# SHARP GRÜSS-TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE OF BOUNDED VARIATION

SEVER S. DRAGOMIR
School of Computer Science and Mathematics
Victoria University
PO Box 14428, Melbourne City
VIC 8001, Australia.
EMail: sever.dragomir@vu.edu.au

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*Abstract:* Sharp Grüss-type inequalities for functions whose derivatives are of bounded variation (Lipschitzian or monotonic) are given. Applications in relation with the well-known Čebyšev, Grüss, Ostrowski and Lupaş inequalities are provided as well.



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## 1. Introduction

In 1998, S.S. Dragomir and I. Fedotov [10] introduced the following *Grüss type error functional*

$$D(f;u) := \int_a^b f(t)\,du(t) - [u(a) - u(b)] \cdot \frac{1}{b-a} \int_a^b f(t)\,dt$$

in order to approximate the *Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ by the simpler quantity

$$[u(a) - u(b)] \cdot \frac{1}{b-a} \int_a^b f(t)\,dt.$$

In the same paper the authors have shown that

$$|D(f;u)| \leq \frac{1}{2} \cdot L(M-m)(b-a), \tag{1.1}$$

provided that $u$ is $L-$*Lipschitzian*, i.e., $|u(t) - u(s)| \leq L|t-x|$ for any $t, s \in [a,b]$ and $f$ is *Riemann integrable* and satisfies the condition

$$-\infty < m \leq f(t) \leq M < \infty \quad \text{for any } t \in [a,b].$$

The constant $\frac{1}{2}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

In [11], the same authors established another result for $D(f;u)$, namely

$$|D(f;u)| \leq \frac{1}{2} K(b-a) \bigvee_a^b (u), \tag{1.2}$$

provided that $u$ is of *bounded variation* on $[a,b]$ with the *total variation* $\bigvee_a^b (u)$ and $f$ is $K-$Lipschitzian. Here $\frac{1}{2}$ is also best possible.


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In [8], by introducing the *kernel* $\Phi_u : [a,b] \to \mathbb{R}$ given by

$$\Phi_u(t) := \frac{1}{b-a}[(t-a)u(b) + (b-t)u(a)] - u(t), \quad t \in [a,b], \tag{1.3}$$

the author has obtained the following *integral representation*

$$D(f;u) = \int_a^b \Phi_u(t)\, df(t), \tag{1.4}$$

where $u, f : [a,b] \to \mathbb{R}$ are bounded functions such that the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ and the Riemann integral $\int_a^b f(t)\, dt$ exist. By the use of this representation he also obtained the following bounds for $D(f;u)$,

$$\text{(1.5)} \quad |D(f;u)| \le \begin{cases} \sup\limits_{t\in[a,b]} |\Phi_u(t)| \cdot \bigvee_a^b(f) & \text{if } u \text{ is continuous and } f \text{ is of bounded variation;} \\ L\int_a^b |\Phi_u(t)|\, dt & \text{if } u \text{ is Riemann integrable and } f \text{ is } L\text{-Lipschitzian;} \\ \int_a^b |\Phi_u(t)|\, dt & \text{if } u \text{ is continuous and } f \text{ is monotonic nondecreasing.} \end{cases}$$

If $u$ is *monotonic nondecreasing* and $K(u)$ is defined by

$$K(u) := \frac{4}{(b-a)^2}\int_a^b \left(t - \frac{a+b}{2}\right) u(t)\, dt \ (\ge 0),$$

then

$$\begin{aligned} |D(f;u)| &\le \frac{1}{2}L(b-a)[u(b) - u(a) - K(u)] \\ &\le \frac{1}{2}L(b-a)[u(b) - u(a)], \end{aligned} \tag{1.6}$$


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provided that $f$ is $L-$Lipschitzian on $[a,b]$.

Here $\frac{1}{2}$ is best possible in both inequalities.

Also, for $u$ monotonic nondecreasing on $[a,b]$ and by defining $Q(u)$ as

$$Q(u) := \frac{1}{b-a}\int_a^b u(t)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\,(\geq 0),$$

we have

$$|D(f;u)| \leq [u(b)-u(a)-Q(u)]\cdot\bigvee_a^b(f) \leq [u(b)-u(a)]\cdot\bigvee_a^b(f), \tag{1.7}$$

provided that $f$ is of bounded variation on $[a,b]$. The first inequality in (1.7) is sharp.

