

# SHARP GRÜSS-TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE OF BOUNDED VARIATION

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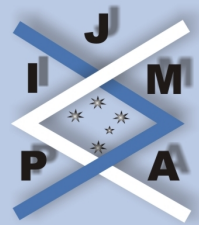
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*Abstract:* Sharp Grüss-type inequalities for functions whose derivatives are of bounded variation (Lipschitzian or monotonic) are given. Applications in relation with the well-known Čebyšev, Grüss, Ostrowski and Lupaş inequalities are provided as well.



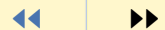
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Grüss-type Inequalities  
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## 1. Introduction

In 1998, S.S. Dragomir and I. Fedotov [10] introduced the following *Grüss type error functional*

$$D(f; u) := \int_a^b f(t) du(t) - [u(a) - u(b)] \cdot \frac{1}{b-a} \int_a^b f(t) dt$$

in order to approximate the *Riemann-Stieltjes integral*  $\int_a^b f(t) du(t)$  by the simpler quantity

$$[u(a) - u(b)] \cdot \frac{1}{b-a} \int_a^b f(t) dt.$$

In the same paper the authors have shown that

$$(1.1) \quad |D(f; u)| \leq \frac{1}{2} \cdot L(M - m)(b - a),$$

provided that  $u$  is  $L$ -Lipschitzian, i.e.,  $|u(t) - u(s)| \leq L|t - s|$  for any  $t, s \in [a, b]$  and  $f$  is *Riemann integrable* and satisfies the condition

$$-\infty < m \leq f(t) \leq M < \infty \quad \text{for any } t \in [a, b].$$

The constant  $\frac{1}{2}$  is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

In [11], the same authors established another result for  $D(f; u)$ , namely

$$(1.2) \quad |D(f; u)| \leq \frac{1}{2} K(b-a) \bigvee_a^b(u),$$

provided that  $u$  is of *bounded variation* on  $[a, b]$  with the *total variation*  $\bigvee_a^b(u)$  and  $f$  is  $K$ -Lipschitzian. Here  $\frac{1}{2}$  is also best possible.



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In [8], by introducing the *kernel*  $\Phi_u : [a, b] \rightarrow \mathbb{R}$  given by

$$(1.3) \quad \Phi_u(t) := \frac{1}{b-a} [(t-a)u(b) + (b-t)u(a)] - u(t), \quad t \in [a, b],$$

the author has obtained the following *integral representation*

$$(1.4) \quad D(f; u) = \int_a^b \Phi_u(t) df(t),$$

where  $u, f : [a, b] \rightarrow \mathbb{R}$  are bounded functions such that the Riemann-Stieltjes integral  $\int_a^b f(t) du(t)$  and the Riemann integral  $\int_a^b f(t) dt$  exist. By the use of this representation he also obtained the following bounds for  $D(f; u)$ ,

$$(1.5) \quad |D(f; u)| \leq \begin{cases} \sup_{t \in [a, b]} |\Phi_u(t)| \cdot \bigvee_a^b(f) & \text{if } u \text{ is continuous and } f \text{ is of bounded variation;} \\ L \int_a^b |\Phi_u(t)| dt & \text{if } u \text{ is Riemann integrable and } f \text{ is } L\text{-Lipschitzian;} \\ \int_a^b |\Phi_u(t)| dt & \text{if } u \text{ is continuous and } f \text{ is monotonic nondecreasing.} \end{cases}$$

If  $u$  is *monotonic nondecreasing* and  $K(u)$  is defined by

$$K(u) := \frac{4}{(b-a)^2} \int_a^b \left(t - \frac{a+b}{2}\right) u(t) dt (\geq 0),$$

then

$$(1.6) \quad \begin{aligned} |D(f; u)| &\leq \frac{1}{2} L(b-a) [u(b) - u(a) - K(u)] \\ &\leq \frac{1}{2} L(b-a) [u(b) - u(a)], \end{aligned}$$



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provided that  $f$  is  $L$ -Lipschitzian on  $[a, b]$ .

Here  $\frac{1}{2}$  is best possible in both inequalities.

Also, for  $u$  monotonic nondecreasing on  $[a, b]$  and by defining  $Q(u)$  as

$$Q(u) := \frac{1}{b-a} \int_a^b u(t) \operatorname{sgn} \left( t - \frac{a+b}{2} \right) dt (\geq 0),$$

we have

$$(1.7) \quad |D(f; u)| \leq [u(b) - u(a) - Q(u)] \cdot \bigvee_a^b(f) \leq [u(b) - u(a)] \cdot \bigvee_a^b(f),$$

provided that  $f$  is of bounded variation on  $[a, b]$ . The first inequality in (1.7) is sharp.

