

GEOMETRIC CONVEXITY OF A FUNCTION INVOLVING GAMMA FUNCTION AND APPLICATIONS TO INEQUALITY THEORY

XIAO-MING ZHANG

Haining TV University,
Haining City, Zhejiang Province,
314400, China
EMail: zjzxm79@sohu.com

TIE-QUAN XU

Qingdao Vocational and Technical College,
Qingdao City, 266071, China

LING-BO SITU

Cangjiang Middle School,
Kaiping City, Guangdong Province,
529300, China

Received: 25 July, 2006

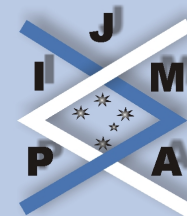
Accepted: 24 February, 2007

Communicated by: F. Qi

2000 AMS Sub. Class.: Primary 33B15, 65R10; Secondary 26A48, 26A51, 26D20.

Key words: Gamma function, Geometrically Convex function, Wallis' inequality, Application, Inequality.

Abstract: In this paper, the geometric convexity of a function involving gamma function is studied, as applications to inequality theory, some important inequalities which improve some known inequalities, including Wallis' inequality, are obtained.



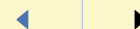
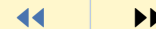
**Geometric Convexity of
a Function Involving
Gamma Function**

Xiao-Ming Zhang, Tie-Quan Xu and
Ling-Bo Situ

vol. 8, iss. 1, art. 17, 2007

[Title Page](#)

[Contents](#)



Page 1 of 19

[Go Back](#)

[Full Screen](#)

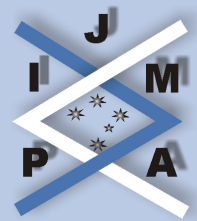
[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

Contents

1	Introduction and main results	3
2	Lemmas	6
3	Proofs of Theorems and Corollaries	8



Geometric Convexity of a Function Involving Gamma Function

Xiao-Ming Zhang, Tie-Quan Xu and

Ling-Bo Situ

vol. 8, iss. 1, art. 17, 2007

Title Page

Contents



Page 2 of 19

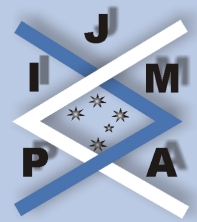
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



1. Introduction and main results

The geometrically convex functions are as defined below.

Definition 1.1 ([10, 11, 12]). Let $f : I \subseteq (0, \infty) \rightarrow (0, \infty)$ be a continuous function. Then f is called a geometrically convex function on I if there exists an integer $n \geq 2$ such that one of the following two inequalities holds:

$$(1.1) \quad f(\sqrt{x_1 x_2}) \leq \sqrt{f(x_1) f(x_2)},$$

$$(1.2) \quad f\left(\prod_{i=1}^n x_i^{\lambda_i}\right) \leq \prod_{i=1}^n [f(x_i)]^{\lambda_i},$$

where $x_1, x_2, \dots, x_n \in I$ and $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$. If inequalities (1.1) and (1.2) are reversed, then f is called a geometrically concave function on I .

For more literature on geometrically convex functions and their properties, see [12, 29, 30, 31, 32] and the references therein.

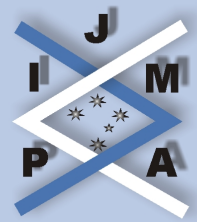
It is well known that Euler's gamma function $\Gamma(x)$ and the psi function $\psi(x)$ are defined for $x > 0$ respectively by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ and $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. For $x > 0$, let

$$(1.3) \quad f(x) = \frac{e^x \Gamma(x)}{x^x}.$$

This function has been studied extensively by many mathematicians, for example, see [6] and the references therein.

In this article, we would like to discuss the geometric convexity of the function f defined by (1.3) and apply this property to obtain, from a new viewpoint, some new inequalities related to the gamma function.

Our main results are as follows.



Theorem 1.1. The function f defined by (1.3) is geometrically convex.

Theorem 1.2. For $x > 0$ and $y > 0$, the double inequality

$$(1.4) \quad \frac{x^x}{y^y} \left(\frac{x}{y}\right)^{y[\psi(y)-\ln y]} e^{y-x} \leq \frac{\Gamma(x)}{\Gamma(y)} \leq \frac{x^x}{y^y} \left(\frac{x}{y}\right)^{x[\psi(x)-\ln x]} e^{y-x}$$

holds.

As consequences of above theorems, the following corollaries can be deduced.

Corollary 1.3. The function f is logarithmically convex.

Remark 1. More generally, the function f is logarithmically completely monotonic in $(0, \infty)$. See [6].