Finally, the case when $u$ is convex and $f$ is of bounded variation produces the bound

$$|D(f;u)| \leq \frac{1}{4}\left[u'_-(b)-u'_+(a)\right](b-a)\bigvee_a^b(f), \tag{1.8}$$

with $\frac{1}{4}$ the best constant (when $u'_-(b)$ and $u'_+(a)$ are finite) and if $f$ is monotonic nondecreasing and $u$ is convex on $[a,b]$, then

$$\begin{aligned} 0 &\leq D(f;u) \\ &\leq 2\cdot\frac{u'_-(b)-u'_+(a)}{b-a}\cdot\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt \\ &\leq \begin{cases} \frac{1}{2}\left[u'_-(b)-u'_+(a)\right]\max\{|f(a)|,|f(b)|\}(b-a) \\ \frac{1}{(q+1)^{1/q}}\left[u'_-(b)-u'_+(a)\right]\|f\|_p(b-a)^{1/q} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[u'_-(b)-u'_+(a)\right]\|f\|_1, \end{cases} \end{aligned} \tag{1.9}$$

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where $2$ and $\frac{1}{2}$ are sharp constants (when $u'_{-}(b)$ and $u'_{+}(a)$ are finite) and $\|\cdot\|_p$ are the usual Lebesgue norms, i.e., $\|f\|_p := \left(\int_a^b |f(t)|^p \, dt\right)^{\frac{1}{p}}$, $p \geq 1$.

The main aim of the present paper is to provide sharp upper bounds for the absolute value of $D(f;u)$ under various conditions for $u'$, the derivative of an absolutely continuous function $u$, and $f$ of bounded variation (Lipschitzian or monotonic). Natural applications for the Čebyšev functional that complement the classical results due to Čebyšev, Grüss, Ostrowski and Lupaş are also given.



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## 2. Preliminary Results

We have the following integral representation of $\Phi_u$.

**Lemma 2.1.** *Assume that* $u : [a, b] \to \mathbb{R}$ *is absolutely continuous on* $[a, b]$ *and such that the derivative* $u'$ *exists on* $[a, b]$ *(eventually except at a finite number of points). If* $u'$ *is Riemann integrable on* $[a, b]$ *, then*

$$\Phi_u(t) := \frac{1}{b-a} \int_a^b K(t, s)\, du'(s), \quad t \in [a, b], \tag{2.1}$$

*where the kernel* $K : [a, b]^2 \to \mathbb{R}$ *is given by*

$$K(t, s) := \begin{cases} (b-t)(s-a) & \text{if } s \in [a, t], \\ (t-a)(b-s) & \text{if } s \in (t, b]. \end{cases} \tag{2.2}$$

*Proof.* We give, for simplicity, a proof only in the case when $u'$ is defined on the entire interval, and for which we have used the usual convention that $u'(a) := u'_+(a), u'(b) := u'_-(b)$ and the lateral derivatives are finite.

Since $u'$ is assumed to be Riemann integrable on $[a, b]$, it follows that the Riemann-Stieltjes integrals $\int_a^t (s-a)\, du'(s)$ and $\int_t^b (b-s)\, du'(s)$ exist for each $t \in [a, b]$. Now, integrating by parts in the Riemann-Stieltjes integral, we have successively

$$\begin{aligned} \int_a^b K(t, s)\, du'(s) &= (b-t) \int_a^t (s-a)\, du'(s) + (t-a) \int_t^b (b-s)\, du'(s) \\ &= (b-t) \left[ (s-a) u'(s) \Big|_a^t - \int_a^t u'(s)\, ds \right] \\ &\quad + (t-a) \left[ (b-s) u'(s) \Big|_t^b - \int_t^b u'(s)\, ds \right] \end{aligned}$$



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$$
\begin{aligned}
&= (b-t)\left[(t-a)u'(t) - (u(t)-u(a))\right] \\
&\qquad + (t-a)\left[-(b-t)u'(t) + u(b) - u(t)\right] \\
&= (t-a)\left[u(b)-u(t)\right] - (b-t)\left[u(t)-u(a)\right] \\
&= (b-a)\Phi_u(t),
\end{aligned}
$$

for any $t \in [a,b]$, and the representation (2.1) is proved. □

The following result provides a sharp bound for $|\Phi_u|$ in the case when $u'$ is of bounded variation.