Finally, the case when  $u$  is convex and  $f$  is of bounded variation produces the bound

$$(1.8) \quad |D(f; u)| \leq \frac{1}{4} [u'_-(b) - u'_+(a)] (b-a) \bigvee_a^b(f),$$

with  $\frac{1}{4}$  the best constant (when  $u'_-(b)$  and  $u'_+(a)$  are finite) and if  $f$  is monotonic nondecreasing and  $u$  is convex on  $[a, b]$ , then

$$(1.9) \quad 0 \leq D(f; u) \leq 2 \cdot \frac{u'_-(b) - u'_+(a)}{b-a} \cdot \int_a^b \left( t - \frac{a+b}{2} \right) f(t) dt$$

$$\leq \begin{cases} \frac{1}{2} [u'_-(b) - u'_+(a)] \max \{ |f(a)|, |f(b)| \} (b-a) \\ \frac{1}{(q+1)^{1/q}} [u'_-(b) - u'_+(a)] \|f\|_p (b-a)^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ [u'_-(b) - u'_+(a)] \|f\|_1, \end{cases}$$

where 2 and  $\frac{1}{2}$  are sharp constants (when  $u'_-(b)$  and  $u'_+(a)$  are finite) and  $\|\cdot\|_p$  are the usual Lebesgue norms, i.e.,  $\|f\|_p := \left( \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}$ ,  $p \geq 1$ .

The main aim of the present paper is to provide sharp upper bounds for the absolute value of  $D(f; u)$  under various conditions for  $u'$ , the derivative of an absolutely continuous function  $u$ , and  $f$  of bounded variation (Lipschitzian or monotonic). Natural applications for the Čebyšev functional that complement the classical results due to Čebyšev, Grüss, Ostrowski and Lupaş are also given.



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## 2. Preliminary Results

We have the following integral representation of  $\Phi_u$ .

**Lemma 2.1.** *Assume that  $u : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous on  $[a, b]$  and such that the derivative  $u'$  exists on  $[a, b]$  (eventually except at a finite number of points). If  $u'$  is Riemann integrable on  $[a, b]$ , then*

$$(2.1) \quad \Phi_u(t) := \frac{1}{b-a} \int_a^b K(t, s) du'(s), \quad t \in [a, b],$$

where the kernel  $K : [a, b]^2 \rightarrow \mathbb{R}$  is given by

$$(2.2) \quad K(t, s) := \begin{cases} (b-t)(s-a) & \text{if } s \in [a, t], \\ (t-a)(b-s) & \text{if } s \in (t, b]. \end{cases}$$

*Proof.* We give, for simplicity, a proof only in the case when  $u'$  is defined on the entire interval, and for which we have used the usual convention that  $u'(a) := u'_+(a)$ ,  $u'(b) := u'_-(b)$  and the lateral derivatives are finite.

Since  $u'$  is assumed to be Riemann integrable on  $[a, b]$ , it follows that the Riemann-Stieltjes integrals  $\int_a^t (s-a) du'(s)$  and  $\int_t^b (b-s) du'(s)$  exist for each  $t \in [a, b]$ . Now, integrating by parts in the Riemann-Stieltjes integral, we have successively

$$\begin{aligned} \int_a^b K(t, s) du'(s) &= (b-t) \int_a^t (s-a) du'(s) + (t-a) \int_t^b (b-s) du'(s) \\ &= (b-t) \left[ (s-a) u'(s) \Big|_a^t - \int_a^t u'(s) ds \right] \\ &\quad + (t-a) \left[ (b-s) u'(s) \Big|_t^b - \int_t^b u'(s) ds \right] \end{aligned}$$

$$\begin{aligned}
&= (b-t) [(t-a)u'(t) - (u(t) - u(a))] \\
&\quad + (t-a) [-(b-t)u'(t) + u(b) - u(t)] \\
&= (t-a) [u(b) - u(t)] - (b-t) [u(t) - u(a)] \\
&= (b-a) \Phi_u(t),
\end{aligned}$$

for any  $t \in [a, b]$ , and the representation (2.1) is proved.  $\square$

The following result provides a sharp bound for  $|\Phi_u|$  in the case when  $u'$  is of bounded variation.

**Theorem 2.2.** *Assume that  $u : [a, b] \rightarrow \mathbb{R}$  is as in Lemma 2.1. If  $u'$  is of bounded variation on  $[a, b]$ , then*

$$(2.3) \quad |\Phi_u(t)| \leq \frac{(t-a)(b-t)}{b-a} \bigvee_a^b(u') \leq \frac{1}{4}(b-a) \bigvee_a^b(u'),$$

where  $\bigvee_a^b(u')$  denotes the total variation of  $u'$  on  $[a, b]$ .