Corollary 1.4 ([7, 13]). For $0 < y < x$ and $0 < s < 1$, inequalities

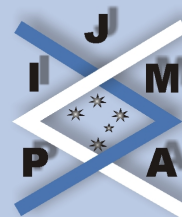
$$(1.5) \quad e^{(x-y)\psi(y)} < \frac{\Gamma(x)}{\Gamma(y)} < e^{(x-y)\psi(x)}$$

and

$$(1.6) \quad \frac{x^{x-1}}{y^{y-1}} e^{y-x} < \frac{\Gamma(x)}{\Gamma(y)} < \frac{x^{x-\frac{1}{2}}}{y^{y-\frac{1}{2}}} e^{y-x}$$

are valid.

Remark 2. Note that inequality (1.4) is better than (1.5) and (1.6). The lower and upper bounds for $\frac{\Gamma(x)}{\Gamma(y)}$ have been established in many papers such as [14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26].



Title Page

Contents

Navigation arrows

Navigation arrows

Page 5 of 19

Go Back

Full Screen

Close

Corollary 1.5. For x > 0 and n in N, the following double inequalities hold:

(1.7) sqrt(ex) (1 + 1/2x)^(-x) < Gamma(x+1)/Gamma(x+1/2) < sqrt(ex) (1 + 1/2x)^(1/12x - x)

and

(1.8) sqrt(e(x+n)) (1 + 1/(2x+2n))^(-x-n) product_{k=1}^n (1 - 1/(2x+2k)) < Gamma(x+1)/Gamma(x+1/2) < sqrt(e(x+n)) (1 + 1/(2x+2n))^(1/(12x+12n) - x - n) product_{k=1}^n (1 - 1/(2x+2k))

Corollary 1.6. For n in N, the double inequality

(1.9) 1/sqrt(e*pi*n) (1 + 1/2n)^(n - 1/12n) < (2n-1)!!/(2n)!! < 1/sqrt(e*pi*n) (1 + 1/2n)^(n - 1/(12n+16))

is valid.

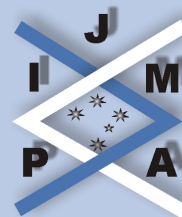
Remark 3. Inequality (1.9) is related to the well known Wallis inequality. If n >= 2, inequality (1.9) is better than

(1.10) 1/sqrt(pi(n + 4/pi - 1)) <= (2n-1)!!/(2n)!! <= 1/sqrt(pi(n + 1/4))

in [3]. For more details, please refer to [2, 8, 33, 34, 35] and the references therein.

Corollary 1.7 ([28]). Let S_n = sum_{k=1}^n 1/k for n in N. Then

(1.11) (2^(n+1)n!)/(2n+1)!! (2n+3)/(2n+2)^(3/2+n) e^(S_n - 1 - gamma)/2 < sqrt(pi)



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 6 of 19

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

2. Lemmas

In order to prove our main results, the following lemmas are necessary.

Lemma 2.1 ([1, 5, 22]). For $x > 0$,

$$(2.1) \quad \ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x},$$
$$\psi(x) > \ln x - \frac{1}{2x} - \frac{1}{12x^2}, \quad \psi'(x) > \frac{1}{x} + \frac{1}{2x^2}.$$

Lemma 2.2. For $x > 0$,

$$(2.2) \quad 2\psi'(x) + x\psi''(x) < \frac{1}{x}.$$

Remark 4. The complete monotonicity of the function $2\psi'(x) + x\psi''(x)$ was obtained in [27].

Proof. It is a well known fact that

$$(2.3) \quad \psi'(x) = \sum_{k=1}^{\infty} \frac{1}{(k-1+x)^2} \quad \text{and} \quad \psi''(x) = -\sum_{k=1}^{\infty} \frac{2}{(k-1+x)^3}.$$

From this, it follows that

$$2\psi'(x) + x\psi''(x) - \frac{1}{x} = 2 \sum_{k=1}^{\infty} \frac{k}{(k+x)^3} - \frac{1}{x}$$
$$< 2 \sum_{k=1}^{\infty} \frac{k}{(k-1+x)(k+x)(k+1+x)} - \frac{1}{x}$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \left[\frac{k}{(k-1+x)(k+x)} - \frac{k}{(k+x)(k+1+x)} \right] - \frac{1}{x} \\
&= \sum_{k=1}^{\infty} \frac{1}{(k-1+x)(k+x)} - \frac{1}{x} \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k-1+x} - \frac{1}{k+x} \right) - \frac{1}{x} = 0.
\end{aligned}$$

