

## A NOTE ON A WEIGHTED OSTROWSKI TYPE INEQUALITY FOR CUMULATIVE DISTRIBUTION FUNCTIONS

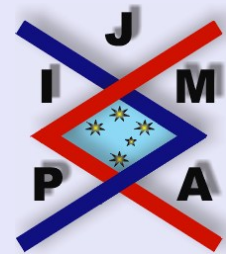
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Abstract

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## Abstract

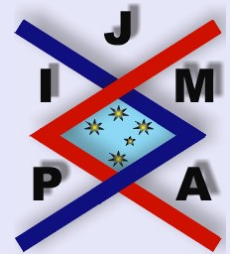
In this note, a weighted Ostrowski type inequality for the cumulative distribution function and expectation of a random variable is established.

*2000 Mathematics Subject Classification:* Primary 65D10, Secondary 65D15.

*Key words:* Weight function, Ostrowski type inequality, Cumulative distribution functions.

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# 1. Introduction

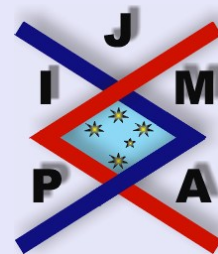
In [1], N. S. Barnett and S. S. Dragomir established the following Ostrowski type inequality for cumulative distribution functions.

**Theorem 1.1.** *Let  $X$  be a random variable taking values in the finite interval  $[a, b]$ , with cumulative distribution function  $F(x) = \Pr(X \leq x)$ , then,*

$$\begin{aligned} & \left| \Pr(X \leq x) - \frac{b - E(X)}{b - a} \right| \\ & \leq \frac{1}{b - a} \left[ [2x - (a + b)] \Pr(X \leq x) + \int_a^b \operatorname{sgn}(t - x) F(t) dt \right] \\ & \leq \frac{1}{b - a} [(b - x) \Pr(X \geq x) + (x - a) \Pr(X \leq x)] \\ (1.1) \quad & \leq \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{(b - a)} \end{aligned}$$

for all  $x \in [a, b]$ . All the inequalities in (1.1) are sharp and the constant  $\frac{1}{2}$  the best possible.

In this paper, we establish a weighted version of this result using similar methods to those used in [1]. The results of [1] are then retrieved by taking the weight function to be 1.



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## 2. Main Results

We assume that the weight function  $w : (a, b) \rightarrow [0, \infty)$ , is integrable, non-negative and

$$\int_a^b w(t)dt < \infty.$$

The domain of  $w$  may be finite or infinite and  $w$  may vanish at the boundary points. We denote

$$m(a, b) = \int_a^b w(t)dt.$$

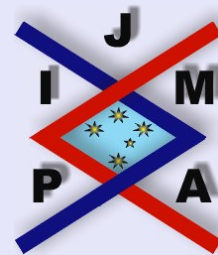
We also know that the expectation of any function  $\varphi(X)$  of the random variable  $X$  is given by:

$$(2.1) \quad E[\varphi(X)] = \int_a^b \varphi(t)dF(t).$$

Taking  $\varphi(X) = \int w(X)dX$ , then from (2.1) and integrating by parts, we get,

$$(2.2) \quad \begin{aligned} E_W &= E \left[ \int w(X)dX \right] \\ &= \int_a^b \left( \int w(t)dt \right) dF(t) \\ &= W(b) - \int_a^b w(t)F(t)dt, \end{aligned}$$

where  $W(b) = \left[ \int w(t)dt \right]_{t=b}$ .



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**Theorem 2.1.** Let  $X$  be a random variable taking values in the finite interval  $[a, b]$ , with cumulative distribution function  $F(x) = \Pr(X \leq x)$ , then,

$$\begin{aligned}
 & \left| \Pr(X \leq x) - \frac{W(b) - E_W}{m(a, b)} \right| \\
 & \leq \frac{1}{m(a, b)} \left[ [m(a, x) - m(x, b)] \Pr(X \leq x) \right. \\
 & \quad \left. + \int_a^b \operatorname{sgn}(t - x) w(t) F(t) dt \right] \\
 & \leq \frac{1}{m(a, b)} [m(a, x) \Pr(X \leq x) + m(x, b) \Pr(X \geq x)] \\
 (2.3) \quad & \leq \frac{1}{2} + \frac{\left| \frac{m(x, b) - m(a, x)}{2} \right|}{m(a, b)}
 \end{aligned}$$

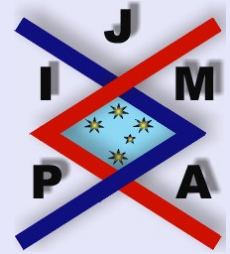
for all  $x \in [a, b]$ . All the inequalities in (2.3) are sharp and the constant  $\frac{1}{2}$  is the best possible.

