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## A NOTE ON WEIGHTED IDENTRIC AND LOGARITHMIC MEANS

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## Abstract

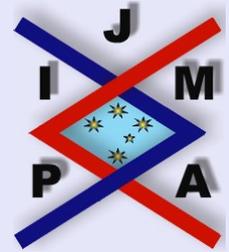
Recently obtained inequalities [12] between the Gaussian hypergeometric function and the power mean are applied to establish new sharp inequalities involving the weighted identric, logarithmic, and power means.

*2000 Mathematics Subject Classification:* 26D07, 26D15, 33C05.

*Key words:* Identric mean, Logarithmic mean, Hypergeometric function.

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# 1. Introduction

For  $x, y > 0$ , the *weighted power mean* of order  $\lambda$  is given by

$$\mathcal{M}_\lambda(\omega; x, y) \equiv [(1 - \omega)x^\lambda + \omega y^\lambda]^{\frac{1}{\lambda}}$$

with  $\omega \in (0, 1)$  and  $\mathcal{M}_0(\omega; x, y) \equiv \lim_{\lambda \rightarrow 0} \mathcal{M}_\lambda(\omega; x, y) = x^{1-\omega}y^\omega$ . Since  $\lambda \mapsto \mathcal{M}_\lambda$  is increasing, it follows that

$$\mathcal{G}(x, y) \leq \mathcal{M}_\lambda\left(\frac{1}{2}; x, y\right) \leq \mathcal{A}(x, y), \quad \text{for } 0 \leq \lambda \leq 1,$$

where  $\mathcal{G}(x, y) \equiv \mathcal{M}_0\left(\frac{1}{2}; x, y\right)$  and  $\mathcal{A}(x, y) \equiv \mathcal{M}_1\left(\frac{1}{2}; x, y\right)$  are the well-known geometric and arithmetic means, respectively (e.g., see [4, p. 203]). Thus,  $\mathcal{M}_\lambda$  provides a refinement of the classical inequality  $\mathcal{G} \leq \mathcal{A}$ . It is natural to seek other bivariate means that separate  $\mathcal{G}$  and  $\mathcal{A}$ . Two such means are the *logarithmic mean* and the *identric mean*. For distinct  $x, y > 0$ , the logarithmic mean  $\mathcal{L}$  is given by

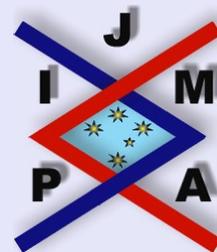
$$\mathcal{L}(x, y) \equiv \frac{x - y}{\ln(x) - \ln(y)},$$

and  $\mathcal{L}(x, x) \equiv x$ . The integral representation

$$(1.1) \quad \mathcal{L}(1, 1 - r) = \left( \int_0^1 (1 - rt)^{-1} dt \right)^{-1}, \quad r < 1$$

is due to Carlson [6]. Similarly, the identric mean  $\mathcal{I}$  is defined by

$$\mathcal{I}(x, y) \equiv \frac{1}{e} \left( \frac{x^x}{y^y} \right)^{\frac{1}{x-y}},$$



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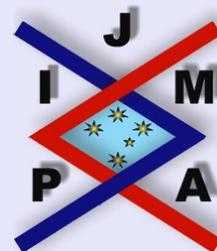
$\mathcal{I}(x, x) \equiv x$ , and has the integral representation

$$(1.2) \quad \mathcal{I}(1, 1 - r) = \exp \left( \int_0^1 \ln(1 - rt) dt \right), \quad r < 1.$$

The inequality  $\mathcal{G} \leq \mathcal{L} \leq \mathcal{A}$  was refined by Carlson [6] who showed that  $\mathcal{L}(x, y) \leq \mathcal{M}_{1/2}(\frac{1}{2}; x, y)$ . Lin [8] then sharpened this by proving  $\mathcal{L}(x, y) \leq \mathcal{M}_{1/3}(\frac{1}{2}; x, y)$ . Shortly thereafter, Stolarsky [14] introduced the generalized logarithmic mean which has since come to bear his name. These and other efforts (e.g., [11, 15]) led to many interesting results, including the following well-known inequalities:

$$(1.3) \quad \mathcal{G} \leq \mathcal{L} \leq \mathcal{M}_{1/3} \leq \mathcal{M}_{2/3} \leq \mathcal{I} \leq \mathcal{A},$$

where each is evaluated at  $(x, y)$ , and the power means have equal weights  $\omega = 1 - \omega = 1/2$ . It also should be noted that the indicated orders of the power means in (1.3), namely 1/3 and 2/3, are sharp. Following the work of Leach and Sholander [7], Páles [10] gave a complete ordering of the general Stolarsky mean which provides an elegant generalization of (1.3). (For a more complete discussion of inequalities involving means, see [4].)