**Theorem 2.2.** *Assume that $u : [a,b] \to \mathbb{R}$ is as in Lemma 2.1. If $u'$ is of bounded variation on $[a,b]$, then*

$$
|\Phi_u(t)| \leq \frac{(t-a)(b-t)}{b-a} \bigvee_a^b (u') \leq \frac{1}{4}(b-a) \bigvee_a^b (u'), \tag{2.3}
$$

*where $\bigvee_a^b (u')$ denotes the total variation of $u'$ on $[a,b]$.*

*The inequalities are sharp and the constant $\frac{1}{4}$ is best possible.*

*Proof.* It is well known that, if $p : [\alpha,\beta] \to \mathbb{R}$ is continuous and $v : [\alpha,\beta] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\,dv(s)$ exists and

$$
\left| \int_\alpha^\beta p(s)\,dv(s) \right| \leq \sup_{s\in[\alpha,\beta]} |p(s)| \bigvee_\alpha^\beta (v).
$$

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Now, utilising the representation (2.1) we have successively:

$$
\begin{aligned}
(2.4)\quad & |\Phi_u(t)| \\
& \leq \frac{1}{b-a}\left[(b-t)\left|\int_a^t (s-a)\,du'(s)\right| + (t-a)\left|\int_t^b (b-s)\,du'(s)\right|\right] \\
& \leq \frac{1}{b-a}\left[(b-t)\sup_{s\in[a,t]}(s-a)\cdot\bigvee_a^t(u') + (t-a)\sup_{s\in[t,b]}(b-s)\cdot\bigvee_t^b(u')\right] \\
& = \frac{(t-a)(b-t)}{b-a}\left[\bigvee_a^t(u') + \bigvee_t^b(u')\right] = \frac{(t-a)(b-t)}{b-a}\bigvee_a^b(u').
\end{aligned}
$$

The second inequality is obvious by the fact that $(t-a)(b-t) \leq \frac{1}{4}(b-a)^2$, $t \in [a,b]$.

For the sharpness of the inequalities, assume that there exist $A, B > 0$ so that

$$
(2.5)\qquad |\Phi_u(t)| \leq A\cdot\frac{(t-a)(b-t)}{b-a}\bigvee_a^b(u') \leq B(b-a)\bigvee_a^b(u'),
$$

with $u$ as in the assumption of the theorem. Then, for $t = \frac{a+b}{2}$, we get from (2.5) that

$$
(2.6)\qquad \left|\frac{u(a)+u(b)}{2} - u\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{4}A(b-a)\bigvee_a^b(u') \leq B(b-a)\bigvee_a^b(u').
$$

Consider the function $u : [a,b] \to \mathbb{R}$, $u(t) = \left|t - \frac{a+b}{2}\right|$. This function is absolutely continuous, $u'(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a,b] \setminus \left\{\frac{a+b}{2}\right\}$ and $\bigvee_a^b(u') = 2$. Then (2.6) becomes $\frac{b-a}{2} \leq \frac{1}{2}A(b-a) \leq 2B(b-a)$, which implies that $A \geq 1$ and $B \geq \frac{1}{4}$. □



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**Corollary 2.3.** *With the assumptions of Theorem 2.2, we have*

$$\left|\frac{u(a)+u(b)}{2}-u\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{4}(b-a)\bigvee_a^b(u')\,. \tag{2.7}$$

*The constant* $\frac{1}{4}$ *is best possible.*

The Lipschitzian case is incorporated in the following result.

**Theorem 2.4.** *Assume that* $u:[a,b]\to\mathbb{R}$ *is absolutely continuous on* $[a,b]$ *with the property that* $u'$ *is* $K-$*Lipschitzian on* $(a,b)$. *Then*

$$|\Phi_u(t)|\leq\frac{1}{2}(t-a)(b-t)K\leq\frac{1}{8}(b-a)^2K. \tag{2.8}$$

*The constants* $\frac{1}{2}$ *and* $\frac{1}{8}$ *are best possible.*

*Proof.* We utilise the fact that, for an $L-$Lipschitzian function $p:[\alpha,\beta]\to\mathbb{R}$ and a Riemann integrable function $v:[\alpha,\beta]\to\mathbb{R}$, the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\,dv(s)$ exists and

$$\left|\int_\alpha^\beta p(s)\,dv(s)\right|\leq L\int_\alpha^\beta|p(s)|\,ds.$$

Then, by (2.1), we have that

$$\begin{aligned}|\Phi_u(t)|&\\ &\leq\frac{1}{b-a}\left[(b-t)\left|\int_a^t(s-a)\,du'(s)\right|+(t-a)\left|\int_t^b(b-s)\,du'(s)\right|\right]\end{aligned} \tag{2.9}$$


