

## ESTIMATES FOR THE GREEN FUNCTION AND CHARACTERIZATION OF A CERTAIN KATO CLASS BY THE GAUSS SEMIGROUP IN THE HALF SPACE

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## Abstract

We establish a 3G-theorem for the Green functions  $G_{m,n}$  of  $(-\Delta)^m$  ( $m \geq 1$ ) on  $\mathbb{R}_+^n := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ ,  $n \geq 2m - 1$ , with Navier boundary conditions  $\Delta^j u|_{\partial\mathbb{R}_+^n} = 0$ ,  $0 \leq j \leq m - 1$ .

We exploit these results to define a certain Kato class of functions that we characterize by means of the Gauss semigroup on  $\mathbb{R}_+^n$ .

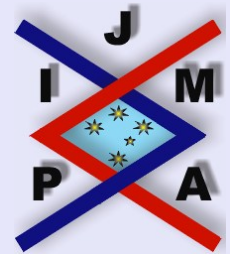
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# 1. Introduction

In [2], for  $n \geq 3$  and [3], for  $n = 2$ , the authors have established interesting estimates for  $G(x, y)$ , the Green function of the Laplace operator corresponding to zero Dirichlet boundary conditions in the half space  $\mathbb{R}_+^n := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_n > 0\}$ . In particular, they have proved the following form of the 3G-Theorem:

**Theorem 1.1.** *There exists a constant  $C > 0$  such that for each  $x, y, z \in \mathbb{R}_+^n$*

$$(1.1) \quad \frac{G(x, z)G(z, y)}{G(x, y)} \leq C \left[ \frac{z_n}{x_n} G(x, z) + \frac{z_n}{y_n} G(y, z) \right].$$

They then introduced a class of functions  $K_{1,n}(\mathbb{R}_+^n)$  as follows:

**Definition 1.1.** *A Borel measurable function  $q$  in  $\mathbb{R}_+^n$  belongs to the class  $K_{1,n}(\mathbb{R}_+^n)$  if  $q$  satisfies the following condition*

$$\limsup_{r \rightarrow 0} \sup_{x \in \mathbb{R}_+^n} \int_{(|x-y| \leq r) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} G(x, y) |q(y)| dy = 0.$$

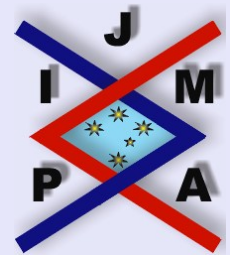
They have studied the properties of functions belonging to this class.

In particular, in [2], the authors have showed the following characterization:

$$(1.2) \quad q \in K_{1,n}(\mathbb{R}_+^n) \iff \lim_{t \rightarrow 0} \left( \sup_{x \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \int_0^t \frac{y_n}{x_n} p(s, x, y) |q(y)| ds dy \right) = 0,$$

where  $p(s, x, y)$  is the density of the Gauss semigroup on  $\mathbb{R}_+^n$ .

Note that similar characterizations have been already established in [1], (see also [5] and [8]) for the classical Kato class  $K_n(\mathbb{R}^n)$  defined as follows:



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**Definition 1.2.** A Borel measurable function  $q$  in  $\mathbb{R}_+^n$  ( $n \geq 3$ ) belongs to the Kato class  $K_n(\mathbb{R}_+^n)$  if  $q$  satisfies the following condition

$$\lim_{\alpha \rightarrow 0} \left( \sup_{x \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n \cap B(x, \alpha)} \frac{1}{|x - y|^{n-2}} |q(y)| dy \right) = 0.$$

For properties of functions in  $K_n(\mathbb{R}_+^n)$  we refer to [1], [5], [8], [10] and [11]).

Throughout this paper, we denote by  $G_{m,n}(x, y)$  the Green's function of the operator  $u \mapsto (-\Delta)^m u$  on  $\mathbb{R}_+^n$  with Navier boundary conditions  $\Delta^j u|_{\partial \mathbb{R}_+^n} = 0$ ,  $0 \leq j \leq m - 1$  for  $m \geq 1$  and  $n \geq \max(3, 2m - 1)$ .

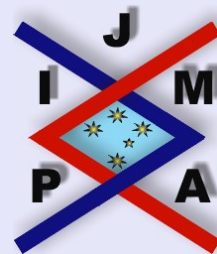
The outline of the paper is as follows. In Section 2, we give explicitly the expression of  $G_{m,n}(x, y)$  and we prove some inequalities on  $G_{m,n}(x, y)$  including a 3G-Theorem of the form (1.1). In Section 3, we introduce a class of functions  $K_{m,n}(\mathbb{R}_+^n)$  defined as follows:

**Definition 1.3.** A Borel measurable function  $q$  in  $\mathbb{R}_+^n$  belongs to the class  $K_{m,n}(\mathbb{R}_+^n)$  if  $q$  satisfies

$$(1.3) \quad \lim_{r \rightarrow 0} \left( \sup_{x \in \mathbb{R}_+^n} \int_{(|x-y| \leq r) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} G_{m,n}(x, y) |q(y)| dy \right) = 0.$$

We then study properties of functions belonging to this class. In particular, we prove the following characterization for  $n > 2m$ :

$$(1.4) \quad q \in K_{m,n}(\mathbb{R}_+^n) \Leftrightarrow \lim_{t \rightarrow 0} \left( \sup_{x \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \int_0^t \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| ds dy \right) = 0,$$



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which extends (1.2).

In order to simplify our statements, we define some convenient notations.

**Notations:**

- $\mathcal{B}(\mathbb{R}_+^n)$  denotes the set of Borel measurable functions in  $\mathbb{R}_+^n$ .
- $s \wedge t = \min(s, t)$  and  $s \vee t = \max(s, t)$  for  $s, t \in \mathbb{R}$ .
- Let  $f$  and  $g$  be two nonnegative functions on a set  $S$ .  
We say  $f \preceq g$  if there exists a constant  $c > 0$ , such that

$$f(x) \leq cg(x) \text{ for all } x \in S.$$

We say  $f \sim g$  if

$$f \preceq g \text{ and } g \preceq f.$$

- Let  $x, y \in \mathbb{R}_+^n$ . Put  $\bar{y} = (y_1, \dots, y_{n-1}, -y_n)$ . Then we have

$$|x - \bar{y}|^2 = |x - y|^2 + 4x_n y_n \text{ and } |x - \bar{y}|^2 \geq (x_n + y_n)^2,$$

which implies that

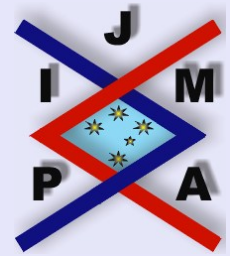
$$(1.5) \quad |x - \bar{y}|^2 \sim |x - y|^2 + x_n y_n$$

$$(1.6) \quad x_n \vee y_n \leq |x - \bar{y}|.$$

The following properties will be used several times.