*Proof.* Consider the Kernel  $p : [a, b]^2 \rightarrow \mathbb{R}$  defined by

$$(2.4) \quad p(x, t) = \begin{cases} \int_a^t w(u) du & \text{if } t \in [a, x] \\ \int_b^t w(u) du & \text{if } t \in (x, b], \end{cases}$$

then the Riemann-Stieltjes integral  $\int_a^b p(x, t) dF(t)$  exists for any  $x \in [a, b]$  and the following identity holds:

$$(2.5) \quad \int_a^b p(x, t) dF(t) = m(a, b)F(x) - \int_a^b w(t)F(t)dt.$$



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Using (2.2) and (2.5), we get (see [2, p. 452]),

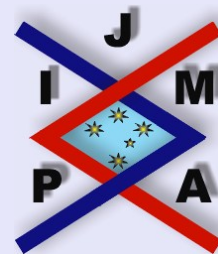
$$(2.6) \quad m(a, b)F(x) + E_W - W(b) = \int_a^b p(x, t)dF(t).$$

As shown in [1], if  $p : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and  $\nu : [a, b] \rightarrow \mathbb{R}$  is monotonic non-decreasing, then the Riemann -Stieltjes integral  $\int_a^b p(x) d\nu(x)$  exists and

$$(2.7) \quad \left| \int_a^b p(x) d\nu(x) \right| \leq \int_a^b |p(x)| d\nu(x).$$

Using (2.7) we have

$$(2.8) \quad \begin{aligned} & \left| \int_a^b p(x, t)dF(t) \right| \\ &= \left| \int_a^x \left( \int_a^t w(u)du \right) dF(t) + \int_x^b \left( \int_b^t w(u)du \right) dF(t) \right| \\ &\leq \int_a^x \left| \int_a^t w(u)du \right| dF(t) + \int_x^b \left| \int_b^t w(u)du \right| dF(t) \\ &= \left( \int_a^t w(u)du \right) F(t) \Big|_a^x - \int_a^x F(t) \frac{d}{dt} \left( \int_a^t w(u)du \right) dt \\ &\quad + \left( \int_t^b w(u)du \right) F(t) \Big|_x^b - \int_x^b F(t) \frac{d}{dt} \left( \int_t^b w(u)du \right) dt \\ &= [m(a, x) - m(x, b)] F(x) + \int_a^b \text{sgn}(t - x)w(t)F(t)dt. \end{aligned}$$



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Using the identity (2.6) and the inequality (2.8), we deduce the first part of (2.3).

We know that

$$\int_a^b \operatorname{sgn}(t-x)w(t)F(t)dt = -\int_a^x w(t)F(t)dt + \int_x^b w(t)F(t)dt.$$

As  $F(\cdot)$  is monotonic non-decreasing on  $[a, b]$ ,

$$\int_a^x w(t)F(t)dt \geq m(a, x)F(a) = 0,$$

$$\int_x^b w(t)F(t)dt \leq m(x, b)F(b) = m(x, b)$$

and

$$\int_a^b \operatorname{sgn}(t-x)w(t)F(t)dt \leq m(x, b) \text{ for all } x \in [a, b].$$

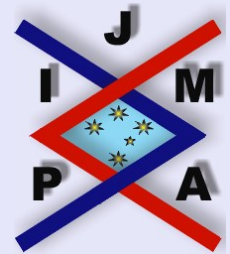
Consequently,

$$\begin{aligned} [m(a, x) - m(x, b)] \Pr(X \leq x) + \int_a^b \operatorname{sgn}(t-x)w(t)F(t)dt \\ \leq [m(a, x) - m(x, b)] \Pr(X \leq x) + m(x, b) \\ = m(a, x) \Pr(X \leq x) + m(x, b) \Pr(X \geq x) \end{aligned}$$

and the second part of (2.3) is proved.