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$$\leq \frac{1}{b-a}\left[\frac{1}{2}K(b-t)(t-a)^2+\frac{1}{2}K(t-a)(b-t)^2\right]$$
$$=\frac{1}{2}(t-a)(b-t)K,$$

which proves the first part of (2.8). The second part is obvious.

Now, for the sharpness of the constants, assume that there exist the constants $C, D > 0$ such that

$$|\Phi_u(t)| \leq C(b-t)(t-a)K \leq D(b-a)^2 K, \tag{2.10}$$

provided that $u$ is as in the hypothesis of the theorem. For $t = \frac{a+b}{2}$, we get from (2.10) that

$$\left|\frac{u(a)+u(b)}{2}-u\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{4}CK(b-a)^2 \leq D(b-a)^2 K. \tag{2.11}$$

Consider $u : [a,b] \to \mathbb{R}$, $u(t) = \frac{1}{2}\left|t-\frac{a+b}{2}\right|^2$. Then $u'(t) = t - \frac{a+b}{2}$ is Lipschitzian with the constant $K = 1$ and (2.11) becomes

$$\frac{1}{8}(b-a)^2 \leq \frac{1}{4}C(b-a)^2 \leq D(b-a)^2,$$

which implies that $C \geq \frac{1}{2}$ and $D \geq \frac{1}{8}$. □

**Corollary 2.5.** *With the assumptions of Theorem 2.4, we have*

$$\left|\frac{u(a)+u(b)}{2}-u\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{8}(b-a)^2 K. \tag{2.12}$$

*The constant $\frac{1}{8}$ is best possible.*


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*Remark* 1. If $u'$ is absolutely continuous and $\|u''\|_\infty := ess\sup_{t\in[a,b]} |u''(t)| < \infty$, then we can take $K = \|u''\|_\infty$, and we have from (2.8) that

$$|\Phi_u(t)| \le \frac{1}{2}(t-a)(b-t)\|u''\|_\infty \le \frac{1}{8}(b-a)^2\|u''\|_\infty . \tag{2.13}$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are best possible in (2.13).

From (2.12) we also get

$$\left|\frac{u(a)+u(b)}{2} - u\left(\frac{a+b}{2}\right)\right| \le \frac{1}{8}(b-a)^2\|u''\|_\infty , \tag{2.14}$$

in which $\frac{1}{8}$ is the best possible constant.


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## 3. Bounds in the Case when $u'$ is of Bounded Variation

We can start with the following result:

**Theorem 3.1.** *Assume that $u : [a,b] \to \mathbb{R}$ is as in Lemma 2.1. If $u'$ and $f$ are of bounded variation on $[a,b]$, then*

$$|D(f;u)| \le \frac{1}{4}(b-a)\bigvee_a^b(u')\cdot\bigvee_a^b(f), \tag{3.1}$$

*and the constant $\frac{1}{4}$ is best possible in (3.1).*

*Proof.* We use the following representation of the functional $D(f;u)$ obtained in [8] (see also [9] or [6]):

$$D(f;u) = \int_a^b \Phi_u(t)\,df(t). \tag{3.2}$$

Then we have the bound

$$\begin{aligned}|D(f;u)| &= \left|\int_a^b \Phi_u(t)\,df(t)\right| \le \sup_{t\in[a,b]}|\Phi_u(t)|\bigvee_a^b(f)\\ &\le \frac{1}{b-a}\bigvee_a^b(u')\sup_{t\in[a,b]}\left[(t-a)(b-t)\right]\cdot\bigvee_a^b(f)\\ &= \frac{1}{4}(b-a)\bigvee_a^b(u')\cdot\bigvee_a^b(f),\end{aligned}$$

where, for the last inequality we have used (2.3).


