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SOME CONSIDERATIONS ON THE MONOTONICITY PROPERTY OF POWER MEANS

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ABSTRACT. If A is an isotonic linear functional and $f : [a, b] \to (0, \infty)$ is a monotone function then $Q(r, f) = (f^r(a) + f^r(b) - A(f^r))^{1/r}$ is increasing in r.

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1. INTRODUCTION

Let $0 < a \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq b$ and w_k $(1 \leq k \leq n)$ be positive weights associated with these x_k and whose sum is unity. A Mc D. Mercer [3] proved the following variant of Jensen's inequality.

Theorem 1.1. *If f is a convex function on an interval containing the points x_k then*

$$f\left(a + b - \sum_{k=1}^{n} w_k x_k\right) \leq f(a) + f(b) - \sum_{k=1}^{n} w_k f(x_k). \tag{1.1}$$

The weighted power means $M_r(x, w)$ of the number x_i with weights w_i are defined as

$$M_r(x, w) = \left(\sum_{k=1}^{n} w_k x_k^r\right)^{\frac{1}{r}} \text{ for } r \neq 0$$

$$M_0(x, w) = \exp\left(\sum_{k=1}^{n} w_k \ln x_k\right).$$

In [2] Mercer defined the family of functions

$$Q_r(a, b, x) = (a^r + b^r - M_r^r(x, w))^{\frac{1}{r}} \text{ for } r \neq 0$$

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$$Q_0(a,b,x) = \frac{ab}{M_0}$$

and proved the following (see also [4]):

Theorem 1.2. *For* $r < s$, $Q_r(a,b,x) \leq Q_s(a,b,x)$.

In [3] are given another proofs of the above theorems.

Let us consider a isotonic linear functional A, i.e., a functional $A : C[a,b] \to \mathbb{R}$ with the properties:

(i) $A(tf + sg) = tA(f) + sA(g)$ for $t, s \in \mathbb{R}$, $f, g \in C[a,b]$;
(ii) $A(f) \geq 0$ of $f(x) \geq 0$ for all $x \in [a,b]$.

In [1] A. Lupaş proved the following result:

"If f is a convex function and A is an isotonic linear functional with $A(e_0) = 1$, then

$$f(a_1) \leq A(f) \leq \frac{(b-a_1)f(a) + (a_1-a)f(b)}{b-a}, \tag{1.2}$$

where $e_i : [a,b] \to \mathbb{R}$, $e_i(x) = x^i$ and $a_1 = A(e_1)$.

Let A be an isotonic linear functional defined on $C[a,b]$ such that $A(e_0) = 1$. For a real number r and positive function f, $f \in C[a,b]$ we define the power mean of order r as

$$M(r,f) = \begin{cases} (A(f^r))^{\frac{1}{r}} & \text{for } r \neq 0 \\ \exp\left(A(\log f)\right) & \text{for } r = 0 \end{cases} \tag{1.3}$$

and for every monotone function $f : [a,b] \to (0,\infty)$

$$Q(r;f) = \begin{cases} \left(f^r(a) + f^r(b) - M^r(r,f)\right)^{\frac{1}{r}}, & r \neq 0 \\ \frac{f(a)f(b)}{\exp(A(\log f))}, & r = 0 \end{cases}. \tag{1.4}$$

2. Main Results

Our main results are given in the following theorems. Let A be an isotonic linear functional defined on $C[a,b]$ such that $A(e_0) = 1$.

Theorem 2.1. *Let f be a convex function on $[a,b]$. Then*

$$\begin{aligned} f(a+b-a_1) &\leq A(g) \leq f(a) + f(b) - f(a)\frac{b-a_1}{b-a} - f(b)\frac{a_1-a}{b-a} \\ &\leq f(a) + f(b) - A(f), \end{aligned} \tag{2.1}$$

where $g = f(a+b-\cdot)$.

Theorem 2.2. *Let $r, s \in \mathbb{R}$ such that $r \leq s$. Then*

$$Q(r,f) \leq Q(s,f), \tag{2.2}$$

for every monotone positive function.

Proof of Theorem 2.1. The function g is a convex function. From inequality (1.2), written for the function g we get:

$$f(a+b-a_1) \leq A(g) \leq \frac{(b-a_1)f(b) + (a_1-a)f(a)}{b-a}. \tag{2.3}$$

Using Hadamard's inequality (1.2) relative to the function f we obtain

$$A(f) \leq f(a)\frac{b-a_1}{b-a} + f(b)\frac{a_1-a}{b-a}. \tag{2.4}$$

However,

$$\frac{(b-a_1)f(b)+(a_1-a)f(a)}{b-a} = f(a)+f(b)-f(a)\frac{b-a_1}{b-a} - f(b)\frac{a_1-a}{b-a}. \tag{2.5}$$

Now (2.1) follows by (2.5), (2.4) and (2.3). □

Proof of Theorem 2.2. Let us denote $\alpha = f^r(a), \quad \beta = f^r(b)$. If $0 < r < s$ then the function $g(x) = x^{s/r}$ is convex. Let us consider the following isotonic linear functional $B : C[\alpha, \beta] \to \mathbb{R}$ defined by $B(h) = A(h \circ f^r)$, where $\alpha = \min(f^r(a), f^r(b))$, $\beta = \max(f^r(a), f^r(b))$. We have:

$$B(e_1) = A(f^r).$$

From (2.1) we get

$$g(\alpha + \beta - B(e_1)) \leq g(\alpha) + g(\beta) - B(g)$$

or

$$(f(a)^r + f^r(b) - A(f)^r)^{s/r} \leq f^s(a) + f^s(b) - A(f^s).$$

The last inequality is equivalent to

$$Q(r, f) \leq A(s, f).$$

For $r < s < 0$, g is concave and we obtain

$$(f^r(a) + f^r(b) - A(f^r))^{s/r} \geq f^s(a) + f^s(b) - A(f^s)$$

which is also equivalent to $Q(r, f) \leq Q(s, f)$. Finally, applying (2.1) to the concave function $\log x$ for the functional

$$B(g) = A(g \circ f^r),$$

we have

$$\log\left(\alpha + \beta - A(f^r)\right) \geq \log \alpha + \log \beta - A\left(\log f^r\right),$$

or

$$r \log(Q(r, f)) \geq r \log Q(0, f),$$

which shows that for $r > 0$

$$Q(-r, f) \leq Q(0, f) \leq Q(r, f).$$

□

Remark 2.3. For the functional A, $A : C[a, b] \to \mathbb{R}$ defined by

$$A(f) = \sum_{k=1}^{n} w_k f(x_k),$$

in the particular case when $f(x) = x^r$ we obtain Theorem 1.2.

REFERENCES

[1] A. LUPAŞ, A generalization of Hadamard's inequalities for convex functions, *Univ. Beograd Publ. Elektro. Fak.*, **544-579** (1976), 115–121.

[2] A. Mc D. MERCER, A monotonicity property of power means, *J. Ineq. Pure and Appl. Math.*, **3**(3) (2002), Article 40. [ONLINE: http://jipam.vu.edu.au/article.php?sid=192]

[3] A. Mc D. MERCER, A variant of Jensen's inequality, *J. Ineq. Pure and Appl. Math.*, **4**(4) (2003), Article 73. [ONLINE: http://jipam.vu.edu.au/article.php?sid=314]

[4] A. WITKOWSKI, A new proof of the monotonicity property of power means, *J. Ineq. Pure and Appl. Math.*, **5**(3) (2004), Article 73. [ONLINE: http://jipam.vu.edu.au/article.php?sid=425]