- (i) For  $s, t \geq 0$ , we have

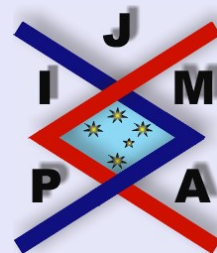
$$(1.7) \quad s \wedge t \sim \frac{st}{s+t}.$$



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(ii) Let  $\lambda, \mu > 0$  and  $0 < \gamma \leq 1$ , then we have

$$(1.8) \quad 1 - t^\lambda \sim 1 - t^\mu, \text{ for } t \in [0, 1].$$

$$(1.9) \quad \log(1 + \lambda t) \sim \log(1 + \mu t), \text{ for } t \geq 0.$$

$$(1.10) \quad \log(1 + t^\lambda) \sim (1 \wedge t^\lambda) \log(2 + t), \text{ for } t \geq 0.$$

$$(1.11) \quad \log(1 + t) \preceq t^\gamma, \text{ for } t \geq 0.$$

(iii) Let  $a > 0$ , then we have

$$(1.12) \quad 1 - e^{-a} \sim \min(1, a).$$

## 2. Inequalities for the Green's Function

In the sequel for  $t > 0$ ,  $x$  and  $y \in \mathbb{R}_+^n$ , we denote by

$$\begin{aligned} p(t, x, y) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \left( \exp\left(-\frac{|x-y|^2}{4t}\right) - \exp\left(-\frac{|x-\bar{y}|^2}{4t}\right) \right) \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) \left(1 - \exp\left(-\frac{x_n y_n}{t}\right)\right), \end{aligned}$$

the density of the Gauss semigroup on  $\mathbb{R}_+^n$ . Then the Green's function of  $\Delta$  with the Dirichlet condition on  $\partial\mathbb{R}_+^n$  is given by

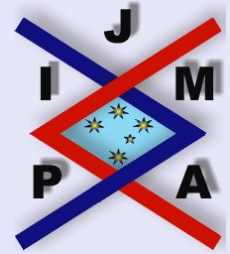
$$(2.1) \quad G(x, y) = \int_0^\infty p(t, x, y) dt.$$

Let  $G_{m,n}(x, y)$  be the Green's function of the operator  $u \mapsto (-\Delta)^m u$  on  $\mathbb{R}_+^n$  with Navier boundary conditions  $\Delta^j u|_{\partial\mathbb{R}_+^n} = 0$ ,  $0 \leq j \leq m-1$ .

Then  $G_{m,n}$  satisfies for  $m \geq 2$ ,

$$G_{m,n}(x, y) = \int_{\mathbb{R}_+^n} \cdots \int_{\mathbb{R}_+^n} G(x, z_1) G(z_1, z_2) \cdots G(z_{m-1}, y) dz_1 \cdots dz_{m-1}.$$

Moreover, using the Fubini theorem, (2.1) and the Chapman-Kolmogorov iden-



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tity we have

$$\begin{aligned}
 G_{m,n}(x, y) &= \int_{\mathbb{R}_+^n} \cdots \int_{\mathbb{R}_+^n} G(x, z_1)G(z_1, z_2)dz_1)G(z_2, z_3) \cdots G(z_{m-1}, y)dz_2 \cdots dz_{m-1} \\
 &= \int_{\mathbb{R}_+^n} \cdots \int_{\mathbb{R}_+^n} \left( \int_0^\infty \int_0^\infty p(t_1+t_2, x, z_2)dt_1dt_2 \right) G(z_2, z_3) \cdots G(z_{m-1}, y)dz_2 \cdots dz_{m-1} \\
 &= \int_0^\infty \cdots \int_0^\infty p(t_1 + t_2 + \cdots + t_m, x, y)dt_1 \cdots dt_m.
 \end{aligned}$$

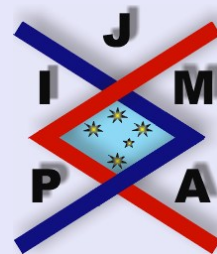
A simple computation shows that for each  $m \geq 1$  and  $x, y \in \mathbb{R}_+^n$

$$(2.2) \quad G_{m,n}(x, y) = \frac{1}{(m-1)!} \int_0^\infty s^{m-1} p(s, x, y) ds.$$

Next, we purpose to give an explicit expression for  $G_{m,n}$ .

Let  $\delta > 0$  and  $x, y \in \mathbb{R}_+^n$  such that  $x \neq y$ . Put  $a = \frac{|x-y|}{2}$  and  $b = \frac{|x-\bar{y}|}{2}$ . Then we have

$$\begin{aligned}
 (2.3) \quad \int_0^\delta s^{m-1} p(s, x, y) ds &= \alpha_{m,n} \left( |x-y|^{2m-n} \int_{\frac{|x-y|^2}{4\delta}}^\infty r^{\left(\frac{n-2m}{2}\right)-1} e^{-r} dr \right. \\
 &\quad \left. - |x-\bar{y}|^{2m-n} \int_{\frac{|x-\bar{y}|^2}{4\delta}}^\infty r^{\left(\frac{n-2m}{2}\right)-1} e^{-r} dr \right) \\
 &= \beta_{m,n} \int_{\frac{1}{\delta}}^\infty \xi^{\left(\frac{n-2m}{2}\right)-1} (e^{-a^2\xi} - e^{-b^2\xi}) d\xi,
 \end{aligned}$$



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where  $\alpha_{m,n}$  and  $\beta_{m,n}$  are some positive constants.

Hence, using this fact and (2.2) it follows that

$$\lim_{\delta \rightarrow \infty} \int_0^\delta s^{m-1} p(s, x, y) ds = (m-1)! G_{m,n}(x, y) < \infty \text{ for } x \neq y \iff 2m-n < 2.$$

Moreover, we deduce from (2.3) by letting  $\delta \rightarrow \infty$ , the following explicit expression of  $G_{m,n}$ .

**Proposition 2.1.** *For each  $x, y \in \mathbb{R}_+^n$ , we have*

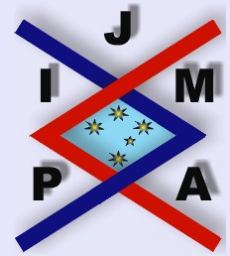
$$G_{m,n}(x, y) = \begin{cases} a_{m,n} \left( \frac{1}{|x-y|^{n-2m}} - \frac{1}{|x-\bar{y}|^{n-2m}} \right), & \text{if } n > 2m, \\ b_{m,n} \log \left( 1 + \frac{4x_n y_n}{|x-y|^2} \right), & \text{if } n = 2m, \\ c_{m,n} (|x-\bar{y}| - |x-y|), & \text{if } n = 2m - 1, \end{cases}$$

where  $a_{m,n}$ ,  $b_{m,n}$  and  $c_{m,n}$  are some positive constants.

