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ESTIMATES FOR THE GREEN FUNCTION AND CHARACTERIZATION OF A CERTAIN KATO CLASS BY THE GAUSS SEMIGROUP IN THE HALF SPACE

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Abstract

We establish a $3G$-theorem for the Green functions $G_{m,n}$ of $(-\Delta)^m$ $(m \geq 1)$ on $\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$, $n \geq 2m-1$, with Navier boundary conditions $\Delta^j u \mid_{\partial\mathbb{R}^n_+} = 0$, $0 \leq j \leq m-1$.

We exploit these results to define a certain Kato class of functions that we characterize by means of the Gauss semigroup on $\mathbb{R}^n_+$.

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1. Introduction

In [2], for $n \geq 3$ and [3], for $n = 2$, the authors have established interesting estimates for $G(x, y)$, the Green function of the Laplace operator corresponding to zero Dirichlet boundary conditions in the half space $\mathbb{R}^n_+ := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_n > 0\}$. In particular, they have proved the following form of the $3G$-Theorem:

Theorem 1.1. *There exists a constant $C > 0$ such that for each $x, y, z \in \mathbb{R}^n_+$*

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C\left[\frac{z_n}{x_n}G(x,z) + \frac{z_n}{y_n}G(y,z)\right]. \tag{1.1}$$

They then introduced a class of functions $K_{1,n}(\mathbb{R}^n_+)$ as follows:

Definition 1.1. *A Borel measurable function q in $\mathbb{R}^n_+$ belongs to the class $K_{1,n}(\mathbb{R}^n_+)$ if q satisfies the following condition*

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n_+} \int_{(|x-y| \leq r) \cap \mathbb{R}^n_+} \frac{y_n}{x_n} G(x,y) \, |q(y)| \, dy = 0.$$

They have studied the properties of functions belonging to this class.

In particular, in [2], the authors have showed the following characterization:

$$q \in K_{1,n}(\mathbb{R}^n_+) \Longleftrightarrow \lim_{t \to 0} \left(\sup_{x \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \int_0^t \frac{y_n}{x_n} p(s,x,y) \, |q(y)| \, ds dy \right) = 0, \tag{1.2}$$

where $p(s, x, y)$ is the density of the Gauss semigroup on $\mathbb{R}^n_+$.

Note that similar characterizations have been already established in [1], (see also [5] and [8]) for the classical Kato class $K_n(\mathbb{R}^n)$ defined as follows:

Definition 1.2. *A Borel measurable function q in $\mathbb{R}^n_+$ $(n \geq 3)$ belongs to the Kato class $K_n\left(\mathbb{R}^n_+\right)$ if q satisfies the following condition*

$$\lim_{\alpha \to 0} \left(\sup_{x \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+ \cap B(x,\alpha)} \frac{1}{|x-y|^{n-2}} |q(y)| dy \right) = 0.$$

For properties of functions in $K_n(\mathbb{R}^n_+)$ we refer to [1], [5], [8], [10] and [11]).

Throughout this paper, we denote by $G_{m,n}(x,y)$ the Green's function of the operator $u \mapsto (-\Delta)^m u$ on $\mathbb{R}^n_+$ with Navier boundary conditions $\Delta^j u \mid_{\partial \mathbb{R}^n_+} = 0$, $0 \leq j \leq m-1$ for $m \geq 1$ and $n \geq \max(3, 2m-1)$.

The outline of the paper is as follows. In Section 2, we give explicitly the expression of $G_{m,n}(x,y)$ and we prove some inequalities on $G_{m,n}(x,y)$ including a $3G$-Theorem of the form (1.1). In Section 3, we introduce a class of functions $K_{m,n}(\mathbb{R}^n_+)$ defined as follows:

Definition 1.3. *A Borel measurable function q in $\mathbb{R}^n_+$ belongs to the class $K_{m,n}(\mathbb{R}^n_+)$ if q satisfies*

$$(1.3) \qquad \lim_{r \to 0} \left(\sup_{x \in \mathbb{R}^n_+} \int_{(|x-y| \leq r) \cap \mathbb{R}^n_+} \frac{y_n}{x_n} G_{m,n}(x,y) \, |q(y)| \, dy \right) = 0.$$

We then study properties of functions belonging to this class. In particular, we prove the following characterization for $n > 2m$:

$$(1.4) \qquad q \in K_{m,n}(\mathbb{R}^n_+) \Leftrightarrow \lim_{t \to 0} \left(\sup_{x \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \int_0^t \frac{y_n}{x_n} s^{m-1} p(s,x,y) \, |q(y)| \, ds dy \right) = 0,$$

which extends (1.2).

In order to simplify our statements, we define some convenient notations.

Notations:

- $\mathcal{B}(\mathbb{R}^n_+)$ denotes the set of Borel measurable functions in $\mathbb{R}^n_+$.
- $s \wedge t = \min(s,t)$ and $s \vee t = \max(s,t)$ for $s,t \in \mathbb{R}$.
- Let f and g be two nonnegative functions on a set S.
  We say $f \preceq g$ if there exists a constant $c > 0$, such that
  $$f(x) \leq cg(x) \text{ for all } x \in S.$$
  We say $f \sim g$ if
  $$f \preceq g \text{ and } g \preceq f.$$
- Let $x, y \in \mathbb{R}^n_+$. Put $\overline{y} = (y_1, \dots, y_{n-1}, -y_n)$. Then we have
  $$|x - \overline{y}|^2 = |x - y|^2 + 4x_n y_n \text{ and } |x - \overline{y}|^2 \geq (x_n + y_n)^2,$$
  which implies that
  $$|x - \overline{y}|^2 \sim |x - y|^2 + x_n y_n \tag{1.5}$$
  $$x_n \vee y_n \leq |x - \overline{y}|. \tag{1.6}$$

  The following properties will be used several times.

