 2\left(1-a^2\right) f\left(\left(1-a\right)x\right) - 2a^2\left(1-a^2\right) f\left(2x\right) \right. \\
&\qquad \left. -4\left(1-a^2\right)^2 f\left(x\right)\right\} + \left\{a^2 f\left(\left(2+a\right)x\right) + a^2 f\left(\left(2-a\right)x\right) - a^4 f\left(3x\right) \right. \\
&\qquad \left. - a^4 f\left(x\right) - 2a^2\left(1-a^2\right) f\left(2x\right)\right\} + \left\{-f\left(\left(1+2a\right)x\right) \right. \\
&\qquad \left. - f\left(\left(1-2a\right)x\right) + a^2 f\left(3x\right) - a^2 f\left(x\right) + 2\left(1-a^2\right) f\left(x\right)\right\} \\
&\qquad + \left\{f\left(\left(1+2a\right)x\right) + f\left(\left(1-2a\right)x\right) + 2f\left(x\right) - a^2 f\left(\left(2+a\right)x\right) \right. \\
&\qquad \left. - a^2 f\left(\left(2-a\right)x\right) - 2\left(1-a^2\right) f\left(\left(1+a\right)x\right) - 2\left(1-a^2\right) f\left(\left(1-a\right)x\right)\right\} \Big\| \\
&\leq 2\left(1-a^2\right)\phi_a\left(x, x\right) + a^2\phi_a\left(2x, x\right) + \phi_a\left(x, 2x\right) \\
&\qquad + \phi_a\left(\left(1+a\right)x, x\right) + \phi_a\left(\left(1-a\right)x, x\right)
\end{aligned}$$

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. Hence from the above inequality, we get

$$\begin{aligned} (3.60) \quad & \left\| f\left(3x\right) - 4f\left(2x\right) + 5f\left(x\right) \right\| \\ & \leq \frac{1}{\left(a^4 - a^2\right)} \left[2\left(1 - a^2\right) \phi_a\left(x, x\right) + a^2 \phi_a\left(2x, x\right) \right. \\ & \qquad \left. + \phi_a\left(x, 2x\right) + \phi_a\left(\left(1 + a\right)x, x\right) + \phi_a\left(\left(1 - a\right)x, x\right) \right] \end{aligned}$$

for all $x \in E_1$. Now multiplying (3.50) by a^2, (3.52) by $2(1-a^2)$ and adding (3.49), (3.53) and (3.59), we have

$$\begin{aligned} & \left(a^4 - a^2\right) \left\| f\left(4x\right) - 2f\left(3x\right) - 2f\left(2x\right) + 6f\left(x\right) \right\| \\ & = \left\| \left\{ -f\left(\left(1 + a\right)x\right) - f\left(\left(1 - a\right)x\right) + a^2 f\left(2x\right) + 2\left(1 - a^2\right) f\left(x\right) \right\} \right. \\ & \qquad + \left\{ a^2 f\left(2\left(1 + a\right)x\right) + a^2 f\left(2\left(1 - a\right)x\right) - a^4 f\left(4x\right) - 2a^2\left(1 - a^2\right) f\left(2x\right) \right\} \\ & \qquad + \left\{ 2\left(1 - a^2\right) f\left(\left(1 + 2a\right)x\right) + 2\left(1 - a^2\right) f\left(\left(1 - 2a\right)x\right) - 2a^2\left(1 - a^2\right) f\left(3x\right) \right. \\ & \qquad \left. + 2a^2\left(1 - a^2\right) f\left(x\right) \; -4\left(1 - a^2\right)^2 f\left(x\right) \right\} + \left\{ -f\left(\left(1 + 3a\right)x\right) \right. \\ & \qquad \left. - f\left(\left(1 - 3a\right)x\right) + a^2 f\left(4x\right) - a^2 f\left(2x\right) \; +2\left(1 - a^2\right) f\left(x\right) \right\} + \left\{ f\left(\left(1 + 3a\right)x\right) \right. \\ & \qquad + f\left(\left(1 - 3a\right)x\right) + f\left(\left(1 + a\right)x\right) + f\left(\left(1 - a\right)x\right) - a^2 f\left(2\left(1 + a\right)x\right) \\ & \qquad \left. \left. -a^2 f\left(2\left(1 - a\right)x\right) - 2\left(1 - a^2\right) f\left(\left(1 + 2a\right)x\right) \; -2\left(1 - a^2\right) f\left(\left(1 - 2a\right)x\right) \right\} \right\| \\ & \leq \phi_a\left(x, x\right) + a^2 \phi_a\left(2x, 2x\right) + 2\left(1 - a^2\right) \phi_a\left(x, 2x\right) \\ & \qquad + \phi_a\left(x, 3x\right) + \phi_a\left(\left(1 + 2a\right)x, x\right) + \phi_a\left(\left(1 - 2a\right)x, x\right) \end{aligned}$$