The inequalities are sharp and the constant  $\frac{1}{4}$  is best possible.

*Proof.* It is well known that, if  $p : [\alpha, \beta] \rightarrow \mathbb{R}$  is continuous and  $v : [\alpha, \beta] \rightarrow \mathbb{R}$  is of bounded variation, then the Riemann-Stieltjes integral  $\int_\alpha^\beta p(s) dv(s)$  exists and

$$\left| \int_\alpha^\beta p(s) dv(s) \right| \leq \sup_{s \in [\alpha, \beta]} |p(s)| \bigvee_\alpha^\beta(v).$$



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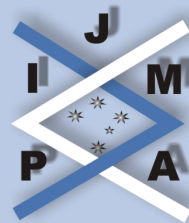
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Now, utilising the representation (2.1) we have successively:

$$\begin{aligned}
 (2.4) \quad |\Phi_u(t)| &\leq \frac{1}{b-a} \left[ (b-t) \left| \int_a^t (s-a) du'(s) \right| + (t-a) \left| \int_t^b (b-s) du'(s) \right| \right] \\
 &\leq \frac{1}{b-a} \left[ (b-t) \sup_{s \in [a,t]} (s-a) \cdot \bigvee_a^t(u') + (t-a) \sup_{s \in [t,b]} (b-s) \cdot \bigvee_t^b(u') \right] \\
 &= \frac{(t-a)(b-t)}{b-a} \left[ \bigvee_a^t(u') + \bigvee_t^b(u') \right] = \frac{(t-a)(b-t)}{b-a} \bigvee_a^b(u').
 \end{aligned}$$

The second inequality is obvious by the fact that  $(t-a)(b-t) \leq \frac{1}{4}(b-a)^2$ ,  $t \in [a, b]$ .

For the sharpness of the inequalities, assume that there exist  $A, B > 0$  so that

$$(2.5) \quad |\Phi_u(t)| \leq A \cdot \frac{(t-a)(b-t)}{b-a} \bigvee_a^b(u') \leq B(b-a) \bigvee_a^b(u'),$$

with  $u$  as in the assumption of the theorem. Then, for  $t = \frac{a+b}{2}$ , we get from (2.5) that

$$(2.6) \quad \left| \frac{u(a) + u(b)}{2} - u\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{4}A(b-a) \bigvee_a^b(u') \leq B(b-a) \bigvee_a^b(u').$$

Consider the function  $u : [a, b] \rightarrow \mathbb{R}$ ,  $u(t) = |t - \frac{a+b}{2}|$ . This function is absolutely continuous,  $u'(t) = \text{sgn}(t - \frac{a+b}{2})$ ,  $t \in [a, b] \setminus \{\frac{a+b}{2}\}$  and  $\bigvee_a^b(u') = 2$ . Then (2.6) becomes  $\frac{b-a}{2} \leq \frac{1}{2}A(b-a) \leq 2B(b-a)$ , which implies that  $A \geq 1$  and  $B \geq \frac{1}{4}$ .  $\square$

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**Corollary 2.3.** *With the assumptions of Theorem 2.2, we have*

$$(2.7) \quad \left| \frac{u(a) + u(b)}{2} - u\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{4} (b-a) \bigvee_a^b (u').$$

The constant  $\frac{1}{4}$  is best possible.

The Lipschitzian case is incorporated in the following result.

**Theorem 2.4.** *Assume that  $u : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous on  $[a, b]$  with the property that  $u'$  is  $K$ -Lipschitzian on  $(a, b)$ . Then*

$$(2.8) \quad |\Phi_u(t)| \leq \frac{1}{2} (t-a)(b-t) K \leq \frac{1}{8} (b-a)^2 K.$$

The constants  $\frac{1}{2}$  and  $\frac{1}{8}$  are best possible.

*Proof.* We utilise the fact that, for an  $L$ -Lipschitzian function  $p : [\alpha, \beta] \rightarrow \mathbb{R}$  and a Riemann integrable function  $v : [\alpha, \beta] \rightarrow \mathbb{R}$ , the Riemann-Stieltjes integral  $\int_{\alpha}^{\beta} p(s) dv(s)$  exists and

$$\left| \int_{\alpha}^{\beta} p(s) dv(s) \right| \leq L \int_{\alpha}^{\beta} |p(s)| ds.$$

Then, by (2.1), we have that

$$(2.9) \quad |\Phi_u(t)| \leq \frac{1}{b-a} \left[ (b-t) \left| \int_a^t (s-a) du'(s) \right| + (t-a) \left| \int_t^b (b-s) du'(s) \right| \right]$$

$$\begin{aligned} &\leq \frac{1}{b-a} \left[ \frac{1}{2} K (b-t)(t-a)^2 + \frac{1}{2} K (t-a)(b-t)^2 \right] \\ &= \frac{1}{2} (t-a)(b-t) K, \end{aligned}$$

which proves the first part of (2.8). The second part is obvious.