  (i) For $s, t \geq 0$, we have
  $$s \wedge t \sim \frac{st}{s+t}. \tag{1.7}$$

(ii) Let $\lambda, \mu > 0$ and $0 < \gamma \leq 1$, then we have

$$1 - t^{\lambda} \sim 1 - t^{\mu}, \text{ for } t \in [0,1]. \tag{1.8}$$

$$\log(1 + \lambda t) \sim \log(1 + \mu t), \text{ for } t \geq 0. \tag{1.9}$$

$$\log\left(1 + t^{\lambda}\right) \sim \left(1 \wedge t^{\lambda}\right) \log\left(2 + t\right), \text{ for } t \geq 0. \tag{1.10}$$

$$\log(1 + t) \preceq t^{\gamma}, \text{ for } t \geq 0. \tag{1.11}$$

(iii) Let $a > 0$, then we have

$$1 - e^{-a} \sim \min(1, a). \tag{1.12}$$

2. Inequalities for the Green's Function

In the sequel for $t > 0$, x and $y \in \mathbb{R}^n_+$, we denote by

$$p(t,x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\left(\exp\left(-\frac{|x-y|^2}{4t}\right) - \exp\left(-\frac{|x-\overline{y}|^2}{4t}\right)\right)$$
$$= \frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-\frac{|x-y|^2}{4t}\right)\left(1-\exp\left(-\frac{x_n y_n}{t}\right)\right),$$

the density of the Gauss semigroup on $\mathbb{R}^n_+$. Then the Green's function of Δ with the Dirichlet condition on $\partial\mathbb{R}^n_+$ is given by

(2.1) $$G(x,y) = \int_0^\infty p(t,x,y)dt.$$

Let $G_{m,n}(x,y)$ be the Green's function of the operator $u \mapsto (-\Delta)^m u$ on $\mathbb{R}^n_+$ with Navier boundary conditions $\Delta^j u \mid_{\partial\mathbb{R}^n_+} = 0$, $0 \leq j \leq m-1$.

Then $G_{m,n}$ satisfies for $m \geq 2$,

$$G_{m,n}(x,y) = \int_{\mathbb{R}^n_+}\cdots\int_{\mathbb{R}^n_+} G(x,z_1)G(z_1,z_2)\cdots G(z_{m-1},y)dz_1\ldots dz_{m-1}.$$

Moreover, using the Fubini theorem, (2.1) and the Chapman-Kolmogorov identity we have

$$\begin{aligned} G_{m,n}(x,y) &= \int_{\mathbb{R}^n_+} \cdots \int_{\mathbb{R}^n_+} G(x,z_1)G(z_1,z_2)dz_1)G(z_2,z_3)\cdots G(z_{m-1},y)dz_2\dots dz_{m-1} \\ &= \int_{\mathbb{R}^n_+} \cdots \int_{\mathbb{R}^n_+} \left(\int_0^\infty \int_0^\infty p(t_1+t_2,x,z_2)dt_1dt_2 \right) G(z_2,z_3)\cdots G(z_{m-1},y)dz_2\dots dz_{m-1} \\ &= \int_0^\infty \cdots \int_0^\infty p(t_1+t_2+\cdots+t_m,x,y)dt_1\dots dt_m. \end{aligned}$$

A simple computation shows that for each $m \geq 1$ and $x, y \in \mathbb{R}^n_+$

$$G_{m,n}(x,y) = \frac{1}{(m-1)!}\int_0^\infty s^{m-1}p(s,x,y)ds. \tag{2.2}$$

Next, we purpose to give an explicit expression for $G_{m,n}$.

Let $\delta > 0$ and $x, y \in \mathbb{R}^n_+$ such that $x \neq y$. Put $a = \frac{|x-y|}{2}$ and $b = \frac{|x-\overline{y}|}{2}$. Then we have

$$\begin{aligned} \int_0^\delta s^{m-1}p(s,x,y)ds &= \alpha_{m,n}\Bigg(|x-y|^{2m-n}\int_{\frac{|x-y|^2}{4\delta}}^\infty r^{(\frac{n-2m}{2})-1}e^{-r}dr \\ &\qquad - |x-\overline{y}|^{2m-n}\int_{\frac{|x-\overline{y}|^2}{4\delta}}^\infty r^{(\frac{n-2m}{2})-1}e^{-r}dr \Bigg) \\ &= \beta_{m,n}\int_{\frac{1}{\delta}}^\infty \xi^{(\frac{n-2m}{2})-1}(e^{-a^2\xi} - e^{-b^2\xi})d\xi, \end{aligned} \tag{2.3}$$

where $\alpha_{m,n}$ and $\beta_{m,n}$ are some positive constants.

Hence, using this fact and (2.2) it follows that

$$\lim_{\delta\to\infty}\int_0^\delta s^{m-1}p(s,x,y)ds = (m-1)!G_{m,n}(x,y) < \infty \text{ for } x \neq y \Longleftrightarrow 2m-n < 2.$$

Moreover, we deduce from (2.3) by letting $\delta \to \infty$, the following explicit expression of $G_{m,n}$.

Proposition 2.1. *For each $x, y \in \mathbb{R}^n_+$, we have*

$$G_{m,n}(x,y) = \begin{cases} a_{m,n}\left(\frac{1}{|x-y|^{n-2m}} - \frac{1}{|x-\overline{y}|^{n-2m}}\right), & \textit{if } n > 2m, \\ b_{m,n}\log\left(1+\frac{4x_n y_n}{|x-y|^2}\right), & \textit{if } n = 2m, \\ c_{m,n}(|x-\overline{y}| - |x-y|\,), & \textit{if } n = 2m-1, \end{cases}$$

where $a_{m,n}$, $b_{m,n}$ and $c_{m,n}$ are some positive constants.