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. Hence from the above inequality, we get

$$\begin{aligned} (3.61) \quad & \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\ & \leq \frac{1}{(a^4 - a^2)} \left[\phi_a(x, x) + a^2 \phi_a(2x, 2x) + 2\left(1 - a^2\right) \phi_a(x, 2x) + \phi_a(x, 3x) \right. \\ & \qquad \left. + \phi_a((1 + 2a)x, x) + \phi_a((1 - 2a)x, x)\right] \end{aligned}$$

for all $x \in E_1$. From (3.60) and (3.61), we arrive at

$$\begin{aligned} (3.62) \quad & \|f(4x) - 10f(2x) + 16f(x)\| \\ & = \|2f(3x) - 8f(2x) + 10f(x) + f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\ & \leq 2\|f(3x) - 4f(2x) + 5f(x)\| \\ & \qquad + \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\ & \leq \frac{1}{(a^4 - a^2)} \left[\left(5 - 4a^2\right) \phi_a(x, x) + a^2 \phi_a(2x, 2x) + 2a^2 \phi_a(2x, x) \right. \\ & \qquad + \left(4 - 2a^2\right) \phi_a(x, 2x) + \phi_a(x, 3x) + 2\phi_a((1 + a)x, x) \\ & \qquad \left. + 2\phi_a((1 - a)x, x) + \phi_a((1 + 2a)x, x) + \phi_a((1 - 2a)x, x)\right] \end{aligned}$$

for all $x \in E_1$. From (3.62), we have

$$(3.63) \qquad \|f(4x) - 10f(2x) + 16f(x)\| \leq \Phi_a(x),$$

where

$$\begin{aligned} \Phi_a(x) = \frac{1}{(a^4 - a^2)} & \left[\left(5 - 4a^2\right) \phi_a(x, x) + a^2 \phi_a(2x, 2x) + 2a^2 \phi_a(2x, x) \right. \\ & + \left(4 - 2a^2\right) \phi_a(x, 2x) + \phi_a(x, 3x) + 2\phi_a((1 + a)x, x) \\ & \left. + 2\phi_a((1 - a)x, x) + \phi_a((1 + 2a)x, x) + \phi_a((1 - 2a)x, x)\right] \end{aligned}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E$. It is easy to see from (3.63)

$$\|f(4x) - 8f(2x) - 2\{f(2x) - 8f(x)\}\| \leq \Phi_a(x) \tag{3.64}$$

for all $x \in E_1$. Define a mapping $\gamma : E_1 \to E_2$ by

$$\gamma(x) = f(2x) - 8f(x) \tag{3.65}$$

for all $x \in E_1$. Using (3.65) in (3.64), we obtain

$$\|\gamma(2x) - 2\gamma(x)\| \leq \Phi_a(x) \tag{3.66}$$

for all $x \in E_1$. From (3.66), we have

$$\left\|\frac{\gamma(2x)}{2} - \gamma(x)\right\| \leq \frac{\Phi_a(x)}{2} \tag{3.67}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 2 in (3.67), we obtain

$$\left\|\frac{\gamma(2^2 x)}{2^2} - \frac{\gamma(2x)}{2}\right\| \leq \frac{\Phi_a(2x)}{2^2} \tag{3.68}$$

for all $x \in E_1$. From (3.67) and (3.68), we arrive at

$$\begin{aligned} \left\|\frac{\gamma(2^2 x)}{2^2} - \gamma(x)\right\| &\leq \left\|\frac{\gamma(2^2 x)}{2^2} - \frac{\gamma(2x)}{2}\right\| + \left\|\frac{\gamma(2x)}{2} - \gamma(x)\right\| \\ &\leq \frac{1}{2}\left[\Phi_a(x) + \frac{\Phi_a(2x)}{2}\right] \end{aligned} \tag{3.69}$$

