GEOMETRIC CONVEXITY OF A FUNCTION INVOLVING GAMMA FUNCTION AND APPLICATIONS TO INEQUALITY THEORY

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Abstract: In this paper, the geometric convexity of a function involving gamma function is studied, as applications to inequality theory, some important inequalities which improve some known inequalities, including Wallis' inequality, are obtained.

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1. Introduction and main results

The geometrically convex functions are as defined below.

Definition 1.1 *([10, 11, 12])*. Let $f : I \subseteq (0,\infty) \to (0,\infty)$ be a continuous function. Then f is called a geometrically convex function on I if there exists an integer $n \geq 2$ such that one of the following two inequalities holds:

$$f(\sqrt{x_1 x_2}) \leq \sqrt{f(x_1) f(x_2)}\,, \tag{1.1}$$

$$f\left(\prod_{i=1}^{n} x_i^{\lambda_i}\right) \leq \prod_{i=1}^{n} [f(x_i)]^{\lambda_i}, \tag{1.2}$$

where $x_1, x_2, \ldots, x_n \in I$ and $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. If inequalities (1.1) and (1.2) are reversed, then f is called a geometrically concave function on I.

For more literature on geometrically convex functions and their properties, see [12, 29, 30, 31, 32] and the references therein.

It is well known that Euler's gamma function $\Gamma(x)$ and the psi function $\psi(x)$ are defined for $x > 0$ respectively by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,\mathrm{d}\,t$ and $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. For $x > 0$, let

$$f(x) = \frac{e^x \Gamma(x)}{x^x}. \tag{1.3}$$

This function has been studied extensively by many mathematicians, for example, see [6] and the references therein.

In this article, we would like to discuss the geometric convexity of the function f defined by (1.3) and apply this property to obtain, from a new viewpoint, some new inequalities related to the gamma function.

Our main results are as follows.

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Theorem 1.1. *The function f defined by (1.3) is geometrically convex.*

Theorem 1.2. *For $x > 0$ and $y > 0$, the double inequality*

$$\frac{x^x}{y^y}\left(\frac{x}{y}\right)^{y[\psi(y)-\ln y]} e^{y-x} \le \frac{\Gamma(x)}{\Gamma(y)} \le \frac{x^x}{y^y}\left(\frac{x}{y}\right)^{x[\psi(x)-\ln x]} e^{y-x} \tag{1.4}$$

holds.

As consequences of above theorems, the following corollaries can be deduced.

Corollary 1.3. *The function f is logarithmically convex.*

Remark 1. More generally, the function f is logarithmically completely monotonic in $(0,\infty)$. See [6].

Corollary 1.4 ([7, 13]). *For $0 < y < x$ and $0 < s < 1$, inequalities*

$$e^{(x-y)\psi(y)} < \frac{\Gamma(x)}{\Gamma(y)} < e^{(x-y)\psi(x)} \tag{1.5}$$

and

$$\frac{x^{x-1}}{y^{y-1}}e^{y-x} < \frac{\Gamma(x)}{\Gamma(y)} < \frac{x^{x-\frac{1}{2}}}{y^{y-\frac{1}{2}}}e^{y-x} \tag{1.6}$$

are valid.

Remark 2. Note that inequality (1.4) is better than (1.5) and (1.6). The lower and upper bounds for $\frac{\Gamma(x)}{\Gamma(y)}$ have been established in many papers such as [14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26].

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Corollary 1.5. *For $x > 0$ and $n \in \mathbb{N}$, the following double inequalities hold:*

$$\sqrt{ex}\left(1+\frac{1}{2x}\right)^{-x} < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} < \sqrt{ex}\left(1+\frac{1}{2x}\right)^{\frac{1}{12x}-x} \tag{1.7}$$

and

$$\sqrt{e(x+n)}\left(1+\frac{1}{2x+2n}\right)^{-x-n}\prod_{k=1}^{n}\left(1-\frac{1}{2x+2k}\right) < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} < \sqrt{e(x+n)}\left(1+\frac{1}{2x+2n}\right)^{\frac{1}{12x+12n}-x-n}\prod_{k=1}^{n}\left(1-\frac{1}{2x+2k}\right). \tag{1.8}$$

Corollary 1.6. *For $n \in \mathbb{N}$, the double inequality*

$$\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+16}} \tag{1.9}$$

is valid.