Corollary 2.2. *For each $x, y \in \mathbb{R}^n_+$, we have*

(i) For $n > 2m$,

$$G_{m,n}(x,y) \sim \frac{x_n y_n}{|x-y|^{n-2m}\,|x-\overline{y}|^2} \sim \frac{1}{|x-y|^{n-2m}}\left(1 \wedge \frac{x_n y_n}{|x-y|^2}\right).$$

(ii) For $n = 2m$,

$$G_{m,n}(x,y) \sim \left(1 \wedge \frac{x_n y_n}{|x-y|^2}\right) \log\left(2 + \frac{x_n y_n}{|x-y|^2}\right)$$
$$\sim \frac{x_n y_n}{|x-\overline{y}|^2} \log\left(1 + \frac{|x-\overline{y}|^2}{|x-y|^2}\right).$$

(iii) For $n = 2m-1$,

$$G_{m,n}(x,y) \sim \frac{x_n y_n}{|x-\overline{y}|} \sim (x_n y_n)^{\frac{1}{2}} \left(1 \wedge \frac{(x_n y_n)^{\frac{1}{2}}}{|x-y|}\right).$$

Proof. The proof follows immediately from Proposition 2.1 and the statements (1.8), (1.5) and (1.7) for $n > 2m$ or $n = 2m-1$ and using further (1.9) – (1.10) for $n = 2m$. □

Corollary 2.3. *For each $x, y \in \mathbb{R}^n_+$ we have*

$$\frac{y_n}{x_n} G_{m,n}(x,y) \preceq \begin{cases} \frac{1}{|x-y|^{n-2m}}, & \text{if } n > 2m, \\ 1 + G_{m,n}(x,y) & \text{if } n = 2m, \\ x_n \wedge y_n & \text{if } n = 2m-1. \end{cases}$$

Remark 1. *For each $x, y \in \mathbb{R}^n_+$ we have*

$$\frac{y_n}{x_n} G_{m,n}(x,y) \preceq \frac{1}{|x-y|^{n-2m}} \left(1 \wedge \left(\frac{y_n}{x_n}\right)^2\right), \quad \text{if } n > 2m.$$

Indeed, from Corollary 2.3, we have

$$\frac{y_n}{x_n} G_{m,n}(x, y) \preceq \frac{1}{|x - y|^{n-2m}}.$$

Interchanging the role of x and y, we get

$$G_{m,n}(x, y) \preceq \frac{y_n}{x_n} \cdot \frac{1}{|x - y|^{n-2m}},$$

which implies the result.

The next lemma is crucial in this work.

Lemma 2.4 (see [7]). *Let $x, y \in \mathbb{R}^n_+$. Then we have the following properties:*

1. *If $x_n y_n \leq |x - y|^2$, then*

$$(x_n \vee y_n) \leq \frac{(\sqrt{5} + 1)}{2} |x - y|.$$

2. *If $|x - y|^2 \leq x_n y_n$, then*

$$\left(\frac{3 - \sqrt{5}}{2}\right) x_n \leq y_n \leq \left(\frac{3 + \sqrt{5}}{2}\right) x_n.$$

Corollary 2.5. *For each $x, y \in \mathbb{R}^n_+$, we have*

$$G_{m,n}(x,y) \preceq \frac{x_n y_n}{|x-y|^{n-2m+2}}, \tag{2.4}$$

$$\frac{x_n y_n}{(|x|+1)^{n-2m+2}(|y|+1)^{n-2m+2}} \preceq G_{m,n}(x,y), \tag{2.5}$$

$$G_{m,n}(x,y) \preceq \frac{x_n \wedge y_n}{|x-y|^{n+1-2m}}. \tag{2.6}$$

Proof. The assertions (2.4) and (2.5) follow from Corollary 2.2 and the fact that

$$|x-y| \leq |\, x - \overline{y}| \leq (|x|+1)(|y|+1)$$

and

$$\frac{t}{1+t} \leq \log(1+t) \leq t,$$

for $t \geq 0$.

To prove (2.6) we claim that

$$G_{m,n}(x,y) \preceq \frac{x_n}{|x-y|^{n+1-2m}}. \tag{2.7}$$

Indeed, we have the following cases:

Case 1. *If $n > 2m$ or $n = 2m-1$, the inequality (2.7) follows from Corollary 2.2, (1.6) and the fact that $|x-\overline{y}| \geq |x-y|$.*

Case 2. *If $n = 2m$, then we have the following subcases:*

1. *If* $|x-y|^2 \leq x_n y_n$*, then by Lemma 2.4, we get* $x_n \sim y_n$*.*
   Using this fact, Proposition 2.1, (1.9) and (1.11) we deduce that

$$G_{m,n}(x,y) \sim \log\left(1+\frac{cx_n^2}{|x-y|^2}\right), \quad \text{(where } c>0\text{)},$$
$$\preceq \frac{x_n}{|x-y|}.$$

2. *If* $x_n y_n \leq |x-y|^2$*, then Lemma 2.4 gives that* $(x_n \vee y_n) \preceq |x-y|$*.*
   Hence from (2.4), we deduce that

$$G_{m,n}(x,y) \preceq \frac{x_n y_n}{|x-y|^2} \preceq \frac{x_n}{|x-y|}.$$

This proves (2.7). Interchange the role of x and y, we obtain (2.6). □

Proposition 2.6. *a) For each* $t>0$*, and all* $x, y \in \mathbb{R}^n_+$*, we have*

$$\int_0^t s^{m-1} p(s,x,y)ds \preceq G_{m,n}(x,y).$$

b) Let $t>0$ *and* $x, y \in \mathbb{R}^n_+$*. Then*

$$G_{m,n}(x,y) \preceq \int_0^t s^{m-1} p(s,x,y)ds,$$

provided

i) $n>2m$ *and* $|x-y| \leq 2\sqrt{t}$*;* *or* *ii)* $n = 2m$ *and* $|x - \overline{y}| \leq 2\sqrt{t}$; *or*

iii) $n = 2m - 1$ *and* $|x - \overline{y}| \leq 2\sqrt{t}$.

Proof. Let $t > 0$ and $x, y \in \mathbb{R}^n_+$. Then $a)$ follows immediately from (2.2). To prove $b)$ we distinguish three cases.