**Corollary 2.2.** *For each  $x, y \in \mathbb{R}_+^n$ , we have*

(i) *For  $n > 2m$ ,*

$$G_{m,n}(x, y) \sim \frac{x_n y_n}{|x-y|^{n-2m} |x-\bar{y}|^2} \sim \frac{1}{|x-y|^{n-2m}} \left( 1 \wedge \frac{x_n y_n}{|x-y|^2} \right).$$



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(ii) For  $n = 2m$ ,

$$\begin{aligned} G_{m,n}(x, y) &\sim \left(1 \wedge \frac{x_n y_n}{|x - y|^2}\right) \log \left(2 + \frac{x_n y_n}{|x - y|^2}\right) \\ &\sim \frac{x_n y_n}{|x - \bar{y}|^2} \log \left(1 + \frac{|x - \bar{y}|^2}{|x - y|^2}\right). \end{aligned}$$

(iii) For  $n = 2m - 1$ ,

$$G_{m,n}(x, y) \sim \frac{x_n y_n}{|x - \bar{y}|} \sim (x_n y_n)^{\frac{1}{2}} \left(1 \wedge \frac{(x_n y_n)^{\frac{1}{2}}}{|x - y|}\right).$$

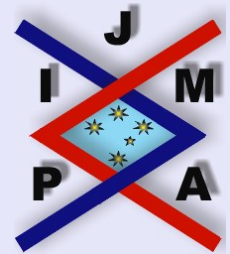
*Proof.* The proof follows immediately from Proposition 2.1 and the statements (1.8), (1.5) and (1.7) for  $n > 2m$  or  $n = 2m - 1$  and using further (1.9) – (1.10) for  $n = 2m$ .  $\square$

**Corollary 2.3.** For each  $x, y \in \mathbb{R}_+^n$  we have

$$\frac{y_n}{x_n} G_{m,n}(x, y) \preceq \begin{cases} \frac{1}{|x - y|^{n-2m}}, & \text{if } n > 2m, \\ 1 + G_{m,n}(x, y) & \text{if } n = 2m, \\ x_n \wedge y_n & \text{if } n = 2m - 1. \end{cases}$$

**Remark 1.** For each  $x, y \in \mathbb{R}_+^n$  we have

$$\frac{y_n}{x_n} G_{m,n}(x, y) \preceq \frac{1}{|x - y|^{n-2m}} \left(1 \wedge \left(\frac{y_n}{x_n}\right)^2\right), \text{ if } n > 2m.$$



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Indeed, from Corollary 2.3, we have

$$\frac{y_n}{x_n} G_{m,n}(x, y) \preceq \frac{1}{|x - y|^{n-2m}}.$$

Interchanging the role of  $x$  and  $y$ , we get

$$G_{m,n}(x, y) \preceq \frac{y_n}{x_n} \cdot \frac{1}{|x - y|^{n-2m}},$$

which implies the result.

The next lemma is crucial in this work.

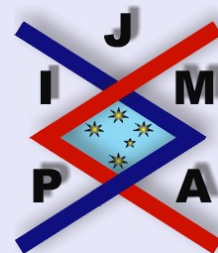
**Lemma 2.4 (see [7]).** Let  $x, y \in \mathbb{R}_+^n$ . Then we have the following properties:

1. If  $x_n y_n \leq |x - y|^2$ , then

$$(x_n \vee y_n) \leq \frac{(\sqrt{5} + 1)}{2} |x - y|.$$

2. If  $|x - y|^2 \leq x_n y_n$ , then

$$\left( \frac{3 - \sqrt{5}}{2} \right) x_n \leq y_n \leq \left( \frac{3 + \sqrt{5}}{2} \right) x_n.$$



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**Corollary 2.5.** For each  $x, y \in \mathbb{R}_+^n$ , we have

$$(2.4) \quad G_{m,n}(x, y) \preceq \frac{x_n y_n}{|x - y|^{n-2m+2}},$$

$$(2.5) \quad \frac{x_n y_n}{(|x| + 1)^{n-2m+2} (|y| + 1)^{n-2m+2}} \preceq G_{m,n}(x, y),$$

$$(2.6) \quad G_{m,n}(x, y) \preceq \frac{x_n \wedge y_n}{|x - y|^{n+1-2m}}.$$

*Proof.* The assertions (2.4) and (2.5) follow from Corollary 2.2 and the fact that

$$|x - y| \leq |x - \bar{y}| \leq (|x| + 1)(|y| + 1)$$

and

$$\frac{t}{1+t} \leq \log(1+t) \leq t,$$

for  $t \geq 0$ .

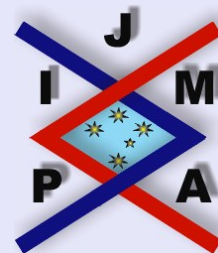
To prove (2.6) we claim that

$$(2.7) \quad G_{m,n}(x, y) \preceq \frac{x_n}{|x - y|^{n+1-2m}}.$$

Indeed, we have the following cases:

**Case 1.** If  $n > 2m$  or  $n = 2m - 1$ , the inequality (2.7) follows from Corollary 2.2, (1.6) and the fact that  $|x - \bar{y}| \geq |x - y|$ .

**Case 2.** If  $n = 2m$ , then we have the following subcases:



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1. If  $|x - y|^2 \leq x_n y_n$ , then by Lemma 2.4, we get  $x_n \sim y_n$ .  
Using this fact, Proposition 2.1, (1.9) and (1.11) we deduce that

$$G_{m,n}(x, y) \sim \log \left( 1 + \frac{cx_n^2}{|x - y|^2} \right), \quad (\text{where } c > 0),$$

$$\preceq \frac{x_n}{|x - y|}.$$

2. If  $x_n y_n \leq |x - y|^2$ , then Lemma 2.4 gives that  $(x_n \vee y_n) \preceq |x - y|$ .  
Hence from (2.4), we deduce that

$$G_{m,n}(x, y) \preceq \frac{x_n y_n}{|x - y|^2} \preceq \frac{x_n}{|x - y|}.$$

This proves (2.7). Interchange the role of  $x$  and  $y$ , we obtain (2.6). □

**Proposition 2.6.** a) For each  $t > 0$ , and all  $x, y \in \mathbb{R}_+^n$ , we have

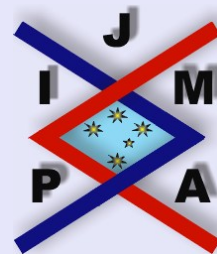
$$\int_0^t s^{m-1} p(s, x, y) ds \preceq G_{m,n}(x, y).$$

b) Let  $t > 0$  and  $x, y \in \mathbb{R}_+^n$ . Then

$$G_{m,n}(x, y) \preceq \int_0^t s^{m-1} p(s, x, y) ds,$$

provided

i)  $n > 2m$  and  $|x - y| \leq 2\sqrt{t}$ ; or



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ii)  $n = 2m$  and  $|x - \bar{y}| \leq 2\sqrt{t}$ ; or

iii)  $n = 2m - 1$  and  $|x - \bar{y}| \leq 2\sqrt{t}$ .

*Proof.* Let  $t > 0$  and  $x, y \in \mathbb{R}_+^n$ . Then a) follows immediately from (2.2).