Remark 3. Inequality (1.9) is related to the well known Wallis inequality. If $n \geq 2$, inequality (1.9) is better than

$$\frac{1}{\sqrt{\pi(n+4/\pi-1)}} \leq \frac{(2n-1)!!}{(2n)!!} \leq \frac{1}{\sqrt{\pi(n+1/4)}} \tag{1.10}$$

in [3]. For more details, please refer to [2, 8, 33, 34, 35] and the references therein.

Corollary 1.7 ([28]). *Let $S_n = \sum_{k=1}^{n}\frac{1}{k}$ for $n \in \mathbb{N}$. Then*

$$\frac{2^{n+1}n!}{(2n+1)!!}\left(\frac{2n+3}{2n+2}\right)^{3/2+n} e^{(S_n-1-\gamma)/2} < \sqrt{\pi}\,. \tag{1.11}$$

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2. Lemmas

In order to prove our main results, the following lemmas are necessary.

Lemma 2.1 ([1, 5, 22]). *For* $x > 0$,

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}, \tag{2.1}$$
$$\psi(x) > \ln x - \frac{1}{2x} - \frac{1}{12x^2}, \qquad \psi'(x) > \frac{1}{x} + \frac{1}{2x^2}.$$

Lemma 2.2. *For* $x > 0$,

$$2\psi'(x) + x\psi''(x) < \frac{1}{x}. \tag{2.2}$$

Remark 4. The complete monotonicity of the function $2\psi'(x)+x\psi''(x)$ was obtained in [27].

Proof. It is a well known fact that

$$\psi'(x) = \sum_{k=1}^{\infty} \frac{1}{(k-1+x)^2} \quad \text{and} \quad \psi''(x) = -\sum_{k=1}^{\infty} \frac{2}{(k-1+x)^3}. \tag{2.3}$$

From this, it follows that

$$\begin{aligned} 2\psi'(x) + x\psi''(x) - \frac{1}{x} &= 2\sum_{k=1}^{\infty} \frac{k}{(k+x)^3} - \frac{1}{x} \\ &< 2\sum_{k=1}^{\infty} \frac{k}{(k-1+x)(k+x)(k+1+x)} - \frac{1}{x} \end{aligned}$$

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$$= \sum_{k=1}^{\infty} \left[\frac{k}{(k-1+x)(k+x)} - \frac{k}{(k+x)(k+1+x)} \right] - \frac{1}{x}$$

$$= \sum_{k=1}^{\infty} \frac{1}{(k-1+x)(k+x)} - \frac{1}{x}$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{k-1+x} - \frac{1}{k+x} \right) - \frac{1}{x} = 0.$$

Thus the proof of Lemma 2.2 is completed. ∎

Lemma 2.3 ([12]). *Let $(a, b) \subset (0, \infty)$ and $f : (a, b) \to (0, \infty)$ be a differentiable function. Then f is a geometrically convex function if and only if the function $\frac{xf'(x)}{f(x)}$ is nondecreasing.*

Lemma 2.4 ([12]). *Let $(a, b) \subset (0, \infty)$ and $f : (a, b) \to (0, \infty)$ be a differentiable function. Then f is a geometrically convex function if and only if $\frac{f(x)}{f(y)} \geq \left(\frac{x}{y}\right)^{yf'(y)/f(y)}$ holds for any $x, y \in (a, b)$.*

Lemma 2.5 ([4, 9]). *Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$ and $C_n = S_n - \ln\left(n + \frac{1}{2}\right) - \gamma$ for $n \in \mathbb{N}$, where $\gamma = 0.5772156\ldots$ is Euler-Mascheroni's constant. Then*

$$\frac{1}{24(n+1)^2} < C_n < \frac{1}{24n^2}. \tag{2.4}$$

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3. Proofs of Theorems and Corollaries

Now we are in a position to prove our main results.