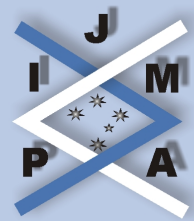
Thus the proof of Lemma 2.2 is completed. ■

Lemma 2.3 ([12]). *Let $(a, b) \subset (0, \infty)$ and $f : (a, b) \rightarrow (0, \infty)$ be a differentiable function. Then f is a geometrically convex function if and only if the function $\frac{xf'(x)}{f(x)}$ is nondecreasing.*

Lemma 2.4 ([12]). *Let $(a, b) \subset (0, \infty)$ and $f : (a, b) \rightarrow (0, \infty)$ be a differentiable function. Then f is a geometrically convex function if and only if $\frac{f(x)}{f(y)} \geq \left(\frac{x}{y}\right)^{yf'(y)/f(y)}$ holds for any $x, y \in (a, b)$.*

Lemma 2.5 ([4, 9]). *Let $S_n = \sum_{k=1}^n \frac{1}{k}$ and $C_n = S_n - \ln\left(n + \frac{1}{2}\right) - \gamma$ for $n \in \mathbb{N}$, where $\gamma = 0.5772156 \dots$ is Euler-Mascheroni's constant. Then*

$$(2.4) \quad \frac{1}{24(n+1)^2} < C_n < \frac{1}{24n^2}.$$



Title Page

Contents

◀◀ ▶▶

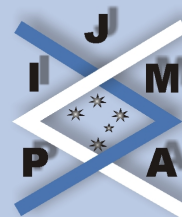
◀ ▶

Page 7 of 19

Go Back

Full Screen

Close



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 8 of 19

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

3. Proofs of Theorems and Corollaries

Now we are in a position to prove our main results.

Proof of Theorem 1.1. Easy calculation yields

$$(3.1) \quad \ln f(x) = \ln \Gamma(x) - x \ln x + x \quad \text{and} \quad \frac{f'(x)}{f(x)} = \psi(x) - \ln x.$$

Let $F(x) = \left[\frac{xf'(x)}{f(x)} \right]'$. Then

$$F(x) = \psi(x) + x\psi'(x) - \ln x - 1, \quad \text{and} \quad F'(x) = 2\psi'(x) + x\psi''(x) - \frac{1}{x}.$$

By virtue of Lemma 2.2, it follows that $F'(x) < 0$, thus F is decreasing in $x > 0$. By Lemma 2.1, we deduce that

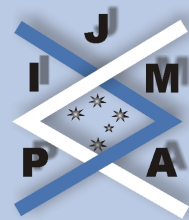
$$F(x) = \psi(x) + x\psi'(x) - \ln x - 1 > \ln x - \frac{1}{x} + x \left(\frac{1}{x} + \frac{1}{2x^2} \right) - \ln x - 1 = -\frac{1}{2x}.$$

Hence $\lim_{x \rightarrow \infty} F(x) \geq 0$. This implies that $F(x) > 0$ and, by Lemma 2.3, the function f is geometrically convex. The proof is completed. ■

Proof of Theorem 1.2. Combining Theorem 1.1, Lemma 2.4 and (3.1) leads to

$$\frac{e^x \Gamma(x)}{x^x} \geq \left(\frac{x}{y} \right)^{y[\psi(y) - \ln y]} \frac{e^y \Gamma(y)}{y^y} \quad \text{and} \quad \frac{e^y \Gamma(y)}{y^y} \geq \left(\frac{y}{x} \right)^{x[\psi(x) - \ln x]} \frac{e^x \Gamma(x)}{x^x}.$$

Inequality (1.4) is established. ■



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 9 of 19

Go Back

Full Screen

Close

Proof of Corollary 1.3. A combination of (3.1) with Lemma 2.1 reveals the decreasing monotonicity of f in $(0, \infty)$. Considering the geometric convexity and the decreasing monotonicity of f and the arithmetic-geometric mean inequality, we have

$$f\left(\frac{x_1 + x_2}{2}\right) \leq f(\sqrt{x_1 x_2}) \leq \sqrt{f(x_1)f(x_2)} \leq \frac{f(x_1) + f(x_2)}{2}.$$

Hence, f is convex and logarithmic convex in $(0, \infty)$. ■

Proof of Corollary 1.4. A property of mean values [9] and direct argument gives

$$(3.2) \quad \frac{1}{x} < \frac{\ln x - \ln y}{x - y} < \frac{1}{y}, \quad \ln x - \ln y > 1 - \frac{y}{x},$$
$$-1 + \ln x + \frac{y}{x} > \psi(y) + y[\ln y - \psi(y)]\frac{1}{y}.$$

