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NOTE ON CERTAIN INEQUALITIES FOR MEANS IN TWO VARIABLES

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ABSTRACT. Given the positive real numbers x and y, let $A(x,y)$, $G(x,y)$, and $I(x,y)$ denote their arithmetic mean, geometric mean, and identric mean, respectively. It is proved that for $p \geq 2$, the double inequality

$$\alpha A^p(x,y) + (1-\alpha)G^p(x,y) < I^p(x,y) < \beta A^p(x,y) + (1-\beta)G^p(x,y)$$

holds true for all positive real numbers $x \neq y$ if and only if $\alpha \leq \left(\frac{2}{e}\right)^p$ and $\beta \geq \frac{2}{3}$. This result complements a similar one established by H. Alzer and S.-L. Qiu [Inequalities for means in two variables, *Arch. Math. (Basel)* **80** (2003), 201–215].

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1. INTRODUCTION AND MAIN RESULT

The means in two variables are special and they have found a number of applications (see, for instance, [1, 5] and the references therein). In this note we focus on certain inequalities involving the arithmetic mean, the geometric mean, and the identric mean of two positive real numbers x and y. Recall that these means are defined by $A(x,y) = \frac{x+y}{2}$, $G(x,y) = \sqrt{xy}$, and

$$I(x,y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}} \qquad \text{if } x \neq y,$$
$$I(x,x) = x,$$

respectively. It is well-known that

$$G(x,y) < I(x,y) < A(x,y) \tag{1.1}$$

for all positive real numbers $x \neq y$. On the other hand, J. Sándor [6] proved that

$$\frac{2}{3}A(x,y) + \frac{1}{3}G(x,y) < I(x,y) \tag{1.2}$$

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for all positive real numbers $x \neq y$. Note that inequality (1.2) is a refinement of the first inequality in (1.1). Also, (1.2) is sharp in the sense that $\frac{2}{3}$ cannot be replaced by any greater constant. An interesting counterpart of (1.2) has been recently obtained by H. Alzer and S.-L. Qiu [1, Theorem 1]. Their result reads as follows:

Theorem 1.1. *The double inequality*

$$\alpha A(x,y) + (1-\alpha)G(x,y) < I(x,y) < \beta A(x,y) + (1-\beta)G(x,y)$$

holds true for all positive real numbers $x \neq y$, if and only if $\alpha \leq \frac{2}{3}$ and $\beta \geq \frac{2}{e}$.

Another counterpart of (1.2) has been obtained by J. Sándor and T. Trif [8, Theorem 2.5]. More precisely, they proved that

$$I^2(x,y) < \frac{2}{3}A^2(x,y) + \frac{1}{3}G^2(x,y) \tag{1.3}$$

for all positive real numbers $x \neq y$. We note that (1.3) is a refinement of the second inequality in (1.1). Moreover, (1.3) is the best possible inequality of the type

$$I^2(x,y) < \beta A^2(x,y) + (1-\beta)G^2(x,y) \tag{1.4}$$

in the sense that (1.4) holds true for all positive real numbers $x \neq y$ if and only if $\beta \geq \frac{2}{3}$.

It should be mentioned that (1.3) was derived in [8] as a consequence of certain power series expansions discovered by J. Sándor [7]. We present here an alternative proof of (1.3), based on the Gauss quadrature formula with two knots (see [2, pp. 343–344] or [3, p. 36])

$$\int_0^1 f(t)dt = \frac{1}{2}f\left(\frac{1}{2}+\frac{1}{2\sqrt{3}}\right) + \frac{1}{2}f\left(\frac{1}{2}-\frac{1}{2\sqrt{3}}\right) + \frac{1}{4320}f^{(4)}(\xi), \qquad 0<\xi<1.$$

Choosing $f(t) = \log(tx+(1-t)y)$ and taking into account that

$$\int_0^1 f(t)dt = \log I(x,y),$$

we get

$$\log I(x,y) = \frac{1}{2}\log\left(\frac{2}{3}A^2(x,y) + \frac{1}{3}G^2(x,y)\right) - \frac{(x-y)^4}{720(\xi x + (1-\xi)y)^4}.$$

Consequently, it holds that

$$\exp\left(\frac{1}{360}\left(\frac{x-y}{\max(x,y)}\right)^4\right) < \frac{\frac{2}{3}A^2(x,y)+\frac{1}{3}G^2(x,y)}{I^2(x,y)} < \exp\left(\frac{1}{360}\left(\frac{x-y}{\min(x,y)}\right)^4\right).$$

This inequality yields (1.3) and estimates the sharpness of (1.3). However, we note that the double inequality (2.33) in [8] provides better bounds for the ratio

$$\left(\frac{2}{3}A^2(x,y) + \frac{1}{3}G^2(x,y)\right) \Big/ I^2(x,y).$$

The next theorem is the main result of this note and it is motivated by (1.3) and Theorem 1.1.