for all $x \in E_1$. In general for any positive integer n, we get

$$\left\|\frac{\gamma(2^n x)}{2^n} - \gamma(x)\right\| \leq \frac{1}{2}\sum_{k=0}^{n-1} \frac{\Phi_a(2^k x)}{2^k} \leq \frac{1}{2}\sum_{k=0}^{\infty} \frac{\Phi_a(2^k x)}{2^k} \tag{3.70}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{\frac{\gamma(2^n x)}{2^n}\right\}$, replace x by $2^m x$ and divide by 2^m in (3.70). Then for any $m, n > 0$, we have

$$\begin{aligned}\left\|\frac{\gamma\left(2^{n+m}x\right)}{2^{n+m}}-\frac{\gamma\left(2^{m}x\right)}{2^{m}}\right\| &= \frac{1}{2^m}\left\|\frac{\gamma\left(2^{n}2^{m}x\right)}{2^{n}}-\gamma\left(2^{m}x\right)\right\| \\ &\le \frac{1}{2}\sum_{k=0}^{n-1}\frac{\Phi_a\left(2^{k+m}x\right)}{2^{k+m}} \\ &\le \frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a\left(2^{k+m}x\right)}{2^{k+m}} \\ &\to 0 \quad \text{as} \quad m\to\infty\end{aligned}$$

for all $x \in E_1$. Hence the sequence $\left\{\frac{\gamma(2^n x)}{2^n}\right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a additive mapping $A : E_1 \to E_2$ such that

$$A(x) = \lim_{n\to\infty}\frac{\gamma\left(2^n x\right)}{2^n} \qquad \forall x \in E_1.$$

Letting $n \to \infty$ in (3.70) and using (3.65) we see that (3.45) holds for all $x \in E_1$. The proof that A satisfies (1.20) and is unique is similar to that of Theorem 3.1. □

The following corollary is the immediate consequence of Theorem 3.7 concerning the stability of (1.20).

Corollary 3.8. *Let ε, p be nonnegative real numbers. Suppose that an odd function*

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality

$$(3.71) \qquad \|Df(x,y)\| \le \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \le p < 2; \\ \varepsilon, & \\ \varepsilon \|x\|^p \|y\|^p, & 0 \le p < \frac{1}{2}; \\ \varepsilon\left(\|x\|^p \|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & 0 \le p < \frac{1}{2} \end{cases}$$

for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ such that

$$(3.72) \qquad \|f(2x) - 8f(x) - A(x)\| \le \begin{cases} \frac{\lambda_5 \|x\|^p}{2-2^p}, \\ \lambda_6, \\ \frac{\lambda_7 \|x\|^{2p}}{2-2^{2p}}, \\ \frac{\lambda_8 \|x\|^{2p}}{2-2^{2p}}, \end{cases}$$

where

$$\lambda_5 = \frac{\varepsilon}{a^4 - a^2}\{21 - 8a^2 + 2^p\left(2a^2 + 4\right) + 3^p + 2(1+a)^p + 2(1-a)^p + (1+2a)^p + (1-2a)^p\},$$

$$\lambda_6 = \frac{\varepsilon\left(16 - 3a^2\right)}{a^4 - a^2},$$

$$\lambda_7 = \frac{\varepsilon}{a^4 - a^2}\{5 - 4a^2 + 2^{2p}a^2 + 4\left(2^p\right) + 3^p + 2(1+a)^p + 2(1-a)^p + (1+2a)^p + (1-2a)^p\}$$

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

and

$$\lambda_8 = \frac{\varepsilon}{a^4 - a^2} \left\{26 - 12a^2 + 2^{2p}\left(3a^2 + 4\right) + 3^{2p} \right.$$
$$+2\left(1+a\right)^{2p} + 2\left(1-a\right)^{2p} + \left(1+2a\right)^{2p} + \left(1-2a\right)^{2p} + 4\left(2^p\right)$$
$$\left. +3^p + 2\left(1+a\right)^p + 2\left(1-a\right)^p + \left(1+2a\right)^p + \left(1-2a\right)^p \right\}$$

for all $x \in E_1$.