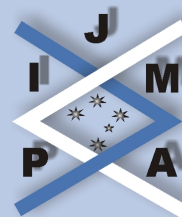
Hence,

$$(3.3) \quad -1 + \ln x + y\frac{\ln x - \ln y}{x - y} > \psi(y) + y[\ln y - \psi(y)]\frac{\ln x - \ln y}{x - y},$$

$$(y - x) + (x - y)\ln x + y(\ln x - \ln y) > (x - y)\psi(y) + y[\ln y - \psi(y)](\ln x - \ln y),$$

$$(y - x) + x \ln x - y \ln y + y[\psi(y) - \ln y](\ln x - \ln y) > (x - y)\psi(y),$$

$$\left(\frac{x}{y}\right)^{y[\psi(y) - \ln y]} \frac{e^y x^x}{e^x y^y} > e^{(x-y)\psi(y)}.$$



Title Page

Contents



Page 10 of 19

Go Back

Full Screen

Close

Similarly,

$$(3.4) \quad -1 + \ln x + y \frac{1}{y} = x[\ln x - \psi(x)] \frac{1}{x} + \psi(x),$$

$$-1 + \ln x + y \frac{\ln x - \ln y}{x - y} < x[\ln x - \psi(x)] \frac{\ln x - \ln y}{x - y} + \psi(x),$$

$$(y - x) + (x - y) \ln x + y(\ln x - \ln y) < x[\ln x - \psi(x)](\ln x - \ln y) + (x - y)\psi(x),$$

$$(y - x) + x \ln x - y \ln y + x[\psi(x) - \ln x](\ln x - \ln y) < (x - y)\psi(x),$$

$$\left(\frac{x}{y}\right)^{x[\psi(x) - \ln x]} \frac{e^y x^x}{e^x y^y} < e^{(x-y)\psi(x)}.$$

Combination of (3.3) and (3.4) leads to (1.5).

By (2.1), it is easy to see that

$$1 < \left(\frac{x}{y}\right)^{y[\ln y - \psi(y)]} \frac{x}{y}, \quad \frac{x^{x-1}}{y^{y-1}} e^{y-x} < \left(\frac{x}{y}\right)^{y[\ln y - \psi(y)]} \frac{e^y x^x}{e^x y^y}.$$

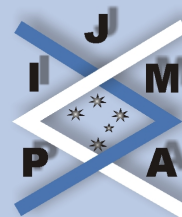
Similarly,

$$\frac{e^y x^x}{e^x y^y} \left(\frac{x}{y}\right)^{x[\ln x - \psi(x)]} < \frac{x^{x-\frac{1}{2}}}{y^{y-\frac{1}{2}}} e^{y-x}.$$

By virtue of (1.4), inequality (1.6) follows. ■

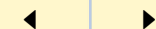
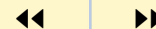
Proof of Corollary 1.5. Let $y = x + \frac{1}{2}$ in inequality (1.4). Then

$$(3.5) \quad \frac{e^{\frac{1}{2}} x^x}{\left(x + \frac{1}{2}\right)^{x+\frac{1}{2}}} \left(\frac{x}{x + \frac{1}{2}}\right)^{\left(x+\frac{1}{2}\right)[\psi\left(x+\frac{1}{2}\right) - \ln\left(x+\frac{1}{2}\right)]}$$



Title Page

Contents



Page 11 of 19

Go Back

Full Screen

Close

$$\begin{aligned} &\leq \frac{\Gamma(x)}{\Gamma(x + \frac{1}{2})} \\ &\leq \frac{e^{\frac{1}{2}x^x}}{(x + \frac{1}{2})^{x + \frac{1}{2}}} \left(\frac{x}{x + \frac{1}{2}}\right)^{x[\psi(x) - \ln x]}, \end{aligned}$$

$$\begin{aligned} \frac{e^{\frac{1}{2}x^{x+1}}}{(x + \frac{1}{2})^{x + \frac{1}{2}}} \left(\frac{x + \frac{1}{2}}{x}\right)^{(x + \frac{1}{2})[\ln(x + \frac{1}{2}) - \psi(x + \frac{1}{2})]} &\leq \frac{x\Gamma(x)}{\Gamma(x + \frac{1}{2})} \\ &\leq \frac{e^{\frac{1}{2}x^{x+1}}}{(x + \frac{1}{2})^{x + \frac{1}{2}}} \left(\frac{x + \frac{1}{2}}{x}\right)^{x[\ln x - \psi(x)]}. \end{aligned}$$