$i)$ For $n > 2m$, using (1.12) and the fact that for $a, b \in (0, \infty)$ we have

$$(1 \wedge ab) \geq (1 \wedge a)(1 \wedge b),$$

then there exists $C > 0$ such that for $|x - y| \leq 2\sqrt{t}$,

$$\begin{aligned}
\int_0^t s^{m-1} p(s, x, y) ds &\geq C \int_0^t \frac{1}{s^{\frac{n}{2}+1-m}} \exp\left(-\frac{|x-y|^2}{4s}\right)\left(1 \wedge \frac{x_n y_n}{s}\right) ds \\
&\geq \frac{C}{|x-y|^{n-2m}} \int_{\frac{|x-y|^2}{4t}}^{\infty} r^{\frac{n}{2}-1-m} e^{-r} \left(1 \wedge \frac{4r x_n y_n}{|x-y|^2}\right) dr \\
&\geq \frac{C}{|x-y|^{n-2m}} \left(1 \wedge \frac{x_n y_n}{|x-y|^2}\right) \int_{\frac{|x-y|^2}{4t}}^{\infty} r^{\frac{n}{2}-1-m} e^{-r} (1 \wedge 4r) dr \\
&\geq \frac{C}{|x-y|^{n-2m}} \left(1 \wedge \frac{x_n y_n}{|x-y|^2}\right) \int_1^{\infty} r^{\frac{n}{2}-1-m} e^{-r} dr \\
&\geq \frac{C}{|x-y|^{n-2m}} \left(1 \wedge \frac{x_n y_n}{|x-y|^2}\right).
\end{aligned}$$

Hence the result follows from Corollary 2.2.

ii) For $n = 2m$, by (2.3), there exists $C > 0$ such that for $|x - \overline{y}| \leq 2\sqrt{t}$

$$\int_0^t s^{m-1} p(s,x,y)ds = \alpha_{m,n} \int_{\frac{|x-y|^2}{4t}}^{\frac{|x-\overline{y}|^2}{4t}} \frac{e^{-r}}{r} dr \geq C \log\left(\frac{|x-\overline{y}|^2}{|x-y|^2}\right).$$

Hence the result follows from Proposition 2.1.

iii) Let $n = 2m-1$, $t > 0$ and $x, y \in \mathbb{R}^n_+$ such that $|x - \overline{y}| \leq 2\sqrt{t}$.
Put $a = \frac{|x-y|}{2\sqrt{t}}$ and $b = \frac{|x-\overline{y}|}{2\sqrt{t}}$. Then using (2.3), we obtain

$$I := \int_0^t s^{m-1} p(s,x,y)ds = 2\alpha_{m,n}\sqrt{t}\left(a \int_{a^2}^{\infty} r^{\frac{-3}{2}} e^{-r} dr - b \int_{b^2}^{\infty} r^{\frac{-3}{2}} e^{-r} dr\right).$$

Now since for $\alpha > 0$, we have

$$\int_{\alpha^2}^{\infty} r^{\frac{-3}{2}} e^{-r} dr = 2\left(\frac{e^{-\alpha^2}}{\alpha} - \int_{\alpha^2}^{\infty} r^{\frac{-1}{2}} e^{-r} dr\right),$$

we deduce that

$$\begin{aligned} I &= 4\alpha_{m,n}\sqrt{t}\left[(e^{-a^2} - e^{-b^2}) + b\int_{b^2}^{\infty} r^{\frac{-1}{2}} e^{-r} dr - a \int_{a^2}^{\infty} r^{\frac{-1}{2}} e^{-r} dr\right] \\ &= 4\alpha_{m,n}\sqrt{t}\left[(b-a)\int_{b^2}^{\infty} r^{\frac{-1}{2}} e^{-r} dr + \int_{a^2}^{b^2} r^{\frac{-1}{2}} e^{-r}(r^{\frac{1}{2}} - a)dr\right]. \end{aligned}$$

Hence

$$I \geq 4\alpha_{m,n}\sqrt{t}(b-a)\int_{b^2}^{\infty} r^{\frac{-1}{2}} e^{-r} dr.$$

That is

$$I \geq 2\alpha_{m,n}(|x-\overline{y}|-|x-y|)\int_{\frac{|x-\overline{y}|^2}{4t}}^{\infty} r^{\frac{-1}{2}}e^{-r}dr$$

$$\geq 2\alpha_{m,n}(|x-\overline{y}|-|x-y|)\int_{1}^{\infty} r^{\frac{-1}{2}}e^{-r}dr.$$

The result follows from Proposition 2.1. □

Next we purpose to prove that $G_{m,n}$ satisfies (1.1).

3G-Theorem. *For $x, y, z \in \mathbb{R}^n_+$, we have*

$$\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \preceq \left[\frac{z_n}{x_n}G_{m,n}(x,z) + \frac{z_n}{y_n}G_{m,n}(y,z)\right].$$

Proof. To prove the inequality, we denote by $A(x,y) := \frac{x_n y_n}{G_{m,n}(x,y)}$ and we claim that A is a quasi-metric, that is for each $x, y, z \in \mathbb{R}^n_+$,

$$A(x,y) \preceq A(x,z) + A(y,z). \tag{2.8}$$

To this end, we observe that by using Corollary 2.2 and Lemma 2.4, the claim can be proved by similar arguments as in [2], for $n > 2m$ and as in [3], for $n = 2m$.

To prove (2.8), for $n = 2m-1$, we derive from Corollary 2.2 that

$$A(x,y) \sim (|x-y|^2 \vee x_n y_n)^{\frac{1}{2}} \sim |x-\overline{y}|.$$

Now since $|\ x - z\ | \leq |\ x - \overline{z}|$, we deduce that

$$\begin{aligned} A(x,y) &\preceq |x - z| + |z - \overline{y}| \\ &\preceq |x - \overline{z}| + |z - \overline{y}| \preceq (A(x,z) + A(y,z)). \end{aligned}$$

□

3. The Class $K_{m,n}(\mathbb{R}^n_+)$

Next we purpose to study and to characterize the class $K_{m,n}(\mathbb{R}^n_+)$ for $n > 2m$.

We recall that for $0 < \alpha < n$, we say that a Borel measurable function q in $\mathbb{R}^n_+$ belongs to the class $\widetilde{K}_{\alpha,n}(\mathbb{R}^n_+)$ (see [6]) if q satisfies the following condition

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n_+} \int_{(|x-y|\leq r)\cap\mathbb{R}^n_+} \frac{|q(y)|}{|x-y|^{n-\alpha}}\, dy = 0. \tag{3.1}$$

The usual Kato class $K_n(\mathbb{R}^n_+)$, corresponds to $\alpha = 2$.