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To prove the sharpness of the constant $\frac{1}{4}$, assume that there is a constant $E > 0$ such that

$$|D(f;u)| \leq E(b-a)\bigvee_a^b(u')\cdot\bigvee_a^b(f). \tag{3.3}$$

Consider $u : [a,b] \to \mathbb{R}$, $u(t) = \left|t - \frac{a+b}{2}\right|$. Then $u'(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a,b] \setminus \left\{\frac{a+b}{2}\right\}$. The total variation on $[a,b]$ is $2$ and

$$D(f;u) = -\int_a^{\frac{a+b}{2}} f(t)\,dt + \int_{\frac{a+b}{2}}^b f(t)\,dt = \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f(t)\,dt.$$

Now, if we choose $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, then we obtain from (3.1) $b - a \leq 4E(b-a)$, which implies that $E \geq \frac{1}{4}$. □

The following result can be stated as well:

**Theorem 3.2.** *Assume that $u : [a,b] \to \mathbb{R}$ is as in Lemma 2.1. If the derivative $u'$ is of bounded variation on $[a,b]$ while $f$ is $L-$Lipschitzian on $[a,b]$, then*

$$|D(f;u)| \leq \frac{1}{6}L(b-a)^2\bigvee_a^b(u'). \tag{3.4}$$

*Proof.* We have

$$\begin{aligned}|D(f;u)| &= \left|\int_a^b \Phi_u(t)\,df(t)\right| \leq L\int_a^b |\Phi_u(t)|\,dt \\ &\leq \frac{L}{b-a}\bigvee_a^b(u')\int_a^b (t-a)(b-t)\,dt = \frac{1}{6}L(b-a)^2\bigvee_a^b(u'),\end{aligned}$$

where for the second inequality we have used the inequality (2.3). □



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*Remark* 2. It is an open problem whether or not the constant $\frac{1}{6}$ is the best possible constant in (3.4).

When the integrand $f$ is monotonic, we can state the following result as well:

**Theorem 3.3.** *Assume that $u$ is as in Theorem 3.1. If $f$ is monotonic nondecreasing on $[a,b]$, then*

$$|D(f;u)| \leq 2 \cdot \frac{\bigvee_a^b (u')}{b-a} \cdot \int_a^b \left| t - \frac{a+b}{2} \right| f(t)\, dt \tag{3.5}$$
$$\leq \begin{cases} \frac{1}{2} \bigvee_a^b (u') \max\{|f(a)|, |f(b)|\} (b-a); \\ \frac{1}{(q+1)^{1/q}} \bigvee_a^b (u') \|f\|_p (b-a)^{1/q} \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \bigvee_a^b (u') \|f\|_1, \end{cases}$$

*where $\|f\|_p := \left( \int_a^b |f(t)|^p\, dt \right)^{\frac{1}{p}}$, $p \geq 1$ are the Lebesgue norms. The constants 2 and $\frac{1}{2}$ are best possible in (3.5).*

*Proof.* It is well known that, if $p : [\alpha,\beta] \to \mathbb{R}$ is continuous and $v : [\alpha,\beta] \to \mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\, dv(t)$ exists and $\left| \int_\alpha^\beta p(t)\, dv(t) \right| \leq \int_\alpha^\beta |p(t)|\, dv(t)$. Then, on applying this property for the integral $\int_a^b \Phi_u(t)\, df(t)$, we have

$$|D(f;u)| = \left| \int_a^b \Phi_u(t)\, df(t) \right| \leq \int_a^b |\Phi_u(t)|\, df(t) \tag{3.6}$$
$$\leq \frac{\bigvee_a^b (u')}{b-a} \cdot \int_a^b (t-a)(b-t)\, df(t),$$


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where for the last inequality we used (2.3).

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\int_a^b (t-a)(b-t)\,df(t) = f(t)(b-t)(t-a)\Big|_a^b - \int_a^b [-2t+(a+b)]\,f(t)\,dt$$
$$= 2\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt,$$

which together with (3.6) produces the first part of (3.5).

The second part is obvious by the Hölder inequality applied for the integral $\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt$ and the details are omitted.