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## 2. Main Results

Our main objective is to present a generalization of (1.3) using the *weighted* logarithmic and identric means. Moreover, sharp power mean bounds are provided. This can be accomplished using the Gaussian hypergeometric function  ${}_2F_1$  which is given by

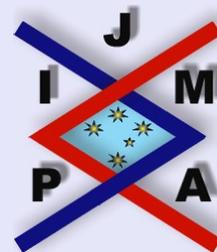
$${}_2F_1(\alpha, \beta; \gamma; r) \equiv \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} r^n, \quad |r| < 1,$$

where  $(\alpha)_n$  is the Pochhammer symbol defined by  $(\alpha)_0 = 1$ ,  $(\alpha)_1 = \alpha$ , and  $(\alpha)_{n+1} = (\alpha)_n(\alpha + n)$ , for  $n \in \mathbb{N}$ . For  $\gamma > \beta > 0$ ,  ${}_2F_1$  has the following integral representation due to Euler (see [2]):

$${}_2F_1(\alpha, \beta; \gamma; r) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \beta)\Gamma(\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-rt)^{-\alpha} dt,$$

which, by continuation, extends the domain of  ${}_2F_1$  to all  $r < 1$ . Here  $\Gamma(z) \equiv \int_0^{\infty} t^{z-1}e^{-t} dt$  for  $z > 0$ ;  $\Gamma(n) = (n-1)!$  for  $n \in \mathbb{N}$ . Inequalities relating the Gaussian hypergeometric function to various means have been widely studied (see [1, 2, 3, 5, 12]). Of particular use here is the *hypergeometric mean* of order  $a$  discussed by Carlson in [5] and defined by

$$\begin{aligned} \mathcal{H}_a(\omega; c; x, y) &\equiv \left[ \frac{\Gamma(c)}{\Gamma(c\omega)\Gamma(c\omega')} \int_0^1 t^{c\omega-1}(1-t)^{c\omega'-1}(x(1-t) + yt)^a dt \right]^{\frac{1}{a}} \\ &= x \cdot \left[ {}_2F_1\left(-a, c\omega; c; 1 - \frac{y}{x}\right) \right]^{\frac{1}{a}} \end{aligned}$$



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with the parameter  $c > 0$  and weights  $\omega, \omega' > 0$  satisfying  $\omega + \omega' = 1$ . Clearly  $\mathcal{H}_a(\omega; c; \rho x, \rho y) = \rho \mathcal{H}_a(\omega; c; x, y)$  for  $\rho > 0$ , so  $\mathcal{H}_a$  is homogeneous. Euler's integral representation and (1.1) together yield

$$\mathcal{H}_{-1}\left(\frac{1}{2}; 2; 1, 1-r\right) = \left(\frac{\Gamma(2)}{\Gamma(1)^2} \int_0^1 (1-rt)^{-1} dt\right)^{-1} = \mathcal{L}(1, 1-r).$$

Multiplying by  $x$ , with  $r = 1 - y/x$ , and applying homogeneity yields  $\mathcal{H}_{-1}\left(\frac{1}{2}; 2; x, y\right) = \mathcal{L}(x, y)$ . This naturally leads to the *weighted logarithmic mean*  $\hat{\mathcal{L}}$  which is defined as

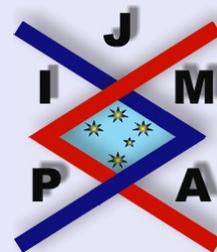
$$\hat{\mathcal{L}}(\omega; c; x, y) \equiv \mathcal{H}_{-1}(\omega; c; x, y).$$