Theorem 3.9. *Let* $\phi_c : E_1 \times E_1 \to [0, \infty)$ *be a function such that*

$$(3.73)\qquad \sum_{n=0}^{\infty} \frac{\phi_c\left(2^n x, 2^n y\right)}{8^n} \quad \text{converges and} \quad \lim_{n\to\infty} \frac{\phi_c\left(2^n x, 2^n y\right)}{8^n} = 0$$

for all $x, y \in E_1$ *and let* $f : E_1 \to E_2$ *be an odd function with* $f(0) = 0$ *that satisfies the inequality*

$$(3.74)\qquad \left\| Df\left(x,y\right)\right\| \leq \phi_c\left(x,y\right)$$

for all $x, y \in E_1$. *Then there exists a unique cubic function* $C : E_1 \to E_2$ *such that*

$$(3.75)\qquad \left\| f\left(2x\right) - 2f\left(x\right) - C\left(x\right)\right\| \leq \frac{1}{8}\sum_{k=0}^{\infty} \frac{\Phi_c\left(2^k x\right)}{8^k}$$

for all $x \in E_1$, *where the mapping* $C(x)$ *and* $\Phi_c(2^k x)$ *are defined by*

$$(3.76)\qquad C\left(x\right) = \lim_{n\to\infty} \frac{1}{8^n}\left\{f\left(2^{n+1}x\right) - 2f\left(2^n x\right)\right\}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$\Phi_c\left(2^k x\right) = \frac{1}{a^4 - a^2}\left[\left(5-4a^2\right)\phi_c\left(2^k x, 2^k x\right) + a^2\phi_c\left(2^{k+1}x, 2^{k+1}x\right) + 2a^2\phi_c\left(2^{k+1}x, 2^k x\right) + \left(4-2a^2\right)\phi_c\left(2^k x, 2^{k+1}x\right) + \phi_c\left(2^k x, 2^k 3x\right) + 2\phi_c\left(2^k\left(1+a\right)x, 2^k x\right) + 2\phi_c\left(2^k\left(1-a\right)x, 2^k x\right) + \phi_c\left(2^k\left(1+2a\right)x, 2^k x\right) + \phi_c\left(2^k\left(1-2a\right)x, 2^k x\right)\right] \tag{3.77}$$

for all $x \in E_1$.

Proof. Following along similar lines to those in the proof of Theorem 3.7, we have

$$\left\|f\left(4x\right) - 10f\left(2x\right) + 16f\left(x\right)\right\| \leq \Phi_c\left(x\right), \tag{3.78}$$

where

$$\Phi_c\left(x\right) = \frac{1}{\left(a^4 - a^2\right)}\left[\left(5-4a^2\right)\phi_c\left(x,x\right) + a^2\phi_c\left(2x,2x\right) + 2a^2\phi_c\left(2x,x\right) + \left(4-2a^2\right)\phi_c\left(x,2x\right) + \phi_c\left(x,3x\right) + 2\phi_c\left(\left(1+a\right)x,x\right) + 2\phi_c\left(\left(1-a\right)x,x\right) + \phi_c\left(\left(1+2a\right)x,x\right) + \phi_c\left(\left(1-2a\right)x,x\right)\right]$$

for all $x \in E_1$. It is easy to see from (3.78) that

$$\left\|f\left(4x\right) - 2f\left(2x\right) - 8\left\{f\left(2x\right) - 2f\left(x\right)\right\}\right\| \leq \Phi_c\left(x\right) \tag{3.79}$$

for all $x \in E_1$. Define a mapping $\delta : E_1 \to E_2$ by

$$\delta\left(x\right) = f\left(2x\right) - 2f\left(x\right) \tag{3.80}$$

for all $x \in E_1$. Using (3.80) in (3.79), we obtain

$$\left\|\delta\left(2x\right) - 8\delta\left(x\right)\right\| \leq \Phi_c\left(x\right) \tag{3.81}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$. From (3.81), we have

$$\left\| \frac{\delta(2x)}{8} - \delta(x) \right\| \leq \frac{\Phi_c(x)}{8} \tag{3.82}$$

for all $x \in E_1$. Now replacing x by $2x$ and dividing by 8 in (3.82), we obtain