Remark 2. *Let $n > 2m$. Using Corollary 2.3, the class $K_{m,n}(\mathbb{R}^n_+)$ obviously includes the class $\widetilde{K}_{2m,n}(\mathbb{R}^n_+)$. In particular, $K_n(\mathbb{R}^n_+) \subset K_{m,n}(\mathbb{R}^n_+)$.*

Example 3.1. *Suppose that for $p > \frac{n}{2m} > 1$, we have*

$$M_0 = \sup_{x\in\mathbb{R}^n_+} \int_{(|x-y|\leq 1)\cap\mathbb{R}^n_+} \min\left(\left(\frac{y_n}{x_n}\right)^{2p}, 1\right) |q(y)|^p\, dy < \infty,$$

then $q \in K_{m,n}(\mathbb{R}^n_+)$.

Indeed, let $0 < r < 1$ and $x \in \mathbb{R}^n_+$, then using Remark 1 and the Hölder inequality we get

$$\int_{(|x-y|\leq r)\cap\mathbb{R}^n_+} \frac{y_n}{x_n} G_{m,n}(x,y)\, |q(y)|\, dy$$

$$\preceq \int_{(|x-y|\leq r)\cap\mathbb{R}^n_+} \min\left(\left(\frac{y_n}{x_n}\right)^2, 1\right) \frac{1}{|x-y|^{n-2m}} |q(y)|\, dy$$

$$\preceq \left(\int_{(|x-y|\leq r)\cap\mathbb{R}^n_+} \min\left(\left(\frac{y_n}{x_n}\right)^{2p}, 1\right) |q(y)|^p\, dy\right)^{\frac{1}{p}}$$

$$\times \left(\int_{(|x-y|\leq r)\cap\mathbb{R}^n_+} \frac{1}{|x-y|^{\frac{p}{p-1}(n-2m)}} dy\right)^{\frac{p-1}{p}}.$$

Hence

$$\sup_{x\in\mathbb{R}^n_+} \int_{(|x-y|\leq r)\cap\mathbb{R}^n_+} \frac{y_n}{x_n} G_{m,n}(x,y)q(y)dy \preceq M_0^{\frac{1}{p}} r^{\frac{2mp-n}{p}} \to 0 \text{ as } r \to 0.$$

Proposition 3.1. *Let* $p > \max\left(\frac{n}{2m}, 1\right)$ *and* $f \in L^p\left(\mathbb{R}^n_+\right)$. *Then*

$$y \mapsto \frac{f(y)}{(|y|+1)^{\mu-\lambda} y_n^\lambda} \in K_{m,n}(\mathbb{R}^n_+)$$

provided

i) $n > 2m$, $\lambda \leq 2$ *and* $\lambda < 2m - \frac{n}{p}$ *and* $\mu \geq \max(0, \lambda)$ *or*

ii) $n = 2m$ *and* $\lambda < \min(2, 2m - \frac{n}{p}) \leq \mu$ *or*

iii) $n = 2m - 1$, $\lambda \leq 2$ *and* $\lambda < 2m - \frac{n}{p}$ *and* $\mu \geq \max(1, \lambda)$.

Proof. Let $p > \max\left(\frac{n}{2m}, 1\right)$ and $q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

For $f \in L^p(\mathbb{R}^n_+)$, $x \in \mathbb{R}^n_+$ and $0 < r < 1$, put

$$I = I(x,r) := \int_{B(x,r)\cap\mathbb{R}^n_+} \frac{y_n}{x_n} G_{m,n}(x,y) \frac{|f(y)|}{(|y|+1)^{\mu-\lambda} y_n^{\lambda}} dy.$$

Note that if $|x-y| \leq r$, then $(|x|+1) \sim (|y|+1)$. So, we distinguish the following cases:

Case 1. $n > 2m$. *Assume that* $\lambda \leq 2$ *and* $\lambda < 2m - \frac{n}{p}$. *Let* $\mu \geq \max(0,\lambda)$ *and put* $\lambda^+ = \max(\lambda, 0)$. *Then using Corollary 2.2, (1.6) and the fact that* $|x-y| \leq |x-\overline{y}|$, *we deduce by the Hölder inequality that*

$$I \preceq \int_{B(x,r)\cap\mathbb{R}^n_+} \frac{|f(y)|}{|x-y|^{n+\lambda^+-2m}} dy \preceq \|f\|_p \left(\int_{B(x,r)\cap\mathbb{R}^n_+} \frac{1}{|x-y|^{(n+\lambda^+-2m)q}} dy \right)^{\frac{1}{q}}$$
$$\preceq r^{2m-\frac{n}{p}-\lambda^+},$$ *which tends to zero if* $r \to 0$.

Case 2. $n = 2m$. *Assume that* $\lambda < \min\left(2, 2m - \frac{n}{p}\right) \leq \mu$.
Using Proposition 2.1 and the Hölder inequality, we deduce that

$$I \preceq \|f\|_p \left(\int_{(|x-y|\leq r)\cap\mathbb{R}^n_+} \left(\frac{y_n}{x_n}\right)^q \left(\log\left(1 + \frac{4x_n y_n}{|x-y|^2}\right)\right)^q \frac{1}{(|y|+1)^{(\mu-\lambda)q} y_n^{\lambda q}} dy \right)^{\frac{1}{q}}$$

$$\preceq \left(\int_{(|x-y|\le r)\cap D_1} \left(\frac{y_n}{x_n}\right)^q \left(\log\left(1+\frac{4x_ny_n}{|x-y|^2}\right)\right)^q \frac{1}{(|y|+1)^{(\mu-\lambda)q}y_n^{\lambda q}}dy\right)^{\frac{1}{q}}$$
$$+\left(\int_{(|x-y|\le r)\cap D_2} \left(\frac{y_n}{x_n}\right)^q \left(\log\left(1+\frac{4x_ny_n}{|x-y|^2}\right)\right)^q \frac{1}{(|y|+1)^{(\mu-\lambda)q}y_n^{\lambda q}}dy\right)^{\frac{1}{q}}$$
$$= I_1 + I_2,$$

where

$$D_1 = \{y \in \mathbb{R}^n_+ : x_ny_n \le |x-y|^2\} \text{ and } D_2 = \{y \in \mathbb{R}^n_+ : |x-y|^2 \le x_ny_n\}.$$