From inequality (2.2), we obtain

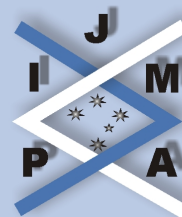
$$\begin{aligned} \frac{\sqrt{ex} x^{x + \frac{1}{2}}}{(x + \frac{1}{2})^{x + \frac{1}{2}}} \left(1 + \frac{1}{2x}\right)^{\frac{1}{2}} &< \frac{\Gamma(x + 1)}{\Gamma(x + \frac{1}{2})} < \frac{\sqrt{ex} x^{x + \frac{1}{2}}}{(x + \frac{1}{2})^{x + \frac{1}{2}}} \left(1 + \frac{1}{2x}\right)^{\frac{1}{2} + \frac{1}{12x}}, \\ \sqrt{ex} \left(1 + \frac{1}{2x}\right)^{-x} &< \frac{\Gamma(x + 1)}{\Gamma(x + \frac{1}{2})} < \sqrt{ex} \left(1 + \frac{1}{2x}\right)^{\frac{1}{12x} - x}. \end{aligned}$$

The proof of inequality (1.7) is completed.

Substituting

$$\frac{\Gamma(x + n + 1)}{\Gamma(x + n + \frac{1}{2})} = \frac{(x + n)\Gamma(x + n)}{(x + n - \frac{1}{2})\Gamma(x + n - \frac{1}{2})} = \cdots = \frac{\Gamma(x + 1)\prod_{k=1}^n (x + k)}{\Gamma(x + \frac{1}{2})\prod_{k=1}^n (x + k - \frac{1}{2})}$$

into (1.7) shows that inequality (1.8) is valid. ■



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 12 of 19

Go Back

Full Screen

Close

Proof of Corollary 1.6. For $n = 1, 2$, inequality (1.9) can be verified readily.

For $n \geq 3$, in view of formulas $\Gamma(n+1) = n!$, $\Gamma(n + \frac{1}{2}) = \frac{(2n-1)!!}{2^n} \sqrt{\pi}$ and inequality (1.7), we have

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} < \sqrt{en} \left(1 + \frac{1}{2n}\right)^{\frac{1}{12n}-n}, \quad \frac{2^n n!}{(2n-1)!!} < \sqrt{e\pi n} \left(1 + \frac{1}{2n}\right)^{\frac{1}{12n}-n},$$

and

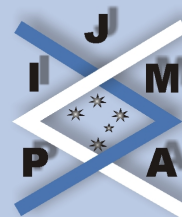
$$(3.6) \quad \frac{(2n-1)!!}{(2n)!!} > \frac{1}{\sqrt{e\pi n}} \left(1 + \frac{1}{2n}\right)^{n-\frac{1}{12n}}.$$

Further, taking $x = n$ in inequality (3.5) reveals

$$\begin{aligned} \frac{e^{\frac{1}{2}n^{n+1}}}{(n+\frac{1}{2})^{n+\frac{1}{2}}} \left(\frac{n+\frac{1}{2}}{n}\right)^{(n+\frac{1}{2})(\ln(n+\frac{1}{2})-\psi(n+\frac{1}{2}))} &\leq \frac{n\Gamma(n)}{\Gamma(n+\frac{1}{2})}, \\ \frac{2^n n!}{(2n-1)!!} &\geq \sqrt{e\pi n} \left(1 + \frac{1}{2n}\right)^{(n+\frac{1}{2})[\ln(n+\frac{1}{2})-\psi(n+\frac{1}{2})-1]}, \\ \frac{2^n n!}{(2n-1)!!} &\geq \sqrt{e\pi n} \left(1 + \frac{1}{2n}\right)^{(n+\frac{1}{2})[\ln(n+\frac{1}{2})-\psi(n+\frac{1}{2})-1]}. \end{aligned}$$

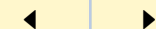
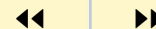
Employing formulas

$$(3.7) \quad \psi(x+1) = \psi(x) + \frac{1}{x}, \quad \psi\left(\frac{1}{2}\right) = -\gamma - 2 \ln 2, \quad C_n = S_n - \ln\left(n + \frac{1}{2}\right) - \gamma$$