Now, for the sharpness of the constants, assume that there exist the constants  $C, D > 0$  such that

$$(2.10) \quad |\Phi_u(t)| \leq C(b-t)(t-a)K \leq D(b-a)^2K,$$

provided that  $u$  is as in the hypothesis of the theorem. For  $t = \frac{a+b}{2}$ , we get from (2.10) that

$$(2.11) \quad \left| \frac{u(a) + u(b)}{2} - u\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{4}CK(b-a)^2 \leq D(b-a)^2K.$$

Consider  $u : [a, b] \rightarrow \mathbb{R}$ ,  $u(t) = \frac{1}{2} \left| t - \frac{a+b}{2} \right|^2$ . Then  $u'(t) = t - \frac{a+b}{2}$  is Lipschitzian with the constant  $K = 1$  and (2.11) becomes

$$\frac{1}{8}(b-a)^2 \leq \frac{1}{4}C(b-a)^2 \leq D(b-a)^2,$$

which implies that  $C \geq \frac{1}{2}$  and  $D \geq \frac{1}{8}$ . □

**Corollary 2.5.** *With the assumptions of Theorem 2.4, we have*

$$(2.12) \quad \left| \frac{u(a) + u(b)}{2} - u\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{8}(b-a)^2K.$$

The constant  $\frac{1}{8}$  is best possible.





*Remark 1.* If  $u'$  is absolutely continuous and  $\|u''\|_\infty := \operatorname{ess\,sup}_{t \in [a,b]} |u''(t)| < \infty$ , then we can take  $K = \|u''\|_\infty$ , and we have from (2.8) that

$$(2.13) \quad |\Phi_u(t)| \leq \frac{1}{2}(t-a)(b-t)\|u''\|_\infty \leq \frac{1}{8}(b-a)^2\|u''\|_\infty.$$

The constants  $\frac{1}{2}$  and  $\frac{1}{8}$  are best possible in (2.13).

From (2.12) we also get

$$(2.14) \quad \left| \frac{u(a) + u(b)}{2} - u\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{8}(b-a)^2\|u''\|_\infty,$$

in which  $\frac{1}{8}$  is the best possible constant.

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### 3. Bounds in the Case when $u'$ is of Bounded Variation

We can start with the following result:

**Theorem 3.1.** Assume that  $u : [a, b] \rightarrow \mathbb{R}$  is as in Lemma 2.1. If  $u'$  and  $f$  are of bounded variation on  $[a, b]$ , then

$$(3.1) \quad |D(f; u)| \leq \frac{1}{4} (b - a) \bigvee_a^b (u') \cdot \bigvee_a^b (f),$$

and the constant  $\frac{1}{4}$  is best possible in (3.1).

*Proof.* We use the following representation of the functional  $D(f; u)$  obtained in [8] (see also [9] or [6]):

$$(3.2) \quad D(f; u) = \int_a^b \Phi_u(t) df(t).$$

Then we have the bound

$$\begin{aligned} |D(f; u)| &= \left| \int_a^b \Phi_u(t) df(t) \right| \leq \sup_{t \in [a, b]} |\Phi_u(t)| \bigvee_a^b (f) \\ &\leq \frac{1}{b-a} \bigvee_a^b (u') \sup_{t \in [a, b]} [(t-a)(b-t)] \cdot \bigvee_a^b (f) \\ &= \frac{1}{4} (b-a) \bigvee_a^b (u') \cdot \bigvee_a^b (f), \end{aligned}$$

where, for the last inequality we have used (2.3).



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To prove the sharpness of the constant  $\frac{1}{4}$ , assume that there is a constant  $E > 0$  such that

$$(3.3) \quad |D(f; u)| \leq E(b-a) \bigvee_a^b(u') \cdot \bigvee_a^b(f).$$

Consider  $u : [a, b] \rightarrow \mathbb{R}$ ,  $u(t) = \left|t - \frac{a+b}{2}\right|$ . Then  $u'(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ ,  $t \in [a, b] \setminus \left\{\frac{a+b}{2}\right\}$ . The total variation on  $[a, b]$  is 2 and

$$D(f; u) = - \int_a^{\frac{a+b}{2}} f(t) dt + \int_{\frac{a+b}{2}}^b f(t) dt = \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f(t) dt.$$

Now, if we choose  $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ , then we obtain from (3.1)  $b - a \leq 4E(b-a)$ , which implies that  $E \geq \frac{1}{4}$ .  $\square$