To prove b) we distinguish three cases.

i) For  $n > 2m$ , using (1.12) and the fact that for  $a, b \in (0, \infty)$  we have

$$(1 \wedge ab) \geq (1 \wedge a)(1 \wedge b),$$

then there exists  $C > 0$  such that for  $|x - y| \leq 2\sqrt{t}$ ,

$$\begin{aligned} \int_0^t s^{m-1} p(s, x, y) ds &\geq C \int_0^t \frac{1}{s^{\frac{n}{2}+1-m}} \exp\left(-\frac{|x-y|^2}{4s}\right) \left(1 \wedge \frac{x_n y_n}{s}\right) ds \\ &\geq \frac{C}{|x-y|^{n-2m}} \int_{\frac{|x-y|^2}{4t}}^{\infty} r^{\frac{n}{2}-1-m} e^{-r} \left(1 \wedge \frac{4r x_n y_n}{|x-y|^2}\right) dr \\ &\geq \frac{C}{|x-y|^{n-2m}} \left(1 \wedge \frac{x_n y_n}{|x-y|^2}\right) \int_{\frac{|x-y|^2}{4t}}^{\infty} r^{\frac{n}{2}-1-m} e^{-r} (1 \wedge 4r) dr \\ &\geq \frac{C}{|x-y|^{n-2m}} \left(1 \wedge \frac{x_n y_n}{|x-y|^2}\right) \int_1^{\infty} r^{\frac{n}{2}-1-m} e^{-r} dr \\ &\geq \frac{C}{|x-y|^{n-2m}} \left(1 \wedge \frac{x_n y_n}{|x-y|^2}\right). \end{aligned}$$

Hence the result follows from Corollary 2.2.



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ii) For  $n = 2m$ , by (2.3), there exists  $C > 0$  such that for  $|x - \bar{y}| \leq 2\sqrt{t}$

$$\int_0^t s^{m-1} p(s, x, y) ds = \alpha_{m,n} \int_{\frac{|x-\bar{y}|^2}{4t}}^{\frac{|x-\bar{y}|^2}{4t}} \frac{e^{-r}}{r} dr \geq C \log \left( \frac{|x-\bar{y}|^2}{|x-y|^2} \right).$$

Hence the result follows from Proposition 2.1.

iii) Let  $n = 2m - 1$ ,  $t > 0$  and  $x, y \in \mathbb{R}_+^n$  such that  $|x - \bar{y}| \leq 2\sqrt{t}$ .

Put  $a = \frac{|x-y|}{2\sqrt{t}}$  and  $b = \frac{|x-\bar{y}|}{2\sqrt{t}}$ . Then using (2.3), we obtain

$$I := \int_0^t s^{m-1} p(s, x, y) ds = 2\alpha_{m,n} \sqrt{t} \left( a \int_{a^2}^{\infty} r^{-\frac{3}{2}} e^{-r} dr - b \int_{b^2}^{\infty} r^{-\frac{3}{2}} e^{-r} dr \right).$$

Now since for  $\alpha > 0$ , we have

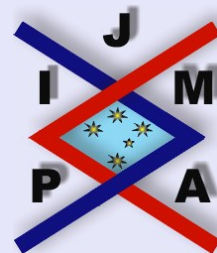
$$\int_{\alpha^2}^{\infty} r^{-\frac{3}{2}} e^{-r} dr = 2 \left( \frac{e^{-\alpha^2}}{\alpha} - \int_{\alpha^2}^{\infty} r^{-\frac{1}{2}} e^{-r} dr \right),$$

we deduce that

$$\begin{aligned} I &= 4\alpha_{m,n} \sqrt{t} \left[ (e^{-a^2} - e^{-b^2}) + b \int_{b^2}^{\infty} r^{-\frac{1}{2}} e^{-r} dr - a \int_{a^2}^{\infty} r^{-\frac{1}{2}} e^{-r} dr \right] \\ &= 4\alpha_{m,n} \sqrt{t} \left[ (b-a) \int_{b^2}^{\infty} r^{-\frac{1}{2}} e^{-r} dr + \int_{a^2}^{b^2} r^{-\frac{1}{2}} e^{-r} (r^{\frac{1}{2}} - a) dr \right]. \end{aligned}$$

Hence

$$I \geq 4\alpha_{m,n} \sqrt{t} (b-a) \int_{b^2}^{\infty} r^{-\frac{1}{2}} e^{-r} dr.$$



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That is

$$\begin{aligned}
 I &\geq 2\alpha_{m,n}(|x - \bar{y}| - |x - y|) \int_{\frac{|x-\bar{y}|^2}{4t}}^{\infty} r^{-\frac{1}{2}} e^{-r} dr \\
 &\geq 2\alpha_{m,n}(|x - \bar{y}| - |x - y|) \int_1^{\infty} r^{-\frac{1}{2}} e^{-r} dr.
 \end{aligned}$$

The result follows from Proposition 2.1. □

Next we purpose to prove that  $G_{m,n}$  satisfies (1.1).

**3G-Theorem.** For  $x, y, z \in \mathbb{R}_+^n$ , we have

$$\frac{G_{m,n}(x, z)G_{m,n}(z, y)}{G_{m,n}(x, y)} \preceq \left[ \frac{z_n}{x_n} G_{m,n}(x, z) + \frac{z_n}{y_n} G_{m,n}(y, z) \right].$$

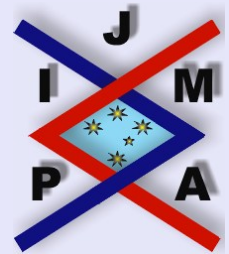
*Proof.* To prove the inequality, we denote by  $A(x, y) := \frac{x_n y_n}{G_{m,n}(x, y)}$  and we claim that  $A$  is a quasi-metric, that is for each  $x, y, z \in \mathbb{R}_+^n$ ,

$$(2.8) \quad A(x, y) \preceq A(x, z) + A(y, z).$$

To this end, we observe that by using Corollary 2.2 and Lemma 2.4, the claim can be proved by similar arguments as in [2], for  $n > 2m$  and as in [3], for  $n = 2m$ .

To prove (2.8), for  $n = 2m - 1$ , we derive from Corollary 2.2 that

$$A(x, y) \sim (|x - y|^2 \vee x_n y_n)^{\frac{1}{2}} \sim |x - \bar{y}|.$$



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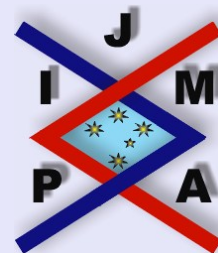
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Now since  $|x - z| \leq |x - \bar{z}|$ , we deduce that

$$\begin{aligned} A(x, y) &\leq |x - z| + |z - \bar{y}| \\ &\leq |x - \bar{z}| + |z - \bar{y}| \leq (A(x, z) + A(y, z)). \end{aligned}$$

□



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### 3. The Class $K_{m,n}(\mathbb{R}_+^n)$

Next we purpose to study and to characterize the class  $K_{m,n}(\mathbb{R}_+^n)$  for  $n > 2m$ .

We recall that for  $0 < \alpha < n$ , we say that a Borel measurable function  $q$  in  $\mathbb{R}_+^n$  belongs to the class  $\tilde{K}_{\alpha,n}(\mathbb{R}_+^n)$  (see [6]) if  $q$  satisfies the following condition

$$(3.1) \quad \lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}_+^n} \int_{(|x-y| \leq r) \cap \mathbb{R}_+^n} \frac{|q(y)|}{|x-y|^{n-\alpha}} dy = 0.$$

The usual Kato class  $K_n(\mathbb{R}_+^n)$ , corresponds to  $\alpha = 2$ .