So, using that $\log(1+t) \le t$*, for* $t \ge 0$ *and Lemma 2.4, we obtain*

$$I_1 \preceq \left(\int_{(|x-y|\le r)\cap D_1} \frac{y_n^{(2-\lambda)q}}{|x-y|^{2q}}dy\right)^{\frac{1}{q}}$$
$$\preceq \left(\int_{(|x-y|\le r)\cap D_1} \frac{1}{|x-y|^{\lambda q}}dy\right)^{\frac{1}{q}}$$
$$\preceq r^{2m-\frac{n}{p}-\lambda}, \text{ which converges to zero as } r \to 0.$$

On the other hand, from Lemma 2.4 and the fact that $(|x|+1) \sim (|y|+1)$*, we obtain*

$$I_2 \preceq \frac{1}{x_n^\lambda(|x|+1)^{(\mu-\lambda)}} \left(\int_{(|x-y|\le r)\cap D_2} \left(\log\left(1+\frac{(cx_n)^2}{|x-y|^2}\right)\right)^q dy\right)^{\frac{1}{q}},$$

where $c = 1+\sqrt{5}$. *Let* $\gamma \in \left]\max(0,\lambda), \min(2, 2m-\frac{n}{p})\right[$.
Since $\log(1+t^2) \preceq t^\gamma$, *for* $t \geq 0$, *then*

$$\begin{aligned} I_2 &\preceq \frac{x_n^{\gamma-\lambda}}{(|x|+1)^{\mu-\lambda}} \left(\int_{(|x-y|\leq r)\cap D_2} \frac{1}{|x-y|^{\gamma q}} dy \right)^{\frac{1}{q}} \\ &\preceq \left(\int_{(|x-y|\leq r)\cap D_2} \frac{1}{|x-y|^{\gamma q}} dy \right)^{\frac{1}{q}} \\ &\preceq r^{2m-\frac{n}{p}-\gamma}, \text{ which converges to zero as } r \to 0. \end{aligned}$$

Case 3. $n = 2m-1$. *Assume that* $\lambda \leq 2$ *and* $\lambda < 2m - \frac{n}{p}$. *Let* $\mu \geq \max(1,\lambda)$. *Using Corollary 2.2 and the Hölder inequality, we obtain*

$$\begin{aligned} I &\preceq \left[\left(\int_{B(x,r)\cap D_1} \frac{y_n^{(2-\lambda)q}}{|x-\overline{y}|^q \left(|y|+1\right)^{(\mu-\lambda)q}} dy \right)^{\frac{1}{q}} \right. \\ &\qquad \left. + \left(\int_{B(x,r)\cap D_2} \frac{y_n^{(2-\lambda)q}}{|x-\overline{y}|^q \left(|y|+1\right)^{(\mu-\lambda)q}} dy \right)^{\frac{1}{q}} \right] \\ &= I_1 + I_2. \end{aligned}$$

Now, if $y \in D_1$, *then* $|x-\overline{y}| \sim |x-y|$ *and so*

$$I_1 \preceq \left(\int_{B(x,r)\cap D_1} \frac{1}{|x-y|^{(\lambda-1)q}} dy \right)^{\frac{1}{q}} \preceq r^{2m-\frac{n}{p}-\lambda}, \text{ which tends to zero as } r \to 0.$$

On the other hand, if $y \in D_2$, *then* $|x-\overline{y}|^2 \sim x_n y_n$ *and by Lemma 2.4, we have further* $x_n \sim y_n$. *This implies that*

$$I_2 \preceq \frac{x_n^{2-\lambda}}{x_n(|x|+1)^{\mu-\lambda}} \left(\int_{B(x,r)\cap D_2} dy \right)^{\frac{1}{q}} \preceq \frac{x_n^{1-\lambda}}{(|x|+1)^{\mu-\lambda}} (r \wedge c x_n)^{\frac{n}{q}}$$

$$\preceq r^{\frac{n}{q}}, \quad \text{which converges to zero as } r \to 0.$$

□

The proof of the next results are similar to the case $m = 1$ and $n \geq 3$, which has been considered in [2]. Since reference [2] is not available, I have chosen to reproduce it here.

Proposition 3.2. *Let* $q \in K_{m,n}(\mathbb{R}^n_+)$, *then for each compact* $L \subseteq \mathbb{R}^n$ *we have*

$$\sup_{x\in\mathbb{R}^n_+} \int_{(x+L)\cap\mathbb{R}^n_+} \frac{y_n^2}{1+x_n y_n} |q(y)|\, dy < \infty.$$

Proof. Let $q \in K_{m,n}(\mathbb{R}^n_+)$, then by (1.3) there exists $r > 0$ such that

$$\sup_{x\in\mathbb{R}^n_+} \int_{(|x-y|\leq r)\cap\mathbb{R}^n_+} \frac{y_n}{x_n} G_{m,n}(x,y)\, |q(y)|\, dy \leq 1.$$

Let $a_1, a_2, \ldots, a_p \in \mathbb{R}^n_+ \cap L$ such that $\mathbb{R}^n_+ \cap L \subseteq \bigcup_{1\leq i\leq p} B(a_i, r)$.