The following result can be stated as well:

**Theorem 3.2.** Assume that  $u : [a, b] \rightarrow \mathbb{R}$  is as in Lemma 2.1. If the derivative  $u'$  is of bounded variation on  $[a, b]$  while  $f$  is  $L$ -Lipschitzian on  $[a, b]$ , then

$$(3.4) \quad |D(f; u)| \leq \frac{1}{6}L(b-a)^2 \bigvee_a^b(u').$$

*Proof.* We have

$$\begin{aligned} |D(f; u)| &= \left| \int_a^b \Phi_u(t) df(t) \right| \leq L \int_a^b |\Phi_u(t)| dt \\ &\leq \frac{L}{b-a} \bigvee_a^b(u') \int_a^b (t-a)(b-t) dt = \frac{1}{6}L(b-a)^2 \bigvee_a^b(u'), \end{aligned}$$

where for the second inequality we have used the inequality (2.3).  $\square$

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*Remark 2.* It is an open problem whether or not the constant  $\frac{1}{6}$  is the best possible constant in (3.4).

When the integrand  $f$  is monotonic, we can state the following result as well:

**Theorem 3.3.** Assume that  $u$  is as in Theorem 3.1. If  $f$  is monotonic nondecreasing on  $[a, b]$ , then

$$(3.5) \quad |D(f; u)| \leq 2 \cdot \frac{V_a^b(u')}{b-a} \cdot \int_a^b \left| t - \frac{a+b}{2} \right| f(t) dt \\ \leq \begin{cases} \frac{1}{2} V_a^b(u') \max\{|f(a)|, |f(b)|\} (b-a); \\ \frac{1}{(q+1)^{1/q}} V_a^b(u') \|f\|_p (b-a)^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ V_a^b(u') \|f\|_1, \end{cases}$$

where  $\|f\|_p := \left( \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}$ ,  $p \geq 1$  are the Lebesgue norms. The constants 2 and  $\frac{1}{2}$  are best possible in (3.5).

*Proof.* It is well known that, if  $p : [\alpha, \beta] \rightarrow \mathbb{R}$  is continuous and  $v : [\alpha, \beta] \rightarrow \mathbb{R}$  is monotonic nondecreasing, then the Riemann-Stieltjes integral  $\int_\alpha^\beta p(t) dv(t)$  exists and  $\left| \int_\alpha^\beta p(t) dv(t) \right| \leq \int_\alpha^\beta |p(t)| dv(t)$ . Then, on applying this property for the integral  $\int_a^b \Phi_u(t) df(t)$ , we have

$$(3.6) \quad |D(f; u)| = \left| \int_a^b \Phi_u(t) df(t) \right| \leq \int_a^b |\Phi_u(t)| df(t) \\ \leq \frac{V_a^b(u')}{b-a} \cdot \int_a^b (t-a)(b-t) df(t),$$



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where for the last inequality we used (2.3).

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned} \int_a^b (t-a)(b-t) df(t) &= f(t)(b-t)(t-a) \Big|_a^b - \int_a^b [-2t + (a+b)] f(t) dt \\ &= 2 \int_a^b \left( t - \frac{a+b}{2} \right) f(t) dt, \end{aligned}$$

which together with (3.6) produces the first part of (3.5).

The second part is obvious by the Hölder inequality applied for the integral  $\int_a^b \left( t - \frac{a+b}{2} \right) f(t) dt$  and the details are omitted.

For the sharpness of the constants we use as examples  $u(t) = \left| t - \frac{a+b}{2} \right|$  and  $f(t) = \operatorname{sgn} \left( t - \frac{a+b}{2} \right)$ ,  $t \in [a, b]$ . The details are omitted.  $\square$





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## 4. Bounds in the Case when $u'$ is Lipschitzian

The following result can be stated as well:

**Theorem 4.1.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  be absolutely continuous on  $[a, b]$  with the property that  $u'$  is  $K$ -Lipschitzian on  $(a, b)$ . If  $f$  is of bounded variation, then*

$$(4.1) \quad |D(f; u)| \leq \frac{1}{8} (b-a)^2 K \bigvee_a^b(f).$$

The constant  $\frac{1}{8}$  is best possible in (4.1).

*Proof.* Utilising (2.8), we have successively:

$$\begin{aligned} |D(f; u)| &= \left| \int_a^b \Phi_u(t) df(t) \right| \leq \sup_{t \in [a, b]} |\Phi_u(t)| \bigvee_a^b(f) \\ &\leq \frac{1}{2} K \sup_{t \in [a, b]} [(b-t)(t-a)] \bigvee_a^b(f) \\ &= \frac{1}{8} (b-a)^2 K \bigvee_a^b(f), \end{aligned}$$

and the inequality (4.1) is proved.