$$\left\| \frac{\delta(2^2x)}{8^2} - \frac{\delta(2x)}{8} \right\| \leq \frac{\Phi_c(2x)}{8^2} \tag{3.83}$$

for all $x \in E_1$. From (3.82) and (3.83), we arrive at

$$\begin{aligned} \left\| \frac{\delta(2^2x)}{8^2} - \delta(x) \right\| &\leq \left\| \frac{\delta(2^2x)}{8^2} - \frac{\delta(2x)}{8} \right\| + \left\| \frac{\delta(2x)}{8} - \delta(x) \right\| \\ &\leq \frac{1}{8}\left[\Phi_c(x) + \frac{\Phi_c(2x)}{8} \right] \end{aligned} \tag{3.84}$$

for all $x \in E_1$. In general for any positive integer n, we get

$$\begin{aligned} \left\| \frac{\delta(2^nx)}{8^n} - \delta(x) \right\| &\leq \frac{1}{8} \sum_{k=0}^{n-1} \frac{\Phi_c(2^kx)}{8^k} \\ &\leq \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c(2^kx)}{8^k} \end{aligned} \tag{3.85}$$

for all $x \in E_1$. In order to prove the convergence of the sequence $\left\{ \frac{\delta(2^nx)}{8^n} \right\}$, replace

vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

x by $2^m x$ and divide by 8^m in (3.85). Then for any $m, n > 0$, we have

$$\begin{aligned}\left\| \frac{\delta\left(2^{n+m}x\right)}{8^{n+m}} - \frac{\delta\left(2^m x\right)}{8^m} \right\| &= \frac{1}{8^m} \left\| \frac{\delta\left(2^n 2^m x\right)}{8^n} - \delta\left(2^m x\right) \right\| \\ &\leq \frac{1}{8} \sum_{k=0}^{n-1} \frac{\Phi_c\left(2^{k+m}x\right)}{8^{k+m}} \\ &\leq \frac{1}{8} \sum_{k=0}^{\infty} \frac{\Phi_c\left(2^{k+m}x\right)}{8^{k+m}} \\ &\to 0 \quad \text{as} \quad m \to \infty\end{aligned}$$

for all $x \in E_1$. Hence the sequence $\left\{\frac{\delta(2^n x)}{8^n}\right\}$ is a Cauchy sequence. Since E_2 is complete, there exists a cubic mapping $C : E_1 \to E_2$ such that

$$C(x) = \lim_{n\to\infty} \frac{\delta\left(2^n x\right)}{8^n} \quad \forall x \in E_1.$$

Letting $n \to \infty$ in (3.84) and using (3.80) we see that (3.75) holds for all $x \in E_1$. The rest of the proof, which proves that C satisfies (1.20) and is unique, is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 3.9 concerning the stability of (1.20).

Corollary 3.10. *Let ε, p be nonnegative real numbers. Suppose that an odd function*

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x,y)\| \leq \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \leq p < 3; \\ \varepsilon, & \\ \varepsilon \|x\|^p \|y\|^p, & 0 \leq p < \frac{3}{2}; \\ \varepsilon\left(\|x\|^p \|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & 0 \leq p < \frac{3}{2} \end{cases} \tag{3.86}$$

for all $x, y \in E_1$. Then there exists a unique cubic function $C : E_1 \to E_2$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \begin{cases} \frac{\lambda_5 \|x\|^p}{8-2^p}, \\ \frac{\lambda_6}{7}, \\ \frac{\lambda_7 \|x\|^{2p}}{8-2^{2p}}, \\ \frac{\lambda_8 \|x\|^{2p}}{8-2^{2p}}, \end{cases} \tag{3.87}$$

for all $x \in E_1$, where $\lambda_i \ (i = 5, 6, 7, 8)$ are given in Corollary 3.8.