**Remark 2.** Let  $n > 2m$ . Using Corollary 2.3, the class  $K_{m,n}(\mathbb{R}_+^n)$  obviously includes the class  $\tilde{K}_{2m,n}(\mathbb{R}_+^n)$ . In particular,  $K_n(\mathbb{R}_+^n) \subset K_{m,n}(\mathbb{R}_+^n)$ .

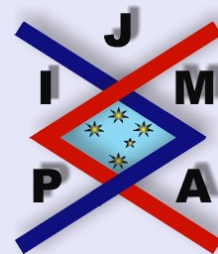
**Example 3.1.** Suppose that for  $p > \frac{n}{2m} > 1$ , we have

$$M_0 = \sup_{x \in \mathbb{R}_+^n} \int_{(|x-y| \leq 1) \cap \mathbb{R}_+^n} \min \left( \left( \frac{y_n}{x_n} \right)^{2p}, 1 \right) |q(y)|^p dy < \infty,$$

then  $q \in K_{m,n}(\mathbb{R}_+^n)$ .

Indeed, let  $0 < r < 1$  and  $x \in \mathbb{R}_+^n$ , then using Remark 1 and the Hölder inequality we get

$$\int_{(|x-y| \leq r) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} G_{m,n}(x, y) |q(y)| dy$$



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$$\begin{aligned} & \leq \int_{(|x-y| \leq r) \cap \mathbb{R}_+^n} \min \left( \left( \frac{y_n}{x_n} \right)^2, 1 \right) \frac{1}{|x-y|^{n-2m}} |q(y)| dy \\ & \leq \left( \int_{(|x-y| \leq r) \cap \mathbb{R}_+^n} \min \left( \left( \frac{y_n}{x_n} \right)^{2p}, 1 \right) |q(y)|^p dy \right)^{\frac{1}{p}} \\ & \quad \times \left( \int_{(|x-y| \leq r) \cap \mathbb{R}_+^n} \frac{1}{|x-y|^{\frac{p-1}{p}(n-2m)}} dy \right)^{\frac{p-1}{p}}. \end{aligned}$$

Hence

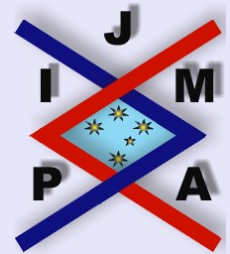
$$\sup_{x \in \mathbb{R}_+^n} \int_{(|x-y| \leq r) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} G_{m,n}(x,y) q(y) dy \leq M_0^{\frac{1}{p}} r^{\frac{2mp-n}{p}} \rightarrow 0 \text{ as } r \rightarrow 0.$$

**Proposition 3.1.** Let  $p > \max \left( \frac{n}{2m}, 1 \right)$  and  $f \in L^p(\mathbb{R}_+^n)$ . Then

$$y \mapsto \frac{f(y)}{(|y| + 1)^{\mu-\lambda} y_n^\lambda} \in K_{m,n}(\mathbb{R}_+^n)$$

provided

- i)  $n > 2m$ ,  $\lambda \leq 2$  and  $\lambda < 2m - \frac{n}{p}$  and  $\mu \geq \max(0, \lambda)$  or
- ii)  $n = 2m$  and  $\lambda < \min(2, 2m - \frac{n}{p}) \leq \mu$  or
- iii)  $n = 2m - 1$ ,  $\lambda \leq 2$  and  $\lambda < 2m - \frac{n}{p}$  and  $\mu \geq \max(1, \lambda)$ .



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*Proof.* Let  $p > \max\left(\frac{n}{2m}, 1\right)$  and  $q \geq 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

For  $f \in L^p(\mathbb{R}_+^n)$ ,  $x \in \mathbb{R}_+^n$  and  $0 < r < 1$ , put

$$I = I(x, r) := \int_{B(x,r) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} G_{m,n}(x, y) \frac{|f(y)|}{(|y| + 1)^{\mu - \lambda} y_n^\lambda} dy.$$

Note that if  $|x - y| \leq r$ , then  $(|x| + 1) \sim (|y| + 1)$ . So, we distinguish the following cases:

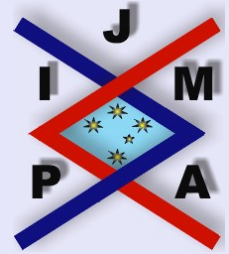
**Case 1.**  $n > 2m$ . Assume that  $\lambda \leq 2$  and  $\lambda < 2m - \frac{n}{p}$ . Let  $\mu \geq \max(0, \lambda)$  and put  $\lambda^+ = \max(\lambda, 0)$ . Then using Corollary 2.2, (1.6) and the fact that  $|x - y| \leq |x - \bar{y}|$ , we deduce by the Hölder inequality that

$$I \preceq \int_{B(x,r) \cap \mathbb{R}_+^n} \frac{|f(y)|}{|x - y|^{n + \lambda^+ - 2m}} dy \preceq \|f\|_p \left( \int_{B(x,r) \cap \mathbb{R}_+^n} \frac{1}{|x - y|^{(n + \lambda^+ - 2m)q}} dy \right)^{\frac{1}{q}} \\ \preceq r^{2m - \frac{n}{p} - \lambda^+}, \text{ which tends to zero if } r \rightarrow 0.$$

**Case 2.**  $n = 2m$ . Assume that  $\lambda < \min\left(2, 2m - \frac{n}{p}\right) \leq \mu$ .

Using Proposition 2.1 and the Hölder inequality, we deduce that

$$I \preceq \|f\|_p \left( \int_{(|x-y| \leq r) \cap \mathbb{R}_+^n} \left(\frac{y_n}{x_n}\right)^q \left(\log\left(1 + \frac{4x_n y_n}{|x - y|^2}\right)\right)^q \frac{1}{(|y| + 1)^{(\mu - \lambda)q} y_n^{\lambda q}} dy \right)^{\frac{1}{q}}$$



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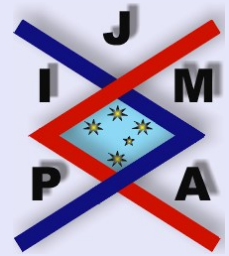


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$$\begin{aligned} & \lesssim \left( \int_{(|x-y|\leq r)\cap D_1} \left(\frac{y_n}{x_n}\right)^q \left(\log\left(1 + \frac{4x_n y_n}{|x-y|^2}\right)\right)^q \frac{1}{(|y|+1)^{(\mu-\lambda)q} y_n^{\lambda q}} dy \right)^{\frac{1}{q}} \\ & + \left( \int_{(|x-y|\leq r)\cap D_2} \left(\frac{y_n}{x_n}\right)^q \left(\log\left(1 + \frac{4x_n y_n}{|x-y|^2}\right)\right)^q \frac{1}{(|y|+1)^{(\mu-\lambda)q} y_n^{\lambda q}} dy \right)^{\frac{1}{q}} \\ & = I_1 + I_2, \end{aligned}$$

where

$$D_1 = \{y \in \mathbb{R}_+^n : x_n y_n \leq |x-y|^2\} \text{ and } D_2 = \{y \in \mathbb{R}_+^n : |x-y|^2 \leq x_n y_n\}.$$