Now, for the sharpness of the constant, assume that the inequality holds with a constant  $G > 0$ , i.e.,

$$(4.2) \quad |D(f; u)| \leq G (b-a)^2 K \bigvee_a^b(f).$$



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for  $u$  and  $f$  as in the statement of the theorem.

Consider  $u(t) := \frac{1}{2} \left(t - \frac{a+b}{2}\right)^2$  and  $f(t) = \operatorname{sgn} \left(t - \frac{a+b}{2}\right)$ ,  $t \in [a, b]$ . Then  $u'(t) = t - \frac{a+b}{2}$  is  $K$ -Lipschitzian with the constant  $K = 1$  and

$$D(f; u) = \int_a^b \operatorname{sgn} \left(t - \frac{a+b}{2}\right) \cdot \left(t - \frac{a+b}{2}\right) dt = \frac{(b-a)^2}{4}.$$

Since  $\bigvee_a^b(f) = 2$ , hence from (4.2) we get  $\frac{(b-a)^2}{4} \leq 2G(b-a)^2$ , which implies that  $G \geq \frac{1}{8}$ .  $\square$

The following result may be stated as well:

**Theorem 4.2.** *Let  $v : [a, b] \rightarrow \mathbb{R}$  be as in Theorem 4.1. If  $f$  is  $L$ -Lipschitzian on  $[a, b]$ , then*

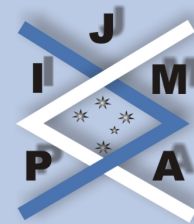
$$(4.3) \quad |D(f; u)| \leq \frac{1}{12} (b-a)^3 KL.$$

The constant  $\frac{1}{12}$  is best possible in (4.3).

*Proof.* We have by (2.8), that:

$$\begin{aligned} |D(f; u)| &= \left| \int_a^b \Phi_u(t) df(t) \right| \\ &\leq L \int_a^b |\Phi_u(t)| dt \\ &\leq \frac{1}{2} LK \int_a^b (b-t)(t-a) dt = \frac{1}{12} LK (b-a)^3, \end{aligned}$$

and the inequality is proved.



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For the sharpness, assume that (4.3) holds with a constant  $F > 0$ . Then

$$(4.4) \quad |D(f; u)| \leq F(b-a)^3 KL,$$

provided  $f$  and  $u$  are as in the hypothesis of the theorem.

Consider  $f(t) = t - \frac{a+b}{2}$  and  $u(t) = \frac{1}{2} \left(t - \frac{a+b}{2}\right)^2$ . Then  $u'$  is Lipschitzian with the constant  $K = 1$  and  $f$  is Lipschitzian with the constant  $L = 1$ . Also,

$$D(f; u) = \int_a^b \left(t - \frac{a+b}{2}\right)^2 dt = \frac{(b-a)^3}{12},$$

and by (4.4) we get  $\frac{(b-a)^3}{12} \leq F(b-a)^3$  which implies that  $F \geq \frac{1}{2}$ . □

Finally, the case of monotonic integrands is enclosed in the following result.

**Theorem 4.3.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  be as in Theorem 4.1. If  $f$  is monotonic nondecreasing, then*

$$(4.5) \quad |D(f; u)| \leq K \int_a^b \left(t - \frac{a+b}{2}\right) f(t) dt \leq \begin{cases} \frac{1}{4} K \max\{|f(a)|, |f(b)|\} (b-a)^2; \\ \frac{1}{2^{(q+1)^{1/q}} K \|f\|_p (b-a)^{1+1/q} \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} (b-a) K \|f\|_1. \end{cases}$$

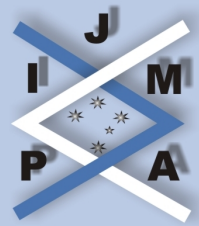
The first inequality is sharp. The constant  $\frac{1}{4}$  is best possible.

*Proof.* We have

$$\begin{aligned} |D(f; u)| &\leq \int_a^b |\Phi_u(t)| df(t) \\ &\leq \frac{1}{2} K \int_a^b (b-t)(t-a) df(t) \\ &= K \int_a^b \left(t - \frac{a+b}{2}\right) f(t) dt \end{aligned}$$

and the first inequality is proved. The second part follows by the Hölder inequality.