Theorem 3.11. *Let $\phi : E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi_a(2^n x, 2^n y)}{2^n}, \quad \sum_{n=0}^{\infty} \frac{\phi_c(2^n x, 2^n y)}{8^n} \quad \text{converges} \tag{3.88}$$

and

$$\lim_{n\to\infty} \frac{\phi_a(2^n x, 2^n y)}{2^n} = 0 = \lim_{n\to\infty} \frac{\phi_c(2^n x, 2^n y)}{8^n} \tag{3.89}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x, y \in E_1$. Suppose that an odd function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequalities (3.44) and (3.74) for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ and a unique cubic function $C : E_1 \to E_2$ such that

$$\|f(x) - A(x) - C(x)\| \le \frac{1}{6}\left\{\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a(2^k x)}{2^k} + \frac{1}{8}\sum_{k=0}^{\infty}\frac{\Phi_c(2^k x)}{8^k}\right\} \tag{3.90}$$

for all $x \in E_1$, where $\Phi_a(2^k x)$ and $\Phi_c(2^k x)$ are defined by (3.47) and (3.77), respectively for all $x \in E_1$.

Proof. By Theorems 3.7 and 3.9, there exists a unique additive function $A_1 : E_1 \to E_2$ and a unique cubic function $C_1 : E_1 \to E_2$ such that

$$\|f(2x) - 8f(x) - A_1(x)\| \le \frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a(2^k x)}{2^k} \tag{3.91}$$

and

$$\|f(2x) - 2f(x) - C_1(x)\| \le \frac{1}{8}\sum_{k=0}^{\infty}\frac{\Phi_c(2^k x)}{8^k} \tag{3.92}$$

for all $x \in E_1$. Now from (3.91) and (3.92), one can see that

$$\begin{aligned}
&\left\|f(x) + \frac{1}{6}A_1(x) - \frac{1}{6}C_1(x)\right\| \\
&\qquad = \left\|\left\{-\frac{f(2x)}{6} + \frac{8f(x)}{6} + \frac{A_1(x)}{6}\right\} + \left\{\frac{f(2x)}{6} - \frac{2f(x)}{6} - \frac{C_1(x)}{6}\right\}\right\| \\
&\qquad \le \frac{1}{6}\left\{\|f(2x) - 8f(x) - A_1(x)\| + \|f(2x) - 2f(x) - C_1(x)\|\right\}
\end{aligned}$$

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$\leq \frac{1}{6}\left\{\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a\left(2^k x\right)}{2^k}+\frac{1}{8}\sum_{k=0}^{\infty}\frac{\Phi_c\left(2^k x\right)}{8^k}\right\}$$

for all $x \in E_1$. Thus we obtain (3.90) by defining $A(x) = \frac{-1}{6}A_1(x)$ and $C(x) = \frac{1}{6}C_1(x)$, where $\Phi_a\left(2^k x\right)$ and $\Phi_c\left(2^k x\right)$ are defined in (3.47) and (3.77), respectively for all $x \in E_1$. □

The following corollary is an immediate consequence of Theorem 3.11 concerning the stability of (1.20).

Corollary 3.12. *Let ε, p be nonnegative real numbers. Suppose that an odd function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x,y)\| \leq \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \leq p < 1; \\ \varepsilon, & \\ \varepsilon\|x\|^p\|y\|^p, & 0 \leq p < \frac{1}{2}; \\ \varepsilon\left(\|x\|^p\|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & 0 \leq p < \frac{1}{2} \end{cases} \tag{3.93}$$

for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ and a unique cubic function $C : E_1 \to E_2$ such that

$$\|f(x) - A(x) - C(x)\| \leq \begin{cases} \frac{\lambda_5\|x\|^p}{6}\left\{\frac{1}{2-2^p} + \frac{1}{8-2^p}\right\}, \\ \frac{4\lambda_6}{21}, \\ \frac{\lambda_7\|x\|^{2p}}{6}\left\{\frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}}\right\}, \\ \frac{\lambda_8\|x\|^p}{6}\left\{\frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}}\right\}, \end{cases} \tag{3.94}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x \in E_1$, where $\lambda_i \;\; (i = 5, 6, 7, 8)$ are given in Corollary 3.8.