So, using that  $\log(1+t) \leq t$ , for  $t \geq 0$  and Lemma 2.4, we obtain

$$\begin{aligned} I_1 & \lesssim \left( \int_{(|x-y|\leq r)\cap D_1} \frac{y_n^{(2-\lambda)q}}{|x-y|^{2q}} dy \right)^{\frac{1}{q}} \\ & \lesssim \left( \int_{(|x-y|\leq r)\cap D_1} \frac{1}{|x-y|^{\lambda q}} dy \right)^{\frac{1}{q}} \\ & \lesssim r^{2m - \frac{n}{p} - \lambda}, \text{ which converges to zero as } r \rightarrow 0. \end{aligned}$$

On the other hand, from Lemma 2.4 and the fact that  $(|x|+1) \sim (|y|+1)$ , we obtain

$$I_2 \lesssim \frac{1}{x_n^\lambda (|x|+1)^{(\mu-\lambda)}} \left( \int_{(|x-y|\leq r)\cap D_2} \left(\log\left(1 + \frac{(cx_n)^2}{|x-y|^2}\right)\right)^q dy \right)^{\frac{1}{q}},$$

where  $c = 1 + \sqrt{5}$ . Let  $\gamma \in ]\max(0, \lambda), \min(2, 2m - \frac{n}{p})[$ .

Since  $\log(1 + t^2) \leq t^\gamma$ , for  $t \geq 0$ , then

$$\begin{aligned} I_2 &\leq \frac{x_n^{\gamma-\lambda}}{(|x| + 1)^{\mu-\lambda}} \left( \int_{(|x-y|\leq r) \cap D_2} \frac{1}{|x-y|^{\gamma q}} dy \right)^{\frac{1}{q}} \\ &\leq \left( \int_{(|x-y|\leq r) \cap D_2} \frac{1}{|x-y|^{\gamma q}} dy \right)^{\frac{1}{q}} \\ &\leq r^{2m-\frac{n}{p}-\gamma}, \text{ which converges to zero as } r \rightarrow 0. \end{aligned}$$

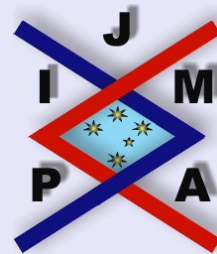
**Case 3.**  $n = 2m - 1$ . Assume that  $\lambda \leq 2$  and  $\lambda < 2m - \frac{n}{p}$ . Let  $\mu \geq \max(1, \lambda)$ .

Using Corollary 2.2 and the Hölder inequality, we obtain

$$\begin{aligned} I &\leq \left[ \left( \int_{B(x,r) \cap D_1} \frac{y_n^{(2-\lambda)q}}{|x-\bar{y}|^q (|y| + 1)^{(\mu-\lambda)q}} dy \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left( \int_{B(x,r) \cap D_2} \frac{y_n^{(2-\lambda)q}}{|x-\bar{y}|^q (|y| + 1)^{(\mu-\lambda)q}} dy \right)^{\frac{1}{q}} \right] \\ &= I_1 + I_2. \end{aligned}$$

Now, if  $y \in D_1$ , then  $|x - \bar{y}| \sim |x - y|$  and so

$$I_1 \leq \left( \int_{B(x,r) \cap D_1} \frac{1}{|x-y|^{(\lambda-1)q}} dy \right)^{\frac{1}{q}} \leq r^{2m-\frac{n}{p}-\lambda}, \text{ which tends to zero as } r \rightarrow 0.$$



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On the other hand, if  $y \in D_2$ , then  $|x - \bar{y}|^2 \sim x_n y_n$  and by Lemma 2.4, we have further  $x_n \sim y_n$ . This implies that

$$\begin{aligned} I_2 &\preceq \frac{x_n^{2-\lambda}}{x_n(|x|+1)^{\mu-\lambda}} \left( \int_{B(x,r) \cap D_2} dy \right)^{\frac{1}{q}} \preceq \frac{x_n^{1-\lambda}}{(|x|+1)^{\mu-\lambda}} (r \wedge cx_n)^{\frac{n}{q}} \\ &\preceq r^{\frac{n}{q}}, \text{ which converges to zero as } r \rightarrow 0. \end{aligned}$$

□

The proof of the next results are similar to the case  $m = 1$  and  $n \geq 3$ , which has been considered in [2]. Since reference [2] is not available, I have chosen to reproduce it here.

**Proposition 3.2.** *Let  $q \in K_{m,n}(\mathbb{R}_+^n)$ , then for each compact  $L \subseteq \mathbb{R}^n$  we have*

$$\sup_{x \in \mathbb{R}_+^n} \int_{(x+L) \cap \mathbb{R}_+^n} \frac{y_n^2}{1+x_n y_n} |q(y)| dy < \infty.$$

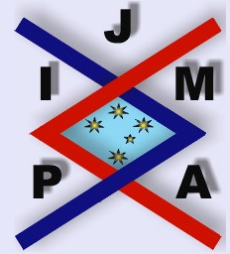
*Proof.* Let  $q \in K_{m,n}(\mathbb{R}_+^n)$ , then by (1.3) there exists  $r > 0$  such that

$$\sup_{x \in \mathbb{R}_+^n} \int_{(|x-y| \leq r) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} G_{m,n}(x, y) |q(y)| dy \leq 1.$$

Let  $a_1, a_2, \dots, a_p \in \mathbb{R}_+^n \cap L$  such that  $\mathbb{R}_+^n \cap L \subseteq \bigcup_{1 \leq i \leq p} B(a_i, r)$ .

Since for  $a, b \in (0, \infty)$ , we have  $\frac{b}{1+ab} \leq 1 + |a - b|$ , then for each  $x, y, z \in \mathbb{R}_+^n$  it follows that

$$\frac{1 + (x_n + z_n)y_n}{1 + x_n y_n} \leq [1 + z_n(1 + |x_n - y_n|)].$$



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Using this fact and Corollary 2.2, we obtain:

For  $n > 2m$ ,

$$\frac{y_n^2}{1 + x_n y_n} \preceq \frac{[1 + z_n(1 + |x_n - y_n|)]}{1 + (x_n + z_n)y_n} |x + z - y|^{n-2m} \\ \times [|x + z - y|^2 + 4(x_n + z_n)y_n] \frac{y_n}{(x_n + z_n)} G_{m,n}(x + z, y).$$

For  $n = 2m$ , using further that  $\frac{t}{1+t} \leq \log(1+t)$ ,  $\forall t \geq 0$ , we have

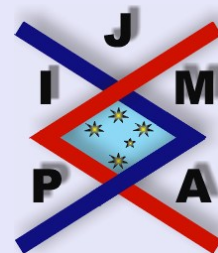
$$\frac{y_n^2}{1 + x_n y_n} \preceq \frac{[1 + z_n(1 + |x_n - y_n|)]}{1 + (x_n + z_n)y_n} \\ \times [|x + z - y|^2 + 4(x_n + z_n)y_n] \frac{y_n}{(x_n + z_n)} G_{m,n}(x + z, y).$$