The sharpness of the first inequality and of the constant  $\frac{1}{4}$  follows by choosing  $u(t) = \left|t - \frac{a+b}{2}\right|$  and  $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ . The details are omitted.  $\square$



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## 5. Applications for the Čebyšev Functional

The above result can naturally be applied in obtaining various sharp upper bounds for the absolute value of the Čebyšev functional  $C(f, g)$  defined by

$$(5.1) \quad C(f, g) := \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt,$$

where  $f, g : [a, b] \rightarrow \mathbb{R}$  are Lebesgue integrable functions such that  $fg$  is also Lebesgue integrable.

There are various sharp upper bounds for  $|C(f, g)|$  and in the following we will recall just a few of them.

In 1934, Grüss [13] showed that

$$(5.2) \quad |C(f, g)| \leq \frac{1}{4} (M - m)(N - n)$$

under the assumptions that  $f$  and  $g$  satisfy the bounds

$$(5.3) \quad -\infty < m \leq f(t) \leq M < \infty \quad \text{and} \quad -\infty < n \leq g(t) \leq N < \infty$$

for almost every  $t \in [a, b]$ , where  $m, M, n, N$  are real numbers. The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller quantity.

Another less known result, even though it was established by Čebyšev in 1882 [1], states that

$$(5.4) \quad |C(f, g)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b - a)^2,$$

provided that  $f', g'$  exist and are continuous in  $[a, b]$  and  $\|f'\|_\infty = \sup_{t \in [a, b]} |f'(t)|$ . The constant  $\frac{1}{12}$  cannot be replaced by a smaller quantity. The Čebyšev inequality

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also holds if  $f, g$  are absolutely continuous on  $[a, b]$ ,  $f', g' \in L_\infty[a, b]$  and  $\|\cdot\|_\infty$  is replaced by the *ess sup* norm  $\|f'\|_\infty = \text{ess sup}_{t \in [a, b]} |f'(t)|$ .

In 1970, A. Ostrowski [16] considered a mixture between Grüss and Čebyšev inequalities by proving that

$$(5.5) \quad |C(f, g)| \leq \frac{1}{8} (b - a) (M - m) \|g'\|_\infty,$$

provided that  $f$  satisfies (5.3) and  $g$  is absolutely continuous and  $g' \in L_\infty[a, b]$ .

Three years after Ostrowski, A. Lupaş [14] obtained another bound for  $C(f, g)$  in terms of the Euclidean norms of the derivatives. Namely, he proved that

$$(5.6) \quad |C(f, g)| \leq \frac{1}{\pi^2} (b - a) \|f'\|_2 \|g'\|_2,$$

provided that  $f$  and  $g$  are absolutely continuous and  $f', g' \in L_2[a, b]$ . Here  $\frac{1}{\pi^2}$  is also best possible.

Recently, Cerone and Dragomir [2], proved the following result:

$$(5.7) \quad |C(f, g)| \leq \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b - a} \int_a^b \left| f(t) - \frac{1}{b - a} \int_a^b f(s) ds \right| dt,$$

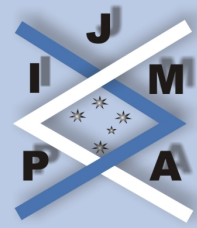
provided  $f \in L[a, b]$  and  $g \in C[a, b]$ .

As particular cases of (5.7), we can state the results:

$$(5.8) \quad |C(f, g)| \leq \|g\|_\infty \frac{1}{b - a} \int_a^b \left| f(t) - \frac{1}{b - a} \int_a^b f(s) ds \right| dt$$

if  $g \in C[a, b]$  and  $f \in L[a, b]$  and

$$(5.9) \quad |C(f, g)| \leq \frac{1}{2} (M - m) \frac{1}{b - a} \int_a^b \left| f(t) - \frac{1}{b - a} \int_a^b f(s) ds \right| dt,$$



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where  $m \leq g(x) \leq M$  for  $x \in [a, b]$ . The constants 1 in (5.8) and  $\frac{1}{2}$  in (5.9) are best possible. The inequality (5.9) has been obtained before in a different way in [5].

For generalisations in abstract Lebesgue spaces, best constants and discrete versions, see [3]. For other results on the Čebyšev functional, see [6], [7] and [12].

Now, assume that  $g : [a, b] \rightarrow \mathbb{R}$  is Lebesgue integrable on  $[a, b]$ . Then the function  $u(t) := \int_a^t g(s) ds$  is absolutely continuous on  $[a, b]$  and we can consider the function

$$(5.10) \quad \tilde{\Phi}_g(t) := \Phi_u(t) = \int_a^t g(s) ds - \frac{t-a}{b-a} \int_a^b g(s) ds, \quad t \in [a, b].$$

Utilising Lemma 2.1, we can state the following representation result.

**Lemma 5.1.** *If  $g$  is absolutely continuous, then*

$$(5.11) \quad \tilde{\Phi}_g(t) = \frac{1}{b-a} \int_a^b K(t, s) dg(s), \quad t \in [a, b],$$

where  $K$  is given by (2.2).