Theorem 3.13. *Let $\phi : E_1 \times E_1 \to [0, \infty)$ be a function that satisfies (3.36), (3.37), (3.88) and (3.89) for all $x, y \in E_1$. Suppose that a function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequalities (3.2), (3.23), (3.44) and (3.74) for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$, a unique quadratic function $B : E_1 \to E_2$, a unique cubic function $C : E_1 \to E_2$ and a unique quartic function $D : E_1 \to E_2$ such that*

$$\|f(x) - A(x) - B(x) - C(x) - D(x)\| \leq \frac{1}{2}\left\{\widetilde{\Phi}_a(x) + \widetilde{\Phi}_b(x) + \widetilde{\Phi}_c(x) + \widetilde{\Phi}_d(x)\right\} \tag{3.95}$$

for all $x \in E_1$, where $\widetilde{\Phi}_a(x), \widetilde{\Phi}_b(x), \widetilde{\Phi}_c(x)$ and $\widetilde{\Phi}_d(x)$ are defined by

$$\widetilde{\Phi}_a(x) = \frac{1}{6}\left\{\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a(2^k x)}{2^k} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a(-2^k x)}{2^k}\right\}, \tag{3.96}$$

$$\widetilde{\Phi}_b(x) = \frac{1}{12}\left\{\frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b(2^k x)}{4^k} + \frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b(-2^k x)}{4^k}\right\}, \tag{3.97}$$

$$\widetilde{\Phi}_c(x) = \frac{1}{6}\left\{\frac{1}{8}\sum_{k=0}^{\infty}\frac{\Phi_c(2^k x)}{8^k} + \frac{1}{8}\sum_{k=0}^{\infty}\frac{\Phi_c(-2^k x)}{8^k}\right\}, \tag{3.98}$$

$$\widetilde{\Phi}_d(x) = \frac{1}{12}\left\{\frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d(2^k x)}{16^k} + \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d(-2^k x)}{16^k}\right\}, \tag{3.99}$$

respectively for all $x \in E_1$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. Let $f_e(x) = \frac{1}{2}\{f(x) + f(-x)\}$ for all $x \in E_1$. Then $f_e(0) = 0$, $f_e(x) = f_e(-x)$. Hence

$$\begin{aligned}\|Df_e(x,y)\| &= \frac{1}{2}\{\|Df(x,y) + Df(-x,-y)\|\} \\ &\le \frac{1}{2}\{\|Df(x,y)\| + \|Df(-x,-y)\|\} \\ &\le \frac{1}{2}\{\phi(x,y) + \phi(-x,-y)\}\end{aligned}$$

for all $x \in E_1$. Hence from Theorem 3.5, there exists a unique quadratic function $B : E_1 \to E_2$ and a unique quartic function $D : E_1 \to E_2$ such that

$$\begin{aligned}(3.100)\quad &\|f(x) - B(x) - D(x)\| \\ &\le \frac{1}{2}\left\{\frac{1}{12}\left[\frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b(2^k x)}{4^k} + \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d(2^k x)}{16^k}\right]\right. \\ &\qquad \left. + \frac{1}{12}\left[\frac{1}{4}\sum_{k=0}^{\infty}\frac{\Phi_b(-2^k x)}{4^k} + \frac{1}{16}\sum_{k=0}^{\infty}\frac{\Phi_d(-2^k x)}{16^k}\right]\right\} \\ &\le \frac{1}{2}\left\{\widetilde{\Phi}_b(x) + \widetilde{\Phi}_d(x)\right\},\end{aligned}$$

where $\widetilde{\Phi}_b(x)$ and $\widetilde{\Phi}_d(x)$ are given in (3.97) and (3.99) for all $x \in E_1$. Again $f_o(x) = \frac{1}{2}\{f(x) - f(-x)\}$ for all $x \in E_1$. Then $f_o(0) = 0$, $f_o(x) = -f_o(-x)$. Hence

$$\begin{aligned}\|Df_o(x,y)\| &= \frac{1}{2}\{\|Df(x,y) - Df(-x,-y)\|\} \\ &\le \frac{1}{2}\{\|Df(x,y)\| + \|Df(-x,-y)\|\}\end{aligned}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$\leq \frac{1}{2}\left\{\phi\left(x,y\right)+\phi\left(-x,-y\right)\right\}$$

for all $x \in E_1$. Hence from Theorem 3.11, there exists a unique additive function $A : E_1 \to E_2$ and a unique cubic function $C : E_1 \to E_2$ such that