For  $n = 2m - 1$ ,

$$\frac{y_n^2}{1 + x_n y_n} \preceq \frac{[1 + z_n(1 + |x_n - y_n|)]}{1 + (x_n + z_n)y_n} \\ \times [|x + z - y|^2 + 4(x_n + z_n)y_n]^{\frac{1}{2}} \frac{y_n}{(x_n + z_n)} G_{m,n}(x + z, y).$$

Now, if  $z \in L$  and  $|x + z - y| \leq r$ , then

$$\frac{y_n^2}{1 + x_n y_n} \preceq \frac{y_n}{(x_n + z_n)} G_{m,n}(x + z, y).$$



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Hence

$$\int_{(x+L) \cap \mathbb{R}_+^n} \frac{y_n^2}{1+x_n y_n} |q(y)| dy$$

$$\leq \sum_{i=1}^p \int_{(|x+a_i-y| \leq r) \cap \mathbb{R}_+^n} \frac{y_n}{(x_n + (a_i)_n)} G_{m,n}(x + a_i, y) |q(y)| dy \leq p.$$

So

$$\sup_{x \in \mathbb{R}_+^n} \int_{(x+L) \cap \mathbb{R}_+^n} \frac{y_n^2}{1+x_n y_n} |q(y)| dy < \infty.$$

□

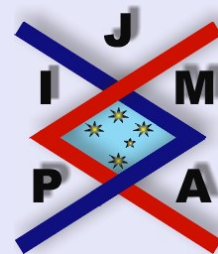
**Corollary 3.3.** *Let  $q \in K_{m,n}(\mathbb{R}_+^n)$ . Then we have for  $M > 0$ ,*

$$\int_{(|y| \leq M) \cap \mathbb{R}_+^n} y_n^2 |q(y)| dy < \infty.$$

**Proposition 3.4.** *Let  $q \in K_{m,n}(\mathbb{R}_+^n)$ , then for each fixed  $\alpha > 0$ , we have*

$$(3.2) \quad \sup_{t \leq 1} \sup_{x \in \mathbb{R}_+^n} \int_{(|x-y| > \alpha) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} p(t, x, y) |q(y)| dy := M(\alpha) < \infty.$$

*Proof.* Let  $q \in K_{m,n}(\mathbb{R}_+^n)$ ,  $0 < t \leq 1$  and without loss of generality assume that



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$0 < \alpha < 1$ . Then by (1.12) and (1.7), it follows that

$$\begin{aligned} & \sup_{x \in \mathbb{R}_+^n} \int_{(|x-y|>\alpha) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} p(t, x, y) |q(y)| dy \\ & \preceq \frac{1}{t^{\frac{n}{2}+1}} e^{-\frac{\alpha^2}{8t}} \sup_{x \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \exp\left(-\frac{|x-y|^2}{8}\right) \frac{y_n^2}{1+x_n y_n} |q(y)| dy. \end{aligned}$$

To conclude, it is sufficient to prove that

$$\sup_{x \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \exp\left(-\frac{|x-y|^2}{8}\right) \frac{y_n^2}{1+x_n y_n} |q(y)| dy < \infty.$$

Indeed, using Proposition 3.2, we have

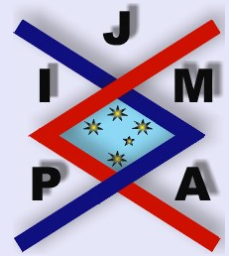
$$\sup_{x \in \mathbb{R}_+^n} \int_{(x+B(0,1)) \cap \mathbb{R}_+^n} \frac{y_n^2}{1+x_n y_n} |q(y)| dy := \widetilde{M} < \infty.$$

Now since any ball  $B(0, k)$ , of radius  $k \geq 1$  in  $\mathbb{R}^n$  can be covered by  $A_n k^n := \alpha(n)$  balls of radius 1, where  $A_n$  is a constant depending only on  $n$  (see [5, p. 67]), then there exists  $a_1, a_2, \dots, a_{\alpha(n)} \in \mathbb{R}_+^n$  such that

$$\mathbb{R}_+^n \cap B(0, k) \subseteq \bigcup_{1 \leq i \leq \alpha(n)} B(a_i, 1).$$

Using the fact that for each  $x, y, z \in \mathbb{R}_+^n$ ,

$$\frac{1 + (x_n + z_n)y_n}{1 + x_n y_n} \leq [1 + z_n(1 + |x_n - y_n|)],$$



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it follows that for all  $x \in \mathbb{R}_+^n$  and  $k \geq 1$  :

$$\begin{aligned} & \int_{(x+B(0,k)) \cap \mathbb{R}_+^n} \frac{y_n^2}{1+x_n y_n} |q(y)| dy \\ & \preceq \sum_{i=1}^{\alpha(n)} \int_{B(x+a_i,1) \cap \mathbb{R}_+^n} \frac{y_n^2}{1+x_n y_n} |q(y)| dy \\ & \preceq \sum_{i=1}^{\alpha(n)} \int_{B(x+a_i,1) \cap \mathbb{R}_+^n} \frac{y_n^2}{1+(x_n+(a_i)_n)y_n} |q(y)| dy \\ & \preceq A_n k^n \widetilde{M}. \end{aligned}$$

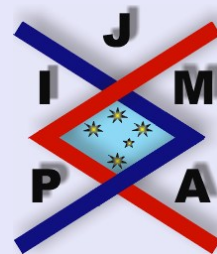
Hence for all  $x \in \mathbb{R}_+^n$ , we have

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \exp\left(-\frac{|x-y|^2}{8}\right) \frac{y_n^2}{1+x_n y_n} |q(y)| dy \\ & \preceq \sum_{k=0}^{\infty} \exp\left(-\frac{\alpha^2 k^2}{8}\right) \int_{[k\alpha \leq |x-y| \leq (k+1)\alpha] \cap \mathbb{R}_+^n} \frac{y_n^2}{1+x_n y_n} |q(y)| dy \\ & \preceq A_n \widetilde{M} \sum_{k=0}^{\infty} (k+1)^n \exp\left(-\frac{\alpha^2 k^2}{8}\right) < \infty. \end{aligned}$$

Thus

$$\sup_{x \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \exp\left(-\frac{|x-y|^2}{8}\right) \frac{y_n^2}{1+x_n y_n} |q(y)| dy < \infty,$$

which completes the proof.  $\square$



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**Theorem 3.5.** Let  $n > 2m$  and  $q \in \mathcal{B}(\mathbb{R}_+^n)$ . Then the following assertions are equivalent:

1.  $q \in K_{m,n}(\mathbb{R}_+^n)$
2.  $\limsup_{t \rightarrow 0} \int_{\mathbb{R}_+^n} \int_0^t \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| ds dy = 0.$

*Proof.* 2)  $\Rightarrow$  1) Assume that

$$\limsup_{t \rightarrow 0} \int_{\mathbb{R}_+^n} \int_0^t \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| ds dy = 0.$$

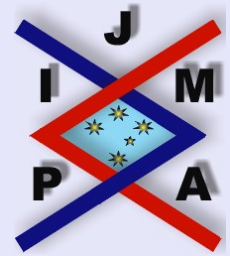
Then by Proposition 2.6, there exists  $c > 0$  such that for  $\alpha > 0$  we have

$$\int_{(|x-y| \leq \alpha) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} G_{m,n}(x, y) |q(y)| dy \leq c \int_{\mathbb{R}_+^n} \int_0^{\frac{\alpha^2}{4}} \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| ds dy,$$

which shows that the function  $q$  satisfies (1.3).