Title Page

Contents



Page 13 of 19

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

yields

$$\begin{aligned}
 \frac{2^n n!}{(2n-1)!!} &\geq \sqrt{e\pi n} \left(1 + \frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)} \left[\ln\left(n+\frac{1}{2}\right) - \psi\left(n-\frac{1}{2}\right) - \frac{1}{n-\frac{1}{2}} - 1\right] \\
 &= \sqrt{e\pi n} \left(1 + \frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)} \left[\ln\left(n+\frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) - \frac{1}{n-\frac{1}{2}} - \dots - \frac{1}{\frac{1}{2}} - 1\right] \\
 (3.8) \quad &= \sqrt{e\pi n} \left(1 + \frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)} \left[\ln\left(n+\frac{1}{2}\right) + 2\ln 2 + \gamma - 2\sum_{k=1}^n \frac{1}{2k-1} - 1\right] \\
 &= \sqrt{e\pi n} \left(1 + \frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)} \left[\ln\left(n+\frac{1}{2}\right) + 2\ln 2 + \gamma - 2\sum_{k=1}^{2n} \frac{1}{k} + \sum_{k=1}^n \frac{1}{k} - 1\right] \\
 &= \sqrt{e\pi n} \left(1 + \frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)} \left[2\ln(2n+1) - 2C_{2n} - 2\ln\left(2n+\frac{1}{2}\right) + C_n - 1\right]
 \end{aligned}$$

Letting $x = \frac{1}{1+4n}$ in $\ln(1+x) > \frac{x}{1+\frac{x}{2}}$ for $x > 0$, we obtain

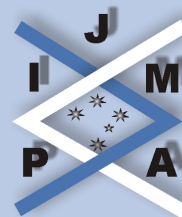
$$(3.9) \quad \ln\left(1 + \frac{1}{1+4n}\right) > \frac{2}{8n+3}.$$

In view of Lemma 2.5 and inequalities (3.8) and (3.9), we have

$$(3.10) \quad \frac{2^n n!}{(2n-1)!!} > \sqrt{e\pi n} \left(1 + \frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)} \left[\frac{4}{8n+3} - \frac{1}{48n^2} + \frac{1}{24(n+1)^2} - 1\right].$$

It is easy to verify that

$$(3.11) \quad \left(n + \frac{1}{2}\right) \left[\frac{4}{8n+3} - \frac{1}{48n^2} + \frac{1}{24(n+1)^2} - 1\right] > -n + \frac{1}{12n+16}$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 14 of 19

Go Back

Full Screen

Close

with $n \geq 3$. By virtue of (3.6), (3.10) and (3.11), Corollary 1.6 is proved. ■

Proof of Corollary 1.7. Letting $x = n + \frac{3}{2}$ and $y = n + 1$ in inequality (1.4) yields

(3.12) 1 / (sqrt(e*pi*(n+1))) * (1 + 1/(2n+2))^{(n+1)[psi(n+1)-ln(n+1)+1]+1/2} <= (2n+1)!! / (2n+2)!!

By using inequality (2.1), psi(n+1) = sum_{k=1}^n 1/k - gamma and 1/sqrt(e) * ((2n+3)/(2n+2))^{n+1} < 1 for n in N, we have

(2n+1)!! / (2n)!! * ((2n+2)/(2n+3))^{3/2+n} * e^{-1/2(Sn-1-gamma)} = (2n+2) * ((2n+1)!! / (2n+2)!!) * ((2n+2)/(2n+3))^{3/2+n} * e^{-1/2[psi(n+1)-1]} > 2*sqrt(n+1) / sqrt(pi) * ((2n+3)/(2n+2))^{-(n+1)ln(n+1)} * [1/sqrt(e) * ((2n+3)/(2n+2))^{n+1}]^{ln(n+1)-1/(2(n+1))} = 2*sqrt(n+1) / sqrt(pi) * sqrt((2n+2)/(2n+3)) * e^{-1/2 ln(n+1) + 1/(4(n+1))} = 2/sqrt(pi) * sqrt((2n+2)/(2n+3)) * e^{1/(4(n+1))} > 2/sqrt(pi)

The proof of Corollary 1.7 is completed. ■

References

- [1] G.D. ANDERSON AND S.L. QIU, A monotoneity property of the gamma function, *Proc. Amer. Math. Soc.*, **125**(11) (1997), 3355–3362.
- [2] J. CAO, D.-W. NIU AND F. QI, A Wallis type inequality and a double inequality for probability integral, *Austral. J. Math. Anal. Appl.*, **4**(1) (2007), Art. 3. [ONLINE: <http://ajmaa.org/cgi-bin/paper.pl?string=v4n1/V4I1P3.tex>].
- [3] C.P. CHEN AND F. QI, The best bounds in Wallis' inequality, *Proc. Amer. Math. Soc.*, **133**(2) (2005), 397–401.
- [4] D.W. DE TEMPLE, A quicker convergence to Euler's constant, *Amer. Math. Monthly*, **100**(5) (1993), 468–470.
- [5] Á. ELBERT AND A. LAFORGIA, On some properties of the gamma function, *Proc. Amer. Math. Soc.*, **128**(9) (2000), 2667–2673.
- [6] S. GUO, Monotonicity and concavity properties of some functions involving the gamma function with applications, *J. Inequal. Pure Appl. Math.*, **7**(2) (2006), Art. 45. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=662>].
- [7] J. D. KEČLIĆ AND P. M. VASIĆ, Some inequalities for the gamma function, *Publ. Inst. Math. (Beograd) (N.S.)*, **11** (1971), 107–114.
- [8] S. KOUMANDOS, Remarks on a paper by Chao-Ping Chen and Feng Qi, *Proc. Amer. Math. Soc.*, **134** (2006), 1365–1367.