Proof of Theorem 1.1. Easy calculation yields

$$\ln f(x) = \ln\Gamma(x) - x\ln x + x \quad \text{and} \quad \frac{f'(x)}{f(x)} = \psi(x) - \ln x. \tag{3.1}$$

Let $F(x) = \left[\frac{xf'(x)}{f(x)}\right]'$. Then

$$F(x) = \psi(x) + x\psi'(x) - \ln x - 1, \quad \text{and} \quad F'(x) = 2\psi'(x) + x\psi''(x) - \frac{1}{x}.$$

By virtue of Lemma 2.2, it follows that $F'(x) < 0$, thus F is decreasing in $x > 0$. By Lemma 2.1, we deduce that

$$F(x) = \psi(x) + x\psi'(x) - \ln x - 1 > \ln x - \frac{1}{x} + x\left(\frac{1}{x} + \frac{1}{2x^2}\right) - \ln x - 1 = -\frac{1}{2x}.$$

Hence $\lim_{x\to\infty} F(x) \geq 0$. This implies that $F(x) > 0$ and, by Lemma 2.3, the function f is geometrically convex. The proof is completed. ∎

Proof of Theorem 1.2. Combining Theorem 1.1, Lemma 2.4 and (3.1) leads to

$$\frac{e^x\Gamma(x)}{x^x} \geq \left(\frac{x}{y}\right)^{y[\psi(y)-\ln y]}\frac{e^y\Gamma(y)}{y^y} \quad \text{and} \quad \frac{e^y\Gamma(y)}{y^y} \geq \left(\frac{y}{x}\right)^{x[\psi(x)-\ln x]}\frac{e^x\Gamma(x)}{x^x}.$$

Inequality (1.4) is established. ∎

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Proof of Corollary 1.3. A combination of (3.1) with Lemma 2.1 reveals the decreasing monotonicity of f in $(0,\infty)$. Considering the geometric convexity and the decreasing monotonicity of f and the arithmetic-geometric mean inequality, we have

$$f\left(\frac{x_1+x_2}{2}\right) \le f(\sqrt{x_1x_2}\,) \le \sqrt{f(x_1)f(x_2)} \le \frac{f(x_1)+f(x_2)}{2}.$$

Hence, f is convex and logarithmic convex in $(0,\infty)$. ∎

Proof of Corollary 1.4. A property of mean values [9] and direct argument gives

$$\frac{1}{x} < \frac{\ln x - \ln y}{x-y} < \frac{1}{y}, \qquad \ln x - \ln y > 1 - \frac{y}{x}, \tag{3.2}$$

$$-1 + \ln x + \frac{y}{x} > \psi(y) + y[\ln y - \psi(y)]\frac{1}{y}.$$

Hence,

$$-1 + \ln x + y\frac{\ln x - \ln y}{x-y} > \psi(y) + y[\ln y - \psi(y)]\frac{\ln x - \ln y}{x-y}, \tag{3.3}$$

$$(y-x) + (x-y)\ln x + y(\ln x - \ln y) > (x-y)\psi(y) + y[\ln y - \psi(y)](\ln x - \ln y),$$

$$(y-x) + x\ln x - y\ln y + y[\psi(y) - \ln y](\ln x - \ln y) > (x-y)\psi(y),$$

$$\left(\frac{x}{y}\right)^{y[\psi(y)-\ln y]} \frac{e^y x^x}{e^x y^y} > e^{(x-y)\psi(y)}.$$

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Similarly,

$$-1+\ln x+y\frac{1}{y}=x[\ln x-\psi(x)]\frac{1}{x}+\psi(x), \tag{3.4}$$

$$-1+\ln x+y\frac{\ln x-\ln y}{x-y}<x[\ln x-\psi(x)]\frac{\ln x-\ln y}{x-y}+\psi(x),$$

$$(y-x)+(x-y)\ln x+y(\ln x-\ln y)<x[\ln x-\psi(x)](\ln x-\ln y)+(x-y)\psi(x),$$

$$(y-x)+x\ln x-y\ln y+x[\psi(x)-\ln x](\ln x-\ln y)<(x-y)\psi(x),$$

$$\left(\frac{x}{y}\right)^{x[\psi(x)-\ln x]}\frac{e^y x^x}{e^x y^y}<e^{(x-y)\psi(x)}.$$

Combination of (3.3) and (3.4) leads to (1.5).