Conversely suppose that  $q \in K_{m,n}(\mathbb{R}_+^n)$ . Let  $\varepsilon > 0$ , then there exists  $0 < \alpha < 1$  such that

$$\sup_{x \in \mathbb{R}_+^n} \int_{(|x-y| \leq \alpha) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} G_{m,n}(x, y) |q(y)| dy \leq \varepsilon.$$



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On the other hand, using Proposition 2.6 and (3.2), we have for  $0 < t < 1$

$$\begin{aligned}
 & \int_{\mathbb{R}_+^n} \int_0^t \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| ds dy \\
 & \quad \preceq \int_{(|x-y| \leq \alpha) \cap \mathbb{R}_+^n} \int_0^t \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| ds dy \\
 & \quad \quad + \int_{(|x-y| > \alpha) \cap \mathbb{R}_+^n} \int_0^t \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| ds dy \\
 & \quad \preceq \int_{(|x-y| \leq \alpha) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} G_{m,n}(x, y) |q(y)| dy \\
 & \quad \quad + \int_0^t \int_{(|x-y| > \alpha) \cap \mathbb{R}_+^n} \frac{y_n}{x_n} p(s, x, y) |q(y)| dy ds \\
 & \quad \preceq \varepsilon + tM(\alpha),
 \end{aligned}$$

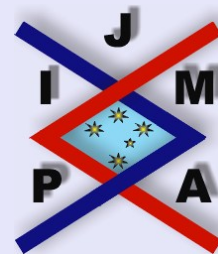
which implies that

$$\limsup_{t \rightarrow 0} \sup_{x \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \int_0^t \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| ds dy = 0.$$

□

**Corollary 3.6.** *Let  $n > 2m$  and  $q \in \mathcal{B}(\mathbb{R}_+^n)$ . For  $\alpha > 0$  and  $x \in \mathbb{R}_+^n$ , put*

$$G_\alpha q(x) := \int_{\mathbb{R}_+^n} \int_0^\infty e^{-\alpha s} \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| ds dy.$$



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Then

$$q \in K_{m,n}(\mathbb{R}_+^n) \Leftrightarrow \lim_{\alpha \rightarrow +\infty} \|G_\alpha q\|_\infty = 0,$$

where  $\|G_\alpha q\|_\infty = \sup_{x \in \mathbb{R}_+^n} |G_\alpha q(x)|$ .

*Proof.* (see [9]). Let  $q \in K_{m,n}(\mathbb{R}_+^n)$ ,  $\alpha > 0$  and put

$$a(\alpha) = \sup_{x \in \mathbb{R}_+^n} \int_0^{\frac{1}{\alpha}} \int_{\mathbb{R}_+^n} \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| dy ds.$$

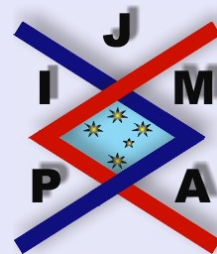
Then we have

$$\begin{aligned} G_\alpha q(x) &= \int_0^\infty \alpha e^{-\alpha t} \left[ \int_0^t \int_{\mathbb{R}_+^n} \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| dy ds \right] dt \\ &= \int_0^\infty e^{-t} \left[ \int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}_+^n} \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| dy ds \right] dt. \end{aligned}$$

It follows that,  $\frac{1}{e} a(\alpha) \leq \|G_\alpha q\|_\infty$ .

On the other hand, for  $t > 0$  and  $k \in \mathbb{N}$  such that  $k \leq t < k + 1$ , we have

$$\begin{aligned} G_\alpha q(x) &\leq \sum_{k=0}^m \int_0^\infty e^{-t} \left[ \int_{\frac{k}{\alpha}}^{\frac{k+1}{\alpha}} \int_{\mathbb{R}_+^n} \frac{y_n}{x_n} s^{m-1} p(s, x, y) |q(y)| dy ds \right] dt \\ &\leq a(\alpha) \int_0^\infty e^{-t} (m + 1) dt \\ &\leq a(\alpha) \int_0^\infty e^{-t} (t + 1) dt = 2a(\alpha), \end{aligned}$$



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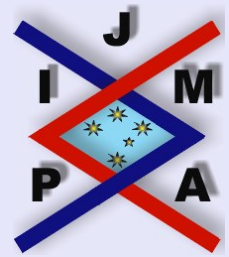
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which gives that  $\frac{1}{e}a(\alpha) \leq \|G_{\alpha}g\|_{\infty} \leq 2a(\alpha)$ .

Hence the results follow from Theorem 3.5. □



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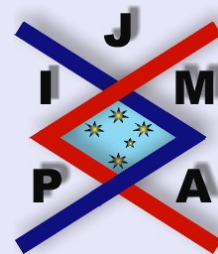
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## References

- [1] M. AIZENMAN AND B. SIMON, Brownian motion and Harnack inequality for Schrödinger operators, *Comm. Pure Appl. Math.*, **XXXV** (1982), 209–273.
- [2] I. BACHAR AND H. MÂAGLI, Estimates on the Green's function and existence of positive solutions of nonlinear singular elliptic equations in the half space, *Positivity*, **9**(2) (2005), 153–192.
- [3] I. BACHAR, H. MÂAGLI AND L. MÂATOUG, Positive solutions of nonlinear elliptic equations in a half space in  $\mathbb{R}^2$ , *E.J.D.E.*, **2002** (2002), No. 41, 1–24.
- [4] I. BACHAR, H. MÂAGLI AND M. ZRIBI, Estimates on the Green function and existence of positive solutions for some polyharmonic nonlinear equations in the half space, *Manuscripta Math.*, **113**, (2004), 269–291.
- [5] K.L. CHUNG AND Z. ZHAO, *From Brownian Motion to Schrödinger's Equation*, Springer Verlag (1995).
- [6] E.B. DAVIES AND A.M. HINZ, Kato class potentials for higher order elliptic operators, *J. London Math. Soc.*, (2) **58** (1998) 669–678.
- [7] H. MÂAGLI, Inequalities for the Riesz potentials, *Archives of Inequalities and Applications*, **1** (2003) 285–294.
- [8] B. SIMON, Schrödinger semi-groups, *Bull. Amer. Math. Soc.*, **7**(3) (1982), 447–526.



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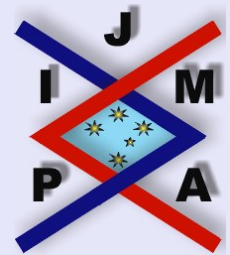
Close

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- [9] J.A. VAN CASTEREN, *Generators Strongly Continuous Semi-groups*, Pitman Advanced Publishing Program, Boston, (1985).
- [10] Z. ZHAO, Subcriticality and gaugeability of the schrödinger operator, *Trans. Amer. Math. Society*, **334**(1) (1992), 75–96.
- [11] Z. ZHAO, On the existence of positive solutions of nonlinear elliptic equations. A probabilistic potential theory approach, *Duke Math. J.*, **69** (1993), 247–258.



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