Finally,

$$m(a, x) \Pr(X \leq x) + m(x, b) \Pr(X \geq x)$$




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$$\begin{aligned} &\leq \max \{m(a, x), m(x, b)\} [\Pr(X \leq x) + \Pr(X \geq x)] \\ &= \frac{m(a, b) + |m(x, b) - m(a, x)|}{2} \end{aligned}$$

and the last part of (2.3) follows. □

**Remark 1.** Since

$$\Pr(X \geq x) = 1 - \Pr(X \leq x),$$

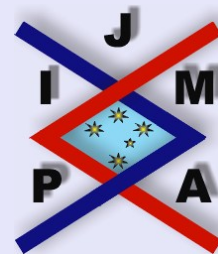
we can obtain an equivalent to (2.3) for

$$\left| \Pr(X \geq x) - \frac{E_W + m(a, b) - W(b)}{m(a, b)} \right|.$$

Following the same style of argument as in Remark 2.3 and Corollary 2.4 of [1], we have the following two corollaries.

**Corollary 2.2.**

$$\begin{aligned} &\left| \Pr \left( X \leq \frac{a+b}{2} \right) - \frac{W(b) - E_W}{m(a, b)} \right| \\ &\leq \frac{1}{m(a, b)} \left[ \left[ m \left( a, \frac{a+b}{2} \right) - m \left( \frac{a+b}{2}, b \right) \right] \Pr(X \leq x) \right. \\ &\quad \left. + \int_a^b \operatorname{sgn} \left( t - \frac{a+b}{2} \right) w(t) F(t) dt \right] \\ &\leq \frac{1}{2} + \frac{\left| \frac{m(\frac{a+b}{2}, b) - m(a, \frac{a+b}{2})}{2} \right|}{m(a, b)} \end{aligned}$$



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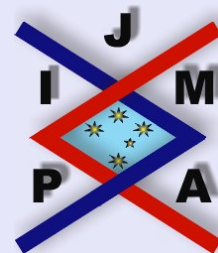
and

$$\begin{aligned} & \left| \Pr \left( X \geq \frac{a+b}{2} \right) - \frac{E_W + m(a, b) - W(b)}{m(a, b)} \right| \\ & \leq \frac{1}{m(a, b)} \left[ \left[ m \left( a, \frac{a+b}{2} \right) - m \left( \frac{a+b}{2}, b \right) \right] \Pr \left( X \leq \frac{a+b}{2} \right) \right. \\ & \quad \left. + \int_a^b \operatorname{sgn} \left( t - \frac{a+b}{2} \right) w(t) F(t) dt \right] \\ & \leq \frac{1}{2} + \frac{\left| \frac{m(\frac{a+b}{2}, b) - m(a, \frac{a+b}{2})}{2} \right|}{m(a, b)}. \end{aligned}$$

**Corollary 2.3.**

$$\begin{aligned} & \frac{1}{m(a, b)} \left[ \frac{2W(b) - m(a, b) - \left| m \left( \frac{a+b}{2}, b \right) - m \left( a, \frac{a+b}{2} \right) \right|}{2} - E_W \right] \\ & \leq \Pr \left( X \leq \frac{a+b}{2} \right) \\ & \leq 1 + \frac{1}{m(a, b)} \left[ \frac{2W(b) - m(a, b) + \left| m \left( \frac{a+b}{2}, b \right) - m \left( a, \frac{a+b}{2} \right) \right|}{2} - E_W \right]. \end{aligned}$$

Additional inequalities for  $\Pr [X \leq x]$  and  $\Pr (X \leq \frac{a+b}{2})$  are obtainable in the style of Corollary 2.6 and Remarks 2.5 and 2.7 of [1] using (2.3).



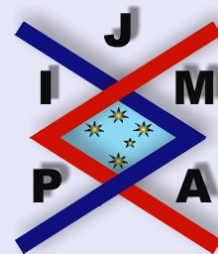
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## References

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