By (2.1), it is easy to see that

$$1<\left(\frac{x}{y}\right)^{y[\ln y-\psi(y)]}\frac{x}{y},\qquad \frac{x^{x-1}}{y^{y-1}}e^{y-x}<\left(\frac{x}{y}\right)^{y[\ln y-\psi(y)]}\frac{e^y x^x}{e^x y^y}.$$

Similarly,

$$\frac{e^y x^x}{e^x y^y}\left(\frac{x}{y}\right)^{x[\ln x-\psi(x)]}<\frac{x^{x-\frac{1}{2}}}{y^{y-\frac{1}{2}}}e^{y-x}.$$

By virtue of (1.4), inequality (1.6) follows. ∎

Proof of Corollary 1.5. Let $y=x+\frac{1}{2}$ in inequality (1.4). Then

$$\frac{e^{\frac{1}{2}}x^x}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}}\left(\frac{x}{x+\frac{1}{2}}\right)^{\left(x+\frac{1}{2}\right)\left[\psi\left(x+\frac{1}{2}\right)-\ln\left(x+\frac{1}{2}\right)\right]} \tag{3.5}$$

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$$\le \frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{2}\right)} \le \frac{e^{\frac{1}{2}} x^{x}}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}} \left(\frac{x}{x+\frac{1}{2}}\right)^{x[\psi(x)-\ln x]},$$

$$\frac{e^{\frac{1}{2}} x^{x+1}}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}} \left(\frac{x+\frac{1}{2}}{x}\right)^{\left(x+\frac{1}{2}\right)\left[\ln\left(x+\frac{1}{2}\right)-\psi\left(x+\frac{1}{2}\right)\right]} \le \frac{x\Gamma(x)}{\Gamma\left(x+\frac{1}{2}\right)} \le \frac{e^{\frac{1}{2}} x^{x+1}}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}} \left(\frac{x+\frac{1}{2}}{x}\right)^{x[\ln x-\psi(x)]}.$$

From inequality (2.2), we obtain

$$\frac{\sqrt{ex}\, x^{x+\frac{1}{2}}}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}} \left(1+\frac{1}{2x}\right)^{\frac{1}{2}} < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \frac{\sqrt{ex}\, x^{x+\frac{1}{2}}}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}} \left(1+\frac{1}{2x}\right)^{\frac{1}{2}+\frac{1}{12x}},$$

$$\sqrt{ex}\left(1+\frac{1}{2x}\right)^{-x} < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{ex}\left(1+\frac{1}{2x}\right)^{\frac{1}{12x}-x}.$$

The proof of inequality (1.7) is completed.

Substituting

$$\frac{\Gamma(x+n+1)}{\Gamma\left(x+n+\frac{1}{2}\right)} = \frac{(x+n)\Gamma(x+n)}{\left(x+n-\frac{1}{2}\right)\Gamma\left(x+n-\frac{1}{2}\right)} = \cdots = \frac{\Gamma(x+1)\prod_{k=1}^{n}(x+k)}{\Gamma\left(x+\frac{1}{2}\right)\prod_{k=1}^{n}\left(x+k-\frac{1}{2}\right)}$$

into (1.7) shows that inequality (1.8) is valid. ∎

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Proof of Corollary 1.6. For $n = 1, 2$, inequality (1.9) can be verified readily.

For $n \geq 3$, in view of formulas $\Gamma(n+1) = n!$, $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$ and inequality (1.7), we have

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} < \sqrt{en}\left(1+\frac{1}{2n}\right)^{\frac{1}{12n}-n}, \qquad \frac{2^n n!}{(2n-1)!!} < \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\frac{1}{12n}-n},$$

and

$$\frac{(2n-1)!!}{(2n)!!} > \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}}. \tag{3.6}$$

Further, taking $x = n$ in inequality (3.5) reveals

$$\frac{e^{\frac{1}{2}}n^{n+1}}{\left(n+\frac{1}{2}\right)^{n+\frac{1}{2}}}\left(\frac{n+\frac{1}{2}}{n}\right)^{\left(n+\frac{1}{2}\right)\left(\ln\left(n+\frac{1}{2}\right)-\psi\left(n+\frac{1}{2}\right)\right)} \leq \frac{n\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)},$$

$$\frac{2^n n!}{(2n-1)!!} \geq \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)-\psi\left(n+\frac{1}{2}\right)-1\right]},$$

$$\frac{2^n n!}{(2n-1)!!} \geq \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)-\psi\left(n+\frac{1}{2}\right)-1\right]}.$$