Title Page

Contents

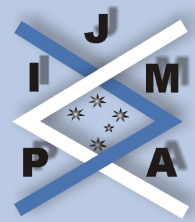


Page 15 of 19

Go Back

Full Screen

Close



**Geometric Convexity of
a Function Involving
Gamma Function**

Xiao-Ming Zhang, Tie-Quan Xu and

Ling-Bo Situ

vol. 8, iss. 1, art. 17, 2007

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 16 of 19

Go Back

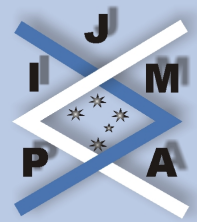
Full Screen

Close

- [9] J.-Ch. KUANG, *Chángyòng Bùděngshì (Applied Inequalities)*, 3rd ed., Shāndōng Kēxué Jìshù Chūbǎn Shè, Jinan City, Shandong Province, China, 2004. (Chinese)
- [10] J. MATKOWSKI, L^p -like paranorms, *Selected topics in functional equations and iteration theory (Graz, 1991)*, 103–138, Grazer Math. Ber., 316, *Karl-Franzens-Univ. Graz, Graz*, 1992.
- [11] P. MONTEL, Sur les fonctions convexes et les fonctions sousharmoniques, *J. de Math.*, **9**(7) (1928), 29–60.
- [12] C.P. NICULESCU, Convexity according to the geometric mean, *Math. Inequal. Appl.*, **2**(2) (2000), 155–167.
- [13] J. PEČARIĆ, G. ALLASIA AND C. GIORDANO, Convexity and the gamma function, *Indian J. Math.*, **41**(1) (1999), 79–93.
- [14] F. QI, A class of logarithmically completely monotonic functions and application to the best bounds in the second Gautschi-Kershaw's inequality, *RGMIA Res. Rep. Coll.*, **9**(4) (2006), Art. 11. [ONLINE: <http://rgmia.vu.edu.au/v9n4.html>].
- [15] F. QI, A class of logarithmically completely monotonic functions and the best bounds in the first Kershaw's double inequality, *J. Comput. Appl. Math.*, (2007), in press. [ONLINE: <http://dx.doi.org/10.1016/j.cam.2006.09.005>]. *RGMIA Res. Rep. Coll.*, **9**(2) (2006), Art. 16. [ONLINE: <http://rgmia.vu.edu.au/v9n2.html>].
- [16] F. QI, A completely monotonic function involving divided difference of psi function and an equivalent inequality involving sum, *RGMIA Res. Rep. Coll.*, **9**(4) (2006), Art. 5. [ONLINE: <http://rgmia.vu.edu.au/v9n4.html>].

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Geometric Convexity of
a Function Involving
Gamma Function

Xiao-Ming Zhang, Tie-Quan Xu and

Ling-Bo Situ

vol. 8, iss. 1, art. 17, 2007

Title Page

Contents

◀▶

◀▶

Page 17 of 19

Go Back

Full Screen

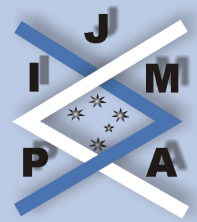
Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

- [17] F. QI, A completely monotonic function involving divided differences of psi and polygamma functions and an application, *RGMA Res. Rep. Coll.*, **9**(4) (2006), Art. 8. [ONLINE: <http://rgmia.vu.edu.au/v9n4.html>].
- [18] F. QI, A new lower bound in the second Kershaw's double inequality, *RGMA Res. Rep. Coll.*, **10**(1) (2007), Art. 9. [ONLINE: <http://rgmia.vu.edu.au/v10n1.html>].
- [19] F. QI, Monotonicity results and inequalities for the gamma and incomplete gamma functions, *Math. Inequal. Appl.*, **5** (1) (2002), 61–67. *RGMA Res. Rep. Coll.*, **2**(7) (1999), Art. 7, 1027–1034. [ONLINE: <http://rgmia.vu.edu.au/v2n7.html>].
- [20] F. QI, Refinements, extensions and generalizations of the second Kershaw's double inequality, *RGMA Res. Rep. Coll.*, **10** (2) (2007), Art. 8. [ONLINE: <http://rgmia.vu.edu.au/v10n2.html>].
- [21] F. QI, The best bounds in Kershaw's inequality and two completely monotonic functions, *RGMA Res. Rep. Coll.*, **9**(4) (2006), Art. 2. [ONLINE: <http://rgmia.vu.edu.au/v9n4.html>].
- [22] F. QI, R.-Q. CUI AND Ch.-P. CHEN, AND B.-N. GUO, Some completely monotonic functions involving polygamma functions and an application, *J. Math. Anal. Appl.*, **310**(1) (2005), 303–308.
- [23] F. QI AND B.-N. GUO, A class of logarithmically completely monotonic functions and the best bounds in the second Kershaw's double inequality, *J. Comput. Appl. Math.*, (2007), in press. [ONLINE: <http://dx.doi.org/10.1016/j.cam.2006.12.022>].