As a consequence of Theorems 2.2 and 2.4, we also have the inequalities:

**Proposition 5.2.** *Assume that  $g$  is Lebesgue integrable on  $[a, b]$ .*

(i) *If  $g$  is of bounded variation on  $[a, b]$ , then*

$$(5.12) \quad \left| \tilde{\Phi}_g(t) \right| \leq \frac{(t-a)(b-t)}{b-a} \bigvee_a^b(g) \leq \frac{1}{4} (b-a) \bigvee_a^b(g).$$

*The inequalities are sharp and  $\frac{1}{4}$  is best possible.*



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(ii) If  $g$  is  $K$ -Lipschitzian on  $[a, b]$ , then

$$(5.13) \quad \left| \tilde{\Phi}_g(t) \right| \leq \frac{1}{2} (b-t)(t-a) K \leq \frac{1}{8} (b-a)^2 K.$$

The constants  $\frac{1}{2}$  and  $\frac{1}{8}$  are best possible.

We notice that the functions  $g_1 : [a, b] \rightarrow \mathbb{R}$ ,  $g_1(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$  and  $g_2 : [a, b] \rightarrow \mathbb{R}$ ,  $g_2(t) = \left(t - \frac{a+b}{2}\right)$  realise equality in (5.12) and (5.13), respectively.

Now, we observe that for  $u(t) = \int_a^t g(s) ds$ ,  $s \in [a, b]$ , we have the identity:

$$(5.14) \quad D(f, u) = (b-a) C(f, g).$$

Utilising this identity and Theorems 3.1 and 3.3, we can state the following result.

**Proposition 5.3.** Assume that  $g$  is of bounded variation on  $[a, b]$ .

(i) If  $f$  is of bounded variation on  $[a, b]$ , then

$$(5.15) \quad |C(f, g)| \leq \frac{1}{4} \bigvee_a^b(g) \cdot \bigvee_a^b(f).$$

The constant  $\frac{1}{4}$  is best possible in (5.15).

(ii) If  $f$  is monotonic nondecreasing, then

$$(5.16) \quad |C(f, g)| \leq 2 \bigvee_a^b(g) \cdot \frac{1}{(b-a)^2} \int_a^b \left(t - \frac{a+b}{2}\right) f(t) dt$$



$$\leq \begin{cases} \frac{1}{2} \cdot V_a^b(g) \max \{|f(a)|, |f(b)|\}; \\ \frac{1}{(q+1)^{1/q}} V_a^b(g) \|f\|_p (b-a)^{-1/p} \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{b-a} V_a^b(g) \|f\|_1. \end{cases}$$

The multiplicative constants 2 and  $\frac{1}{2}$  are best possible in (5.16).

Finally, by Theorems 4.1 – 4.3 we also have the following sharp bounds for the Čebyšev functional  $C(f, g)$ .

**Proposition 5.4.** Assume that  $g$  is  $K$ -Lipschitzian on  $[a, b]$ .

(i) If  $f$  is of bounded variation, then

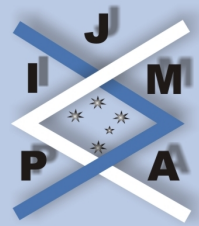
$$(5.17) \quad |C(f, g)| \leq \frac{1}{8} \cdot (b-a) K \bigvee_a^b(f).$$

The constant  $\frac{1}{8}$  is best possible.

(ii) If  $f$  is  $L$ -Lipschitzian, then

$$(5.18) \quad |C(f, g)| \leq \frac{1}{12} (b-a)^2 KL.$$

The constant  $\frac{1}{12}$  is best possible in (5.18).





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(iii) If  $f$  is monotonic nondecreasing, then

$$(5.19) \quad |C(f, g)| \leq K \cdot \frac{1}{b-a} \int_a^b \left( t - \frac{a+b}{2} \right) f(t) dt$$
$$\leq \begin{cases} \frac{1}{4} K (b-a) \max \{ |f(a)|, |f(b)| \}; \\ \frac{1}{2^{(q+1)^{1/q}} K (b-a)^{1/q} \|f\|_p \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} K \|f\|_1. \end{cases}$$

The first inequality is sharp. The constant  $\frac{1}{4}$  is best possible.

*Remark 3.* The inequalities (5.15) and (5.17) were obtained by P. Cerone and S.S. Dragomir in [4, Corollary 3.5]. However, the sharpness of the constants  $\frac{1}{4}$  and  $\frac{1}{8}$  were not discussed there. Inequality (5.18) is similar to the Čebyšev inequality (5.4).

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