Employing formulas

$$\psi(x+1) = \psi(x)+\frac{1}{x}, \quad \psi\left(\frac{1}{2}\right) = -\gamma-2\ln 2, \quad C_n = S_n - \ln\left(n+\frac{1}{2}\right)-\gamma \tag{3.7}$$

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yields

$$
\begin{aligned}
\frac{2^n n!}{(2n-1)!!} &\geq \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)-\psi\left(n-\frac{1}{2}\right)-\frac{1}{n-\frac{1}{2}}-1\right]} \\
&= \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)-\psi\left(\frac{1}{2}\right)-\frac{1}{n-\frac{1}{2}}-\cdots-\frac{1}{\frac{1}{2}}-1\right]} \\
&= \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)+2\ln 2+\gamma-2\sum_{k=1}^{n}\frac{1}{2k-1}-1\right]} \\
&= \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)+2\ln 2+\gamma-2\sum_{k=1}^{2n}\frac{1}{k}+\sum_{k=1}^{n}\frac{1}{k}-1\right]} \\
&= \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[2\ln(2n+1)-2C_{2n}-2\ln\left(2n+\frac{1}{2}\right)+C_n-1\right]}.
\end{aligned} \tag{3.8}
$$

Letting $x=\frac{1}{1+4n}$ in $\ln(1+x)>\frac{x}{1+\frac{x}{2}}$ for $x>0$, we obtain

$$
\ln\left(1+\frac{1}{1+4n}\right)>\frac{2}{8n+3}. \tag{3.9}
$$

In view of Lemma 2.5 and inequalities (3.8) and (3.9), we have

$$
\frac{2^n n!}{(2n-1)!!}>\sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\frac{4}{8n+3}-\frac{1}{48n^2}+\frac{1}{24(n+1)^2}-1\right]}. \tag{3.10}
$$

It is easy to verify that

$$
\left(n+\frac{1}{2}\right)\left[\frac{4}{8n+3}-\frac{1}{48n^2}+\frac{1}{24(n+1)^2}-1\right]>-n+\frac{1}{12n+16} \tag{3.11}
$$

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with $n \geq 3$. By virtue of (3.6), (3.10) and (3.11), Corollary 1.6 is proved. ∎

Proof of Corollary 1.7. Letting $x = n + \frac{3}{2}$ and $y = n + 1$ in inequality (1.4) yields

$$\text{(3.12)} \qquad \frac{1}{\sqrt{e\pi(n+1)}} \left(1 + \frac{1}{2n+2}\right)^{(n+1)[\psi(n+1) - \ln(n+1) + 1] + \frac{1}{2}} \leq \frac{(2n+1)!!}{(2n+2)!!}.$$

By using inequality (2.1), $\psi(n+1) = \sum_{k=1}^{n} \frac{1}{k} - \gamma$ and $\frac{1}{\sqrt{e}} \left(\frac{2n+3}{2n+2}\right)^{n+1} < 1$ for $n \in \mathbb{N}$, we have

$$\begin{aligned} &\frac{(2n+1)!!}{(2n)!!} \left(\frac{2n+2}{2n+3}\right)^{\frac{3}{2}+n} e^{-\frac{1}{2}(S_n - 1 - \gamma)} \\ &= (2n+2) \frac{(2n+1)!!}{(2n+2)!!} \left(\frac{2n+2}{2n+3}\right)^{\frac{3}{2}+n} e^{-\frac{1}{2}[\psi(n+1) - 1]} \\ &> \frac{2\sqrt{n+1}}{\sqrt{\pi}} \left(\frac{2n+3}{2n+2}\right)^{-(n+1)\ln(n+1)} \left[\frac{1}{\sqrt{e}} \left(\frac{2n+3}{2n+2}\right)^{n+1}\right]^{\ln(n+1) - \frac{1}{2(n+1)}} \\ &= \frac{2\sqrt{n+1}}{\sqrt{\pi}} \sqrt{\frac{2n+2}{2n+3}} \, e^{-\frac{1}{2}\ln(n+1) + \frac{1}{4(n+1)}} = \frac{2}{\sqrt{\pi}} \sqrt{\frac{2n+2}{2n+3}} \, e^{\frac{1}{4(n+1)}} > \frac{2}{\sqrt{\pi}}. \end{aligned}$$

The proof of Corollary 1.7 is completed. ∎

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