- [24] F. QI AND B.-N. GUO, Wendel-Gautschi-Kershaw's inequalities and sufficient and necessary conditions that a class of functions involving ratio of gamma functions are logarithmically completely monotonic, *RGMA Res. Rep. Coll.*, **10**(1) (2007), Art 2. [ONLINE: <http://rgmia.vu.edu.au/v10n1.html>].
- [25] F. QI, B.-N. GUO AND Ch.-P. CHEN, *The best bounds in Gautschi-Kershaw inequalities*, *Math. Inequal. Appl.*, **9** (3) (2006), 427–436. *RGMA Res. Rep. Coll.*, **8**(2) (2005), Art. 17. . [ONLINE: <http://rgmia.vu.edu.au/v8n2.html>].
- [26] F. QI AND S. GUO, New upper bounds in the second Kershaw's double inequality and its generalizations, *RGMA Res. Rep. Coll.*, **10**(2) (2007), Art. 1. [ONLINE: <http://rgmia.vu.edu.au/v10n2.html>].
- [27] F. QI, S. GUO AND B.-N. GUO, Note on a class of completely monotonic functions involving the polygamma functions, *RGMA Res. Rep. Coll.*, **10**(1) (2006), Art. 5. [ONLINE: <http://rgmia.vu.edu.au/v10n1.html>].
- [28] Z. STARC, Power product inequalities for the Gamma function, *Kragujevac J. Math.*, **24** (2002), 81–84.
- [29] L. YANG, Some inequalities on geometric convex function, *Héběi Dàxué Xuébào (Zìrán Kēxué Bǎn) (J. Hebei Univ. (Nat. Sci. Ed.))*, **22**(4) (2002), 325–328. (Chinese)
- [30] X.-M. ZHANG, Some theorem on geometric convex function and its applications, *Shōudū Shìfàn Dàxué Xuébào (Zìrán Kēxué Bǎn) (J. Capital Norm. Univ. (Nat. Sci. Ed.))*, **25**(2) (2004), 11–13. (Chinese)

Title Page

Contents



Page 18 of 19

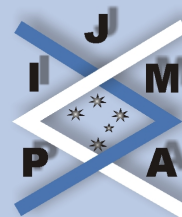
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



**Geometric Convexity of
a Function Involving
Gamma Function**

Xiao-Ming Zhang, Tie-Quan Xu and

Ling-Bo Situ

vol. 8, iss. 1, art. 17, 2007

Title Page

Contents



Page 19 of 19

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

- [31] X.-M. ZHANG AND Y.-D. WU, Geometrically convex functions and solution of a question, *RGMIA Res. Rep. Coll.*, **7**(4) (2004), Art. 11. [ONLINE: <http://rgmia.vu.edu.au/v7n4.html>].
- [32] N.-G. ZHENG AND X.-M. ZHANG, An important property and application of geometrically concave functions, *Shùxué de Shíjiàn yǔ Rènshí (Math. Practice Theory)*, **35**(8) (2005), 200–205. (Chinese)
- [33] D.-J. ZHAO, On a two-sided inequality involving Wallis' formula, *Shùxué de Shíjiàn yǔ Rènshí (Mathematics in Practice and Theory)*, **34**(7) (2004), 166–168. (Chinese)
- [34] Y.-Q. ZHAO AND Q.-B. WU, An improvement of the Wallis inequality, *Zhējiāng Dàxué Xuébào (Lǐxué Bǎn) (Journal of Zhejiang University (Science Edition))*, **33**(2) (2006), 372–375. (Chinese)
- [35] Y.-Q. ZHAO AND Q.-B. WU, Wallis inequality with a parameter, *J. Inequal. Pure Appl. Math.*, **7**(2) (2006), Art. 56. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=673>].