Since for $a, b \in (0,\infty)$, we have $\frac{b}{1+ab} \leq 1 + |a-b|$, then for each $x, y, z \in \mathbb{R}^n_+$ it follows that

$$\frac{1+(x_n+z_n)y_n}{1+x_n y_n} \leq [1 + z_n(1+|x_n-y_n|)].$$

Using this fact and Corollary 2.2, we obtain:
For $n > 2m$,

$$\frac{y_n^2}{1+x_ny_n} \preceq \frac{[1+z_n(1+|x_n-y_n|)]}{1+(x_n+z_n)y_n}|x+z-y|^{n-2m} \times \left[|x+z-y|^2+4(x_n+z_n)y_n\right]\frac{y_n}{(x_n+z_n)}G_{m,n}(x+z,y).$$

For $n = 2m$, using further that $\frac{t}{1+t} \leq \log(1+t)$, $\forall t \geq 0$, we have

$$\frac{y_n^2}{1+x_ny_n} \preceq \frac{[1+z_n(1+|x_n-y_n|)]}{1+(x_n+z_n)y_n} \times \left[|x+z-y|^2+4(x_n+z_n)y_n\right]\frac{y_n}{(x_n+z_n)}G_{m,n}(x+z,y).$$

For $n = 2m-1$,

$$\frac{y_n^2}{1+x_ny_n} \preceq \frac{[1+z_n(1+|x_n-y_n|)]}{1+(x_n+z_n)y_n} \times \left[|x+z-y|^2+4(x_n+z_n)y_n\right]^{\frac{1}{2}}\frac{y_n}{(x_n+z_n)}G_{m,n}(x+z,y).$$

Now, if $z \in L$ and $|x+z-y| \leq r$, then

$$\frac{y_n^2}{1+x_ny_n} \preceq \frac{y_n}{(x_n+z_n)}G_{m,n}(x+z,y).$$

Hence

$$\int_{(x+L)\cap\mathbb{R}^n_+} \frac{y_n^2}{1+x_n y_n} |q(y)|\,dy \preceq \sum_{i=1}^{p} \int_{(|x+a_i-y|\le r)\cap\mathbb{R}^n_+} \frac{y_n}{(x_n+(a_i)_n)} G_{m,n}(x+a_i,y)\,|q(y)|\,dy \preceq p.$$

So

$$\sup_{x\in\mathbb{R}^n_+} \int_{(x+L)\cap\mathbb{R}^n_+} \frac{y_n^2}{1+x_n y_n} |q(y)|\,dy < \infty.$$

□

Corollary 3.3. *Let $q \in K_{m,n}(\mathbb{R}^n_+)$.Then we have for $M > 0$,*

$$\int_{(|y|\le M)\cap\mathbb{R}^n_+} y_n^2\,|q(y)|\,dy < \infty.$$

Proposition 3.4. *Let $q \in K_{m,n}(\mathbb{R}^n_+)$, then for each fixed $\alpha > 0$, we have*

$$\sup_{t\le 1}\sup_{x\in\mathbb{R}^n_+} \int_{(|x-y|>\alpha)\cap\mathbb{R}^n_+} \frac{y_n}{x_n} p(t,x,y)\,|q(y)|\,dy := M(\alpha) < \infty. \tag{3.2}$$

Proof. Let $q \in K_{m,n}(\mathbb{R}^n_+)$, $0 < t \le 1$ and without loss of generality assume that

$0 < \alpha < 1$. Then by (1.12) and (1.7), it follows that

$$\sup_{x\in\mathbb{R}^n_+} \int_{(|x-y|>\alpha)\cap\mathbb{R}^n_+} \frac{y_n}{x_n} p(t,x,y)\,|q(y)|\,dy$$
$$\preceq \frac{1}{t^{\frac{n}{2}+1}} e^{-\frac{\alpha^2}{8t}} \sup_{x\in\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \exp\left(-\frac{|x-y|^2}{8}\right) \frac{y_n^2}{1+x_n y_n}\,|q(y)|\,dy.$$

To conclude, it is sufficient to prove that

$$\sup_{x\in\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \exp\left(-\frac{|x-y|^2}{8}\right) \frac{y_n^2}{1+x_n y_n}\,|q(y)|\,dy < \infty.$$

Indeed, using Proposition 3.2, we have

$$\sup_{x\in\mathbb{R}^n_+} \int_{(x+B(0,1))\cap\mathbb{R}^n_+} \frac{y_n^2}{1+x_n y_n}\,|q(y)|\,dy := \widetilde{M} < \infty.$$

Now since any ball $B(0,k)$, of radius $k \geq 1$ in $\mathbb{R}^n$ can be covered by $A_n k^n := \alpha(n)$ balls of radius 1, where A_n is a constant depending only on n (see [5, p. 67]), then there exists $a_1, a_2, \ldots, a_{\alpha(n)} \in \mathbb{R}^n_+$ such that

$$\mathbb{R}^n_+ \cap B(0,k) \subseteq \bigcup_{1\leq i\leq\alpha(n)} B(a_i,1).$$

Using the fact that for each $x, y, z \in \mathbb{R}^n_+$,

$$\frac{1+(x_n+z_n)y_n}{1+x_n y_n} \leq [1 + z_n(1+|x_n-y_n|)],$$

it follows that for all $x \in \mathbb{R}^n_+$ and $k \geq 1$:

$$\begin{aligned}\int_{(x+B(0,k))\cap\mathbb{R}^n_+} \frac{y_n^2}{1+x_n y_n} |q(y)|\, dy &\preceq \sum_{i=1}^{\alpha(n)} \int_{B(x+a_i,1)\cap\mathbb{R}^n_+} \frac{y_n^2}{1+x_n y_n} |q(y)|\, dy \\ &\preceq \sum_{i=1}^{\alpha(n)} \int_{B(x+a_i,1)\cap\mathbb{R}^n_+} \frac{y_n^2}{1+(x_n+(a_i)_n) y_n} |q(y)|\, dy \\ &\preceq A_n k^n \widetilde{M}.\end{aligned}$$

Hence for all $x \in \mathbb{R}^n_+$, we have

$$\begin{aligned}\int_{\mathbb{R}^n_+} \exp\left(-\frac{|x-y|^2}{8}\right) \frac{y_n^2}{1+x_n y_n} |q(y)|\, dy &\preceq \sum_{k=0}^{\infty} \exp\left(-\frac{\alpha^2 k^2}{8}\right) \int_{[k\alpha\leq|x-y|\leq(k+1)\alpha]\cap\mathbb{R}^n_+} \frac{y_n^2}{1+x_n y_n} |q(y)|\, dy \\ &\preceq A_n \widetilde{M} \sum_{k=0}^{\infty} (k+1)^n \exp\left(-\frac{\alpha^2 k^2}{8}\right) < \infty.\end{aligned}$$