Theorem 1.2. *Given the real number $p \geq 2$, the double inequality*

$$\alpha A^p(x,y) + (1-\alpha)G^p(x,y) < I^p(x,y) < \beta A^p(x,y) + (1-\beta)G^p(x,y) \tag{1.5}$$

holds true for all positive real numbers $x \neq y$ if and only if $\alpha \leq \left(\frac{2}{e}\right)^p$ and $\beta \geq \frac{2}{3}$.

2. PROOF OF THEOREM 1.2

Proof. In order to prove that the first inequality in (1.5) holds true for $\alpha = \left(\frac{2}{e}\right)^p$, we will use the ingenious method of E.B. Leach and M.C. Sholander [4] (see also [1]). More precisely, we show that

$$\left(\frac{2}{e}\right)^p A^p\left(e^t, e^{-t}\right) + \left(1 - \left(\frac{2}{e}\right)^p\right) G^p\left(e^t, e^{-t}\right) < I^p\left(e^t, e^{-t}\right), \qquad \text{for all } t > 0. \tag{2.1}$$

It is easily seen that (2.1) is equivalent to $f_p(t) < 0$ for all $t > 0$, where $f_p : (0,\infty) \to \mathbb{R}$ is the function defined by

$$f_p(t) = (2\cosh t)^p + e^p - 2^p - \exp(pt\coth t).$$

We have

$$f_p'(t) = \frac{4p\sinh^3 t(2\cosh t)^{p-1} - p(\sinh(2t) - 2t)\exp(pt\coth t)}{2\sinh^2 t}.$$

By means of the logarithmic mean of two variables,

$$L(x,y) = \frac{x-y}{\log x - \log y} \qquad \text{if } x \neq y,$$
$$L(x,x) = x,$$

the derivative f_p' may be expressed as

$$f_p'(t) = p\,\frac{L(u(t), v(t))}{2\sinh^2 t}\,g(t), \tag{2.2}$$

where

$$\begin{aligned} u(t) &= 4\sinh^3 t(2\cosh t)^{p-1}, \\ v(t) &= (\sinh(2t) - 2t)\exp(pt\coth t), \\ g(t) &= \log u(t) - \log v(t) \\ &= (p+1)\log 2 + 3\log(\sinh t) + (p-1)\log(\cosh t) - \log(\sinh(2t) - 2t) - pt\coth t. \end{aligned}$$

We have

$$\begin{aligned} g'(t) &= \frac{3\cosh t}{\sinh t} + \frac{(p-1)\sinh t}{\cosh t} - \frac{2\cosh(2t) - 2}{\sinh(2t) - 2t} - \frac{p\cosh t}{\sinh t} + \frac{pt}{\sinh^2 t} \\ &= \frac{3\cosh^2 t - \sinh^2 t - p(\cosh^2 t - \sinh^2 t)}{\sinh t\cosh t} + \frac{pt}{\sinh^2 t} - \frac{2\cosh(2t) - 2}{\sinh(2t) - 2t} \\ &= \frac{\cosh(2t) + 2 - p}{\sinh t\cosh t} + \frac{pt}{\sinh^2 t} - \frac{2\cosh(2t) - 2}{\sinh(2t) - 2t}, \end{aligned}$$

hence

$$g'(t) = g_1(t) + g_2(t), \tag{2.3}$$

where

$$\begin{aligned} g_1(t) &= \frac{\cosh(2t)}{\sinh t\cosh t} + \frac{2t}{\sinh^2 t} - \frac{2\cosh(2t) - 2}{\sinh(2t) - 2t}, \\ g_2(t) &= \frac{(p-2)t}{\sinh^2 t} - \frac{p-2}{\sinh t\cosh t}. \end{aligned}$$

But

$$g_2(t) = \frac{p-2}{\sinh^2 t \cosh t}(t\cosh t - \sinh t)$$
$$= \frac{p-2}{\sinh^2 t \cosh t}\sum_{k=1}^{\infty}\left(\frac{1}{(2k)!} - \frac{1}{(2k+1)!}\right)t^{2k+1}.$$

Taking into account that $p \geq 2$, we deduce that

$$g_2(t) \geq 0 \qquad \text{for all } \ t > 0. \tag{2.4}$$

Further, let $h : (0,\infty) \to \mathbb{R}$ be the function defined by

$$h(t) = \sinh^2 t \cosh t(\sinh(2t) - 2t)g_1(t).$$

Then we have

$$h(t) = 2t\sinh t + \sinh t \sinh 2t - 4t^2$$
$$= 2t\sinh t + \frac{1}{2}\cosh(3t) - \frac{1}{2}\cosh t - 4t^2$$
$$= \sum_{k=2}^{\infty}\left(\frac{2}{(2k-1)!} + \frac{3^{2k}-1}{2(2k)!}\right)t^{2k}.$$