Weighted logarithmic means have been discussed by Pittenger [11] and Neuman [9], among others (see also [4, p. 391-392]). Similarly, the *weighted identric mean*  $\hat{\mathcal{I}}$  is given by

$$\begin{aligned} \hat{\mathcal{I}}(\omega; c; x, y) &\equiv \mathcal{H}_0(\omega; c; x, y) \equiv \lim_{a \rightarrow 0} \mathcal{H}_a(\omega; c; x, y) \\ &= \exp\left(\frac{\Gamma(c)}{\Gamma(c\omega')\Gamma(c\omega)} \int_0^1 t^{c\omega-1}(1-t)^{c\omega'-1} \ln[x(1-t) + yt] dt\right) \end{aligned}$$

(see [5], [13]). Thus,  $\hat{\mathcal{I}}\left(\frac{1}{2}; 2; x, y\right) = \mathcal{I}(x, y)$ .

The following theorem establishes inequalities between the power means and the weighted identric and logarithmic means.



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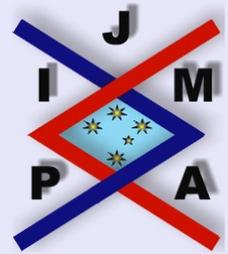


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**Theorem 2.1.** *Suppose  $x > y > 0$  and  $c \geq 1$ .*

*If  $0 < \omega \leq 1/2$ , then the weighted identric mean  $\hat{\mathcal{I}}$  satisfies*

$$(2.1) \quad \mathcal{M}_{\frac{c}{c+1}}(\omega; x, y) \leq \hat{\mathcal{I}}(\omega; c; x, y).$$

*If  $1/2 \leq \omega < 1$  and  $c \leq 3$ , then the weighted logarithmic mean  $\hat{\mathcal{L}}$  satisfies*

$$(2.2) \quad \hat{\mathcal{L}}(\omega; c; x, y) \leq \mathcal{M}_{\frac{c-1}{c+1}}(\omega; x, y).$$

*Moreover, the power mean orders  $c/(c+1)$  and  $(c-1)/(c+1)$  are sharp.*

A key step in the proof will be an application of the following recently obtained result:

**Proposition 2.2.** [12] *Suppose  $1 \geq a$  and  $c > b > 0$ . If  $c \geq \max\{1 - 2a, 2b\}$ , then*

$$(2.3) \quad \mathcal{M}_\lambda \left( \frac{b}{c}; 1, 1 - r \right) \leq [{}_2F_1(-a, b; c; r)]^{\frac{1}{a}} \text{ for all } r \in (0, 1),$$

*if and only if  $\lambda \leq (a + c)/(1 + c)$ . If  $-a \leq c \leq \min\{1 - 2a, 2b\}$ , then the inequality in (2.3) reverses if and only if  $\lambda \geq (a + c)/(1 + c)$ .*

*Proof of Theorem 2.1.* Suppose  $x > y > 0$ ,  $c \geq 1$ ,  $\omega \in (0, 1)$  and define  $b \equiv c\omega$  with  $r \equiv 1 - y/x \in (0, 1)$ . If  $\omega \leq 1/2$  and  $a \in (0, 1)$ , it follows that  $c \geq \max\{1 - 2a, 2b\}$ . Hence the previous proposition implies

$$(2.4) \quad \mathcal{M}_{\frac{a+c}{1+c}}(\omega; 1, 1 - r) \leq [{}_2F_1(-a, b; c; r)]^{\frac{1}{a}}.$$

Taking the limit of both sides of (2.4) as  $a \rightarrow 0^+$  yields

$$(2.5) \quad \mathcal{M}_{\frac{c}{c+1}}(\omega; 1, 1-r) \leq \mathcal{H}_0(\omega; c; 1, 1-r).$$

Now suppose  $\omega \geq 1/2$  and  $c \leq 3$ . Then  $c \leq 2b$  and  $-a = 1 \leq c \leq 3 = 1 - 2a$  for  $a = -1$ . Thus

$$(2.6) \quad \mathcal{H}_{-1}(\omega; c; 1, 1-r) = [{}_2F_1(1, b; c; r)]^{-1} \leq \mathcal{M}_{\frac{c-1}{c+1}}(\omega; 1, 1-r),$$

again by the above proposition. Multiplying both sides of the inequalities in (2.5) and (2.6) by  $x$  and applying homogeneity yields the desired results.  $\square$