For the sharpness of the constants we use as examples $u(t) = \left|t-\frac{a+b}{2}\right|$ and $f(t) = \operatorname{sgn}\left(t-\frac{a+b}{2}\right), t \in [a,b]$. The details are omitted. □



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## 4. Bounds in the Case when $u'$ is Lipschitzian

The following result can be stated as well:

**Theorem 4.1.** *Let $u : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$ with the property that $u'$ is $K-$Lipschitzian on $(a,b)$. If $f$ is of bounded variation, then*

$$|D(f;u)| \le \frac{1}{8}(b-a)^2 K \bigvee_a^b (f). \tag{4.1}$$

*The constant $\frac{1}{8}$ is best possible in (4.1).*

*Proof.* Utilising (2.8), we have successively:

$$\begin{aligned} |D(f;u)| &= \left| \int_a^b \Phi_u(t)\, df(t) \right| \le \sup_{t\in[a,b]} |\Phi_u(t)| \bigvee_a^b (f) \\ &\le \frac{1}{2} K \sup_{t\in[a,b]} [(b-t)(t-a)] \bigvee_a^b (f) \\ &= \frac{1}{8}(b-a)^2 K \bigvee_a^b (f), \end{aligned}$$

and the inequality (4.1) is proved.

Now, for the sharpness of the constant, assume that the inequality holds with a constant $G > 0$, i.e.,

$$|D(f;u)| \le G(b-a)^2 K \bigvee_a^b (f). \tag{4.2}$$

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for $u$ and $f$ as in the statement of the theorem.

Consider $u(t) := \frac{1}{2}\left(t - \frac{a+b}{2}\right)^2$ and $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a,b]$. Then $u'(t) = t - \frac{a+b}{2}$ is $K-$Lipschitzian with the constant $K = 1$ and

$$D(f;u) = \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) \cdot \left(t - \frac{a+b}{2}\right) dt = \frac{(b-a)^2}{4}.$$

Since $\bigvee_a^b (f) = 2$, hence from (4.2) we get $\frac{(b-a)^2}{4} \leq 2G(b-a)^2$, which implies that $G \geq \frac{1}{8}$. □

The following result may be stated as well:

**Theorem 4.2.** *Let $v : [a,b] \to \mathbb{R}$ be as in Theorem 4.1. If $f$ is $L-$Lipschitzian on $[a,b]$, then*

$$|D(f;u)| \leq \frac{1}{12}(b-a)^3 KL. \tag{4.3}$$

*The constant $\frac{1}{12}$ is best possible in (4.3).*

*Proof.* We have by (2.8), that:

$$\begin{aligned} |D(f;u)| &= \left| \int_a^b \Phi_u(t)\, df(t) \right| \\ &\leq L \int_a^b |\Phi_u(t)|\, dt \\ &\leq \frac{1}{2} LK \int_a^b (b-t)(t-a)\, dt = \frac{1}{12} KL(b-a)^3, \end{aligned}$$

and the inequality is proved.


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For the sharpness, assume that (4.3) holds with a constant $F > 0$. Then

$$|D(f;u)| \leq F(b-a)^3 KL, \tag{4.4}$$

provided $f$ and $u$ are as in the hypothesis of the theorem.

Consider $f(t) = t - \frac{a+b}{2}$ and $u(t) = \frac{1}{2}\left(t - \frac{a+b}{2}\right)^2$. Then $u'$ is Lipschitzian with the constant $K = 1$ and $f$ is Lipschitzian with the constant $L = 1$. Also,

$$D(f;u) = \int_a^b \left(t - \frac{a+b}{2}\right)^2 dt = \frac{(b-a)^3}{12},$$

and by (4.4) we get $\frac{(b-a)^3}{12} \leq F(b-a)^3$ which implies that $F \geq \frac{1}{2}$. □

Finally, the case of monotonic integrands is enclosed in the following result.

**Theorem 4.3.** *Let $u : [a,b] \to \mathbb{R}$ be as in Theorem 4.1. If $f$ is monotonic nondecreasing, then*

$$\begin{aligned} |D(f;u)| &\leq K \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\, dt \\ &\leq \begin{cases} \frac{1}{4} K \max\{|f(a)|, |f(b)|\}(b-a)^2; \\ \frac{1}{2(q+1)^{1/q}} K \|f\|_p (b-a)^{1+1/q} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(b-a) K \|f\|_1 . \end{cases} \end{aligned} \tag{4.5}$$

*The first inequality is sharp. The constant $\frac{1}{4}$ is best possible.*

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*Proof.* We have

$$\begin{aligned}|D(f;u)| &\leq \int_a^b |\Phi_u(t)|\, df(t) \\ &\leq \frac{1}{2}K\int_a^b (b-t)(t-a)\, df(t) \\ &= K\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\, dt\end{aligned}$$

and the first inequality is proved. The second part follows by the Hölder inequality.