$$\begin{aligned}
(3.101)\qquad &\left\|f\left(x\right)-A\left(x\right)-C\left(x\right)\right\| \\
&\leq \frac{1}{2}\left\{\frac{1}{6}\left[\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a\left(2^k x\right)}{2^k}+\frac{1}{8}\sum_{k=0}^{\infty}\frac{\Phi_c\left(2^k x\right)}{8^k}\right]\right. \\
&\qquad \left.+\frac{1}{6}\left[\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Phi_a\left(-2^k x\right)}{2^k}+\frac{1}{8}\sum_{k=0}^{\infty}\frac{\Phi_c\left(-2^k x\right)}{8^k}\right]\right\} \\
&\leq \frac{1}{2}\left\{\widetilde{\Phi}_a\left(x\right)+\widetilde{\Phi}_c\left(x\right)\right\},
\end{aligned}$$

where $\widetilde{\Phi}_a(x)$ and $\widetilde{\Phi}_c(x)$ are given in (3.96) and (3.98) for all $x \in E_1$. Since $f(x) = f_e(x) + f_o(x)$, then it follows from (3.100) and (3.101) that

$$\begin{aligned}
&\left\|f\left(x\right)-A\left(x\right)-B\left(x\right)-C\left(x\right)-D\left(x\right)\right\| \\
&\qquad = \left\|\left\{f_e\left(x\right)-B\left(x\right)-D\left(x\right)\right\}+\left\{f_o\left(x\right)-C\left(x\right)-D\left(x\right)\right\}\right\| \\
&\qquad \leq \left\|f_e\left(x\right)-B\left(x\right)-D\left(x\right)\right\|+\left\|f_o\left(x\right)-C\left(x\right)-D\left(x\right)\right\| \\
&\qquad \leq \frac{1}{2}\left\{\widetilde{\Phi}_a\left(x\right)+\widetilde{\Phi}_b\left(x\right)+\widetilde{\Phi}_c\left(x\right)+\widetilde{\Phi}_d\left(x\right)\right\}
\end{aligned}$$

for all $x \in E_1$. Hence the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 3.13 concerning the stability of (1.20).

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Corollary 3.14. *Let ε, p be nonnegative real numbers. Suppose a function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality*

$$\|D_f(x,y)\| \leq \begin{cases} \varepsilon\left(\|x\|^p + \|y\|^p\right), & 0 \leq p < 1; \\ \varepsilon, & \\ \varepsilon \|x\|^p \|y\|^p, & 0 \leq p < \frac{1}{2}; \\ \varepsilon\left(\|x\|^p \|y\|^p + \left\{\|x\|^{2p} + \|y\|^{2p}\right\}\right), & 0 \leq p < \frac{1}{2} \end{cases} \tag{3.102}$$

for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$, a unique quadratic function $B : E_1 \to E_2$, a unique cubic function $C : E_1 \to E_2$ and a unique quartic function $D : E_1 \to E_2$ such that

$$\|f(x) - A(x) - B(x) - C(x) - D(x)\| \leq \begin{cases} \frac{1}{2}\left\{\frac{\lambda_1}{6}\left\{\frac{1}{4-2^p} + \frac{1}{16-2^p}\right\} + \frac{\lambda_5}{3}\left\{\frac{1}{2-2^p} + \frac{1}{8-2^p}\right\}\right\}\|x\|^p; \\ \frac{1}{2}\left\{\lambda_2 + \frac{4\lambda_6}{21}\right\}; \\ \frac{1}{2}\left\{\frac{\lambda_3}{6}\left\{\frac{1}{4-2^{2p}} + \frac{1}{16-2^{2p}}\right\} + \frac{\lambda_7}{3}\left\{\frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}}\right\}\right\}\|x\|^{2p}; \\ \frac{1}{2}\left\{\frac{\lambda_4}{6}\left\{\frac{1}{4-2^{2p}} + \frac{1}{16-2^{2p}}\right\} + \frac{\lambda_8}{3}\left\{\frac{1}{2-2^{2p}} + \frac{1}{8-2^{2p}}\right\}\right\}\|x\|^{2p} \end{cases} \tag{3.103}$$

for all $x \in E_1$, where λ_i $(i = 1, \ldots, 8)$ are respectively, given in Corollaries 3.6 and 3.12.

Stability of Generalized Mixed Type
K. Ravi, J.M. Rassias,
M. Arunkumar and R. Kodandan
vol. 10, iss. 4, art. 114, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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