In the case that  $\omega = 1/2$ , we have

**Corollary 2.3.** *If  $x, y > 0$ ,  $1 \leq c \leq 3$ , and  $\omega = 1/2$  then*

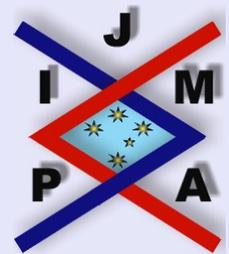
$$(2.7) \quad \mathcal{H}_{-2} \leq \mathcal{H}_{-1} \leq \mathcal{M}_{\frac{c-1}{c+1}} \leq \mathcal{M}_{\frac{c}{c+1}} \leq \mathcal{H}_0 \leq \mathcal{H}_1.$$

*Moreover,  $(c-1)/(c+1)$  and  $c/(c+1)$  are sharp. If  $c = 2$ , then (2.7) reduces to (1.3).*

*Proof.* Suppose  $x > y > 0$ ,  $1 \leq c \leq 3$ , and  $\omega = 1/2$ . Hence (2.2) and (2.1), together with the fact that  $\lambda \mapsto \mathcal{M}_\lambda$  is increasing, imply

$$\mathcal{H}_{-1}\left(\frac{1}{2}; c; x, y\right) \leq \mathcal{M}_{\frac{c-1}{c+1}}\left(\frac{1}{2}; x, y\right) \leq \mathcal{M}_{\frac{c}{c+1}}\left(\frac{1}{2}; x, y\right) \leq \mathcal{H}_0\left(\frac{1}{2}; c; x, y\right).$$

The remaining inequalities follow directly from Carlson's observation [5] that  $a \mapsto \mathcal{H}_a$  is increasing. The condition that  $x > y$  can be relaxed by noting



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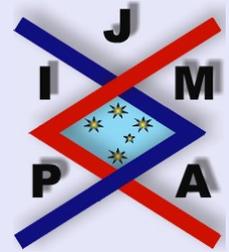
that  $\mathcal{H}_a$  is symmetric in  $(x, y)$  when  $\omega = 1/2$ . This symmetry can be seen by making the substitution  $s = 1 - t$  in Euler's integral representation:

$$\begin{aligned} \mathcal{H}_a \left( \frac{1}{2}; c; x, y \right)^a &= \frac{\Gamma(c)}{\Gamma(c/2)^2} \int_0^1 [t(1-t)]^{c/2-1} ((1-t)x + ty)^a dt \\ &= \frac{\Gamma(c)}{\Gamma(c/2)^2} \int_0^1 [(1-s)s]^{c/2-1} (sx + (1-s)y)^a ds \\ &= \mathcal{H}_a \left( \frac{1}{2}; c; y, x \right)^a. \end{aligned}$$

Finally, note that  $\mathcal{M}_{\frac{c-1}{c+1}} = \mathcal{M}_{\frac{1}{3}}$  and  $\mathcal{M}_{\frac{c}{c+1}} = \mathcal{M}_{\frac{2}{3}}$  when  $c = 2$ . Also,

$$\mathcal{H}_{-2} \left( \frac{1}{2}; 2; 1, 1-r \right)^{-2} = {}_2F_1(2, 1; 2; r) = \frac{1}{1-r},$$

for  $|r| < 1$ . It follows that  $\mathcal{H}_{-2} \left( \frac{1}{2}; 2; x, y \right) = (xy)^{\frac{1}{2}} = \mathcal{G}(x, y)$ . Likewise,  $\mathcal{H}_1 \left( \frac{1}{2}; 2; x, y \right) = x(1 - (1 - y/x)/2) = \mathcal{A}(x, y)$ . Thus (2.7) implies (1.3).  $\square$




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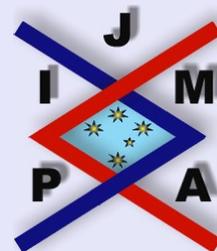
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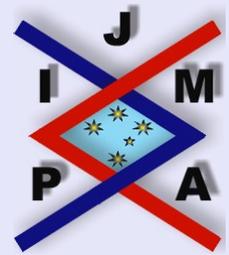
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