The sharpness of the first inequality and of the constant $\frac{1}{4}$ follows by choosing $u(t) = \left|t - \frac{a+b}{2}\right|$ and $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$. The details are omitted. □

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# 5. Applications for the Čebyšev Functional

The above result can naturally be applied in obtaining various sharp upper bounds for the absolute value of the Čebyšev functional $C(f,g)$ defined by

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt, \tag{5.1}$$

where $f,g : [a,b] \to \mathbb{R}$ are Lebesgue integrable functions such that $fg$ is also Lebesgue integrable.

There are various sharp upper bounds for $|C(f,g)|$ and in the following we will recall just a few of them.

In 1934, Grüss [13] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n) \tag{5.2}$$

under the assumptions that $f$ and $g$ satisfy the bounds

$$-\infty < m \le f(t) \le M < \infty \quad \text{and} \quad -\infty < n \le g(t) \le N < \infty \tag{5.3}$$

for almost every $t \in [a,b]$, where $m, M, n, N$ are real numbers. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Another less known result, even though it was established by Čebyšev in 1882 [1], states that

$$|C(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{5.4}$$

provided that $f', g'$ exist and are continuous in $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be replaced by a smaller quantity. The Čebyšev inequality


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also holds if $f, g$ are absolutely continuous on $[a, b]$, $f', g' \in L_\infty[a, b]$ and $\|\cdot\|_\infty$ is replaced by the $ess\sup$ norm $\|f'\|_\infty = ess\sup_{t\in[a,b]} |f'(t)|$.

In 1970, A. Ostrowski [16] considered a mixture between Grüss and Čebyšev inequalities by proving that

$$|C(f, g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{5.5}$$

provided that $f$ satisfies (5.3) and $g$ is absolutely continuous and $g' \in L_\infty[a, b]$.

Three years after Ostrowski, A. Lupaş [14] obtained another bound for $C(f, g)$ in terms of the Euclidean norms of the derivatives. Namely, he proved that

$$|C(f, g)| \leq \frac{1}{\pi^2}(b-a)\|f'\|_2 \|g'\|_2, \tag{5.6}$$

provided that $f$ and $g$ are absolutely continuous and $f', g' \in L_2[a, b]$. Here $\frac{1}{\pi^2}$ is also best possible.

Recently, Cerone and Dragomir [2], proved the following result:

$$|C(f, g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \tag{5.7}$$

provided $f \in L[a, b]$ and $g \in C[a, b]$.

As particular cases of (5.7), we can state the results:

$$|C(f, g)| \leq \|g\|_\infty \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{5.8}$$

if $g \in C[a, b]$ and $f \in L[a, b]$ and

$$|C(f, g)| \leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \tag{5.9}$$


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where $m \leq g(x) \leq M$ for $x \in [a, b]$. The constants $1$ in (5.8) and $\frac{1}{2}$ in (5.9) are best possible. The inequality (5.9) has been obtained before in a different way in [5].

For generalisations in abstract Lebesgue spaces, best constants and discrete versions, see [3]. For other results on the Čebyšev functional, see [6], [7] and [12].

Now, assume that $g : [a, b] \to \mathbb{R}$ is Lebesgue integrable on $[a, b]$. Then the function $u(t) := \int_a^t g(s)\, ds$ is absolutely continuous on $[a, b]$ and we can consider the function

$$\tilde{\Phi}_g(t) := \Phi_u(t) = \int_a^t g(s)\, ds - \frac{t-a}{b-a}\int_a^b g(s)\, ds, \quad t \in [a, b]. \tag{5.10}$$

Utilising Lemma 2.1, we can state the following representation result.