Thus

$$\sup_{x\in\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \exp\left(-\frac{|x-y|^2}{8}\right) \frac{y_n^2}{1+x_n y_n} |q(y)|\, dy < \infty,$$

which completes the proof. □

Theorem 3.5. *Let $n > 2m$ and $q \in \mathcal{B}(\mathbb{R}^n_+)$. Then the following assertions are equivalent:*

1. $q \in K_{m,n}(\mathbb{R}^n_+)$
2. $\lim_{t \to 0} \sup_{x \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \int_0^t \frac{y_n}{x_n} s^{m-1} p(s,x,y) \, |q(y)| \, ds dy = 0.$

Proof. $2) \Rightarrow 1)$ Assume that

$$\lim_{t \to 0} \sup_{x \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \int_0^t \frac{y_n}{x_n} s^{m-1} p(s,x,y) \, |q(y)| \, ds dy = 0.$$

Then by Proposition 2.6, there exists $c > 0$ such that for $\alpha > 0$ we have

$$\int_{(|x-y| \le \alpha) \cap \mathbb{R}^n_+} \frac{y_n}{x_n} G_{m,n}(x,y) \, |q(y)| \, dy \le c \int_{\mathbb{R}^n_+} \int_0^{\frac{\alpha^2}{4}} \frac{y_n}{x_n} s^{m-1} p(s,x,y) \, |q(y)| \, ds dy,$$

which shows that the function q satisfies (1.3).

Conversely suppose that $q \in K_{m,n}(\mathbb{R}^n_+)$. Let $\varepsilon > 0$, then there exists $0 < \alpha < 1$ such that

$$\sup_{x \in \mathbb{R}^n_+} \int_{(|x-y| \le \alpha) \cap \mathbb{R}^n_+} \frac{y_n}{x_n} G_{m,n}(x,y) \, |q(y)| \, dy \le \varepsilon.$$

On the other hand, using Proposition 2.6 and (3.2), we have for $0 < t < 1$

$$\begin{aligned}
\int_{\mathbb{R}^n_+}\int_0^t \frac{y_n}{x_n} s^{m-1} p(s,x,y)\,|q(y)|\,dsdy
&\preceq \int_{(|x-y|\le\alpha)\cap\mathbb{R}^n_+}\int_0^t \frac{y_n}{x_n} s^{m-1} p(s,x,y)\,|q(y)|\,dsdy \\
&\qquad + \int_{(|x-y|>\alpha)\cap\mathbb{R}^n_+}\int_0^t \frac{y_n}{x_n} s^{m-1} p(s,x,y)\,|q(y)|\,dsdy \\
&\preceq \int_{(|x-y|\le\alpha)\cap\mathbb{R}^n_+} \frac{y_n}{x_n} G_{m,n}(x,y)\,|q(y)|\,dy \\
&\qquad + \int_0^t\int_{(|x-y|>\alpha)\cap\mathbb{R}^n_+} \frac{y_n}{x_n} p(s,x,y)\,|q(y)|\,dyds \\
&\preceq \varepsilon + tM(\alpha),
\end{aligned}$$

which implies that

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\int_0^t \frac{y_n}{x_n} s^{m-1} p(s,x,y)\,|q(y)|\,dsdy = 0.$$

□

Corollary 3.6. *Let* $n > 2m$ *and* $q \in \mathcal{B}(\mathbb{R}^n_+)$. *For* $\alpha > 0$ *and* $x \in \mathbb{R}^n_+$, *put*

$$G_\alpha q(x) := \int_{\mathbb{R}^n_+}\int_0^\infty e^{-\alpha s}\frac{y_n}{x_n} s^{m-1} p(s,x,y)\,|q(y)|\,dsdy.$$

Then

$$q \in K_{m,n}(\mathbb{R}^n_+) \Leftrightarrow \lim_{\alpha \to +\infty} \|G_\alpha q\|_\infty = 0,$$

where $\|G_\alpha q\|_\infty = \sup_{x \in \mathbb{R}^n_+} |G_\alpha q(x)|$.

Proof. (see [9]). Let $q \in K_{m,n}(\mathbb{R}^n_+)$, $\alpha > 0$ and put

$$a(\alpha) = \sup_{x \in \mathbb{R}^n_+} \int_0^{\frac{1}{\alpha}} \int_{\mathbb{R}^n_+} \frac{y_n}{x_n} s^{m-1} p(s,x,y) |q(y)| \, dyds.$$

Then we have

$$\begin{aligned} G_\alpha q(x) &= \int_0^\infty \alpha e^{-\alpha t} \left[\int_0^t \int_{\mathbb{R}^n_+} \frac{y_n}{x_n} s^{m-1} p(s,x,y) |q(y)| \, dyds \right] dt \\ &= \int_0^\infty e^{-t} \left[\int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}^n_+} \frac{y_n}{x_n} s^{m-1} p(s,x,y) |q(y)| \, dyds \right] dt. \end{aligned}$$

It follows that, $\frac{1}{e} a(\alpha) \le \|G_\alpha q\|_\infty$.

On the other hand, for $t > 0$ and $k \in \mathbb{N}$ such that $k \le t < k+1$, we have

$$\begin{aligned} G_\alpha q(x) &\le \sum_{k=0}^{m} \int_0^\infty e^{-t} \left[\int_{\frac{k}{\alpha}}^{\frac{k+1}{\alpha}} \int_{\mathbb{R}^n_+} \frac{y_n}{x_n} s^{m-1} p(s,x,y) |q(y)| \, dyds \right] dt \\ &\le a(\alpha) \int_0^\infty e^{-t} (m+1) dt \\ &\le a(\alpha) \int_0^\infty e^{-t} (t+1) dt = 2a(\alpha), \end{aligned}$$

which gives that $\frac{1}{e}a(\alpha) \leq \|G_\alpha q\|_\infty \leq 2a(\alpha)$.

Hence the results follow from Theorem 3.5. □

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