Therefore $h(t) > 0$ for $t > 0$, hence

$$g_1(t) > 0 \qquad \text{for all } \ t > 0. \tag{2.5}$$

By (2.3), (2.4), and (2.5) we conclude that $g'(t) > 0$ for $t > 0$, hence g is increasing on $(0,\infty)$. Taking into account that

$$\lim_{t\to\infty} g(t) = (p+1)\log 2 + \log\left(\lim_{t\to\infty}\frac{\sinh^3 t}{\cosh t(\sinh(2t) - 2t)}\right)$$
$$+ p\lim_{t\to\infty}(\log(\cosh t) - t) + p\lim_{t\to\infty} t(1 - \coth t)$$
$$= (p+1)\log 2 + \log\frac{1}{2} + p\log\frac{1}{2}$$
$$= 0,$$

it follows that $g(t) < 0$ for all $t > 0$. By virtue of (2.2), we deduce that $f_p'(t) < 0$ for all $t > 0$, hence f_p is decreasing on $(0,\infty)$. Since $\lim_{t\searrow 0} f_p(t) = 0$, we conclude that $f_p(t) < 0$ for all $t > 0$. This proves the validity of (2.1).

Now let $x \neq y$ be two arbitrary positive real numbers. Letting $t = \log\sqrt{\frac{x}{y}}$ in (2.1) and multiplying the obtained inequality by $\left(\sqrt{xy}\right)^p$, we obtain

$$\left(\frac{2}{e}\right)^p A^p(x,y) + \left(1 - \left(\frac{2}{e}\right)^p\right)G^p(x,y) < I^p(x,y).$$

Consequently, the first inequality in (1.5) holds true for $\alpha = \left(\frac{2}{e}\right)^p$.

Let us prove now that the second inequality in (1.5) holds true for $\beta = \frac{2}{3}$. Indeed, taking into account (1.3) as well as the convexity of the function $t \in (0,\infty) \mapsto t^{\frac{p}{2}} \in (0,\infty)$ (recall that

$p \geq 2$), we get

$$\begin{aligned} I^p(x,y) &= \left[I^2(x,y)\right]^{\frac{p}{2}} \\ &< \left[\frac{2}{3}A^2(x,y) + \frac{1}{3}G^2(x,y)\right]^{\frac{p}{2}} \\ &\leq \frac{2}{3}\left[A^2(x,y)\right]^{\frac{p}{2}} + \frac{1}{3}\left[G^2(x,y)\right]^{\frac{p}{2}} \\ &= \frac{2}{3}\,A^p(x,y) + \frac{1}{3}\,G^p(x,y). \end{aligned}$$

Conversely, suppose that (1.5) holds true for all positive real numbers $x \neq y$. Then we have

$$\alpha < \frac{I^p(x,y) - G^p(x,y)}{A^p(x,y) - G^p(x,y)} < \beta.$$

The limits

$$\lim_{x\to 0} \frac{I^p(x,1) - G^p(x,1)}{A^p(x,1) - G^p(x,1)} = \left(\frac{2}{e}\right)^p \qquad \text{and} \qquad \lim_{x\to 1} \frac{I^p(x,1) - G^p(x,1)}{A^p(x,1) - G^p(x,1)} = \frac{2}{3}$$

yield $\alpha \leq \left(\frac{2}{e}\right)^p$ and $\beta \geq \frac{2}{3}$. □

REFERENCES

[1] H. ALZER AND S.-L. QIU, Inequalities for means in two variables, *Arch. Math. (Basel)*, **80** (2003), 201–215.

[2] P.J. DAVIS, *Interpolation and Approximation*, Blaisdell, New York, 1963.

[3] P.J. DAVIS AND P. RABINOWITZ, *Numerical Integration*, Blaisdell, Massachusetts–Toronto–London, 1967.

[4] E.B. LEACH AND M.C. SHOLANDER, Extended mean values II, *J. Math. Anal. Appl.*, **92** (1983), 207–223.

[5] G. LORENZEN, Why means in two arguments are special, *Elem. Math.*, **49** (1994), 32–37.

[6] J. SÁNDOR, A note on some inequalities for means, *Arch. Math. (Basel)* **56** (1991), 471–473.

[7] J. SÁNDOR, On certain identities for means, *Studia Univ. Babeş-Bolyai, Ser. Math.*, **38**(4) (1993), 7–14.

[8] J. SÁNDOR AND T. TRIF, Some new inequalities for means of two arguments, *Int. J. Math. Math. Sci.*, **25** (2001), 525–532.