**Lemma 5.1.** *If $g$ is absolutely continuous, then*

$$\tilde{\Phi}_g(t) = \frac{1}{b-a}\int_a^b K(t, s)\, dg(s), \quad t \in [a, b], \tag{5.11}$$

*where $K$ is given by (2.2).*

As a consequence of Theorems 2.2 and 2.4, we also have the inequalities:

**Proposition 5.2.** *Assume that $g$ is Lebesgue integrable on $[a, b]$.*

*(i) If $g$ is of bounded variation on $[a, b]$, then*

$$\left|\tilde{\Phi}_g(t)\right| \leq \frac{(t-a)(b-t)}{b-a}\bigvee_a^b(g) \leq \frac{1}{4}(b-a)\bigvee_a^b(g). \tag{5.12}$$

*The inequalities are sharp and $\frac{1}{4}$ is best possible.*



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*(ii) If $g$ is $K-$Lipschitzian on $[a,b]$, then*

$$\left|\tilde{\Phi}_g(t)\right| \leq \frac{1}{2}(b-t)(t-a)K \leq \frac{1}{8}(b-a)^2 K. \tag{5.13}$$

*The constants $\frac{1}{2}$ and $\frac{1}{8}$ are best possible.*

We notice that the functions $g_1 : [a,b] \to \mathbb{R}$, $g_1(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ and $g_2 : [a,b] \to \mathbb{R}$, $g(t) = \left(t - \frac{a+b}{2}\right)$ realise equality in (5.12) and (5.13), respectively.

Now, we observe that for $u(t) = \int_a^t g(s)\,ds$, $s \in [a,b]$, we have the identity:

$$D(f,u) = (b-a)\,C(f,g). \tag{5.14}$$

Utilising this identity and Theorems 3.1 and 3.3, we can state the following result.

**Proposition 5.3.** *Assume that $g$ is of bounded variation on $[a,b]$.*

*(i) If $f$ is of bounded variation on $[a,b]$, then*

$$|C(f,g)| \leq \frac{1}{4}\bigvee_a^b(g)\cdot\bigvee_a^b(f). \tag{5.15}$$

*The constant $\frac{1}{4}$ is best possible in (5.15).*

*(ii) If $f$ is monotonic nondecreasing, then*

$$|C(f,g)| \leq 2\bigvee_a^b(g)\cdot\frac{1}{(b-a)^2}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt \tag{5.16}$$



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$$\leq \begin{cases} \frac{1}{2} \cdot \bigvee_a^b (g) \max\{|f(a)|, |f(b)|\}; \\ \frac{1}{(q+1)^{1/q}} \bigvee_a^b (g) \|f\|_p (b-a)^{-1/p} & \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{b-a} \bigvee_a^b (g) \|f\|_1 . \end{cases}$$

*The multiplicative constants* $2$ *and* $\frac{1}{2}$ *are best possible in (5.16).*

Finally, by Theorems 4.1 – 4.3 we also have the following sharp bounds for the Čebyšev functional $C(f,g)$.

**Proposition 5.4.** *Assume that* $g$ *is* $K-$*Lipschitzian on* $[a,b]$.

(i) *If* $f$ *is of bounded variation, then*

$$|C(f,g)| \leq \frac{1}{8} \cdot (b-a) K \bigvee_a^b (f). \tag{5.17}$$

*The constant* $\frac{1}{8}$ *is best possible.*

(ii) *If* $f$ *is* $L-$*Lipschitzian, then*

$$|C(f,g)| \leq \frac{1}{12} (b-a)^2 KL. \tag{5.18}$$

*The constant* $\frac{1}{12}$ *is best possible in (5.18).*



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*(iii) If $f$ is monotonic nondecreasing, then*

$$|C(f,g)| \le K \cdot \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt \tag{5.19}$$
$$\le \begin{cases} \frac{1}{4}K(b-a)\max\{|f(a)|,|f(b)|\}; \\ \frac{1}{2(q+1)^{1/q}}K(b-a)^{1/q}\|f\|_p \quad \textit{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}K\|f\|_1 . \end{cases}$$

*The first inequality is sharp. The constant $\frac{1}{4}$ is best possible.*

*Remark* 3. The inequalities (5.15) and (5.17) were obtained by P. Cerone and S.S. Dragomir in [4, Corollary 3.5]. However, the sharpnes of the constants $\frac{1}{4}$ and $\frac{1}{8}$ were not discussed there. Inequality (5.18) is similar to the Čebyšev inequality (5.4).


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