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A SPANNING SET FOR THE SPACE OF SUPER CUSP FORMS

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ABSTRACT. The aim of this article is the construction of a spanning set for the space $sS_k(\Gamma)$ of super cusp forms on a complex bounded symmetric super domain $\mathcal{B}$ of rank 1 with respect to a lattice Γ. The main ingredients are a generalization of the ANOSOV closing lemma for partially hyperbolic diffeomorphisms and an unbounded realization $\mathcal{H}$ of $\mathcal{B}$, in particular FOURIER decomposition at the cusps of the quotient $\Gamma\backslash B$ mapped to ∞ via a partial CAYLEY transformation. The elements of the spanning set are in finite-to-one correspondence with closed geodesics of the body $\Gamma\backslash B$ of $\Gamma\backslash\mathcal{B}$, the number of elements corresponding to a geodesic growing linearly with its length.

Key words and phrases: Automorphic and cusp forms, super symmetry, semisimple LIE groups, partially hyperbolic flows, unbounded realization of a complex bounded symmetric domain.

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1. INTRODUCTION

Automorphic and cusp forms on a complex bounded symmetric domain B are already a well established field of research in mathematics. They play a fundamental role in representation theory of semisimple LIE groups of Hermitian type, and they have applications to number theory, especially in the simplest case where B is the unit disc in $\mathbb{C}$, biholomorphic to the upper half plane H via a CAYLEY transform, $G = SL(2,\mathbb{R})$ acting on H via MÖBIUS transformations and $\Gamma \sqsubset SL(2,\mathbb{Z})$ of finite index. The aim of the present paper is to generalize an approach used by Tatyana FOTH and Svetlana KATOK in [4] and [8] for the construction of spanning sets for the space of cusp forms on a complex bounded symmetric domain B of rank 1, which then by classification is (biholomorphic to) the unit ball of some $\mathbb{C}^n$, $n \in \mathbb{N}$, and a lattice $\Gamma \sqsubset G = \mathrm{Aut}_1(B)$ for sufficiently high weight k. This is done in Theorem 4.3, which is the main theorem of this article, again for sufficiently large weight k.

The new idea in [4] and [8] is to use the concept of a hyperbolic (or ANOSOV) diffeomorphism resp. flow on a Riemannian manifold and an appropriate version of the ANOSOV closing lemma. This concept originally comes from the theory of dynamical systems, see for example

in [7]. Roughly speaking a flow $(\varphi_t)_{t\in\mathbb{R}}$ on a Riemannian manifold M is called hyperbolic if there exists an orthogonal and $(\varphi_t)_{t\in\mathbb{R}}$-stable splitting $TM = T^+ \oplus T^- \oplus T^0$ of the tangent bundle TM such that the differential of the flow $(\varphi_t)_{t\in\mathbb{R}}$ is uniformly expanding on T^+, uniformly contracting on T^- and isometric on T^0, and finally T^0 is one-dimensional, generated by $\partial_t\varphi_t$. In this situation the ANOSOV closing lemma says that given an 'almost' closed orbit of the flow $(\varphi_t)_{t\in\mathbb{R}}$ there exists a closed orbit 'nearby'. Indeed given a complex bounded symmetric domain B of rank 1, $G = \mathrm{Aut}_1(B)$ is a semisimple LIE group of real rank 1, and the root space decomposition of its LIE algebra $\mathfrak{g}$ with respect to a CARTAN subalgebra $\mathfrak{a} \sqsubset \mathfrak{g}$ shows that the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ on the unit tangent bundle $S(B)$, which is at the same time the left-invariant flow on $S(B)$ generated by $\mathfrak{a} \simeq \mathbb{R}$, is hyperbolic. The final result in this direction is Theorem 5.3 (i).

For the super case, first it is necessary to develop the theory of super automorphic resp. cusp forms, while the general theory of ($\mathbb{Z}_2$-) graded structures and super manifolds is already well established, see for example [3]. It was first developed by F. A. BEREZIN as a mathematical method for describing super symmetry in physics of elementary particles. However, even for mathematicians the elegance within the theory of super manifolds is really amazing and satisfying. Here I deal with a simple case of super manifolds, namely complex super domains. Roughly speaking a complex super domain $\mathcal{B}$ is an object which has a super dimension $(n, r) \in \mathbb{N}^2$ and the characteristics:

(i) it has a body $B = \mathcal{B}^{\#}$ being an ordinary domain in $\mathbb{C}^n$,
(ii) the complex unital graded commutative algebra $\mathcal{O}(\mathcal{B})$ of holomorphic super functions on $\mathcal{B}$ is (isomorphic to) $\mathcal{O}(B) \otimes \bigwedge(\mathbb{C}^r)$, where $\bigwedge(\mathbb{C}^r)$ denotes the exterior algebra of $\mathbb{C}^r$. Furthermore $\mathcal{O}(\mathcal{B})$ naturally embeds into the first two factors of the complex unital graded commutative algebra $\mathcal{D}(\mathcal{B}) \simeq \mathcal{C}^\infty(B)^{\mathbb{C}} \otimes \bigwedge(\mathbb{C}^r) \boxtimes \bigwedge(\mathbb{C}^r) \simeq \mathcal{C}^\infty(B)^{\mathbb{C}} \otimes \bigwedge(\mathbb{C}^{2r})$ of 'smooth' super functions on $\mathcal{B}$, where $\mathcal{C}^\infty(B)^{\mathbb{C}} = \mathcal{C}^\infty(B, \mathbb{C})$ denotes the algebra of ordinary smooth functions with values in $\mathbb{C}$, which is at the same time the complexification of $\mathcal{C}^\infty(B)$, and '$\boxtimes$' denotes the graded tensor product.

We see that for each pair (B, r) where $B \subset \mathbb{C}^n$ is an ordinary domain and $r \in \mathbb{N}$ there exists exactly one (n, r)-dimensional complex super domain $\mathcal{B}$ of super dimension (n, r) with body B, and we denote it by $B^{|r}$. Now let $\zeta_1, \ldots, \zeta_n \in \mathbb{C}^r$ denote the standard basis vectors of $\mathbb{C}^r$. Then they are the standard generators of $\bigwedge(\mathbb{C}^r)$, and so we get the standard even (commuting) holomorphic coordinate functions $z_1, \ldots, z_n \in \mathcal{O}(B) \hookrightarrow \mathcal{O}\left(B^{|r}\right)$ and odd (anticommuting) coordinate functions $\zeta_1, \ldots, \zeta_r \in \bigwedge(\mathbb{C}^r) \hookrightarrow \mathcal{O}\left(B^{|r}\right)$. So omitting the tensor products, as there is no danger of confusion, we can decompose every $f \in \mathcal{O}\left(B^{|r}\right)$ uniquely as

$$f = \sum_{I\in\wp(r)} f_I \zeta^I,$$

where $\wp(r)$ denotes the power set of $\{1, \ldots, r\}$, all $f_I \in \mathcal{O}(B)$, $I \in \wp(r)$, and $\zeta^I := \zeta_{i_1} \cdots \zeta_{i_s}$ for all $I = \{i_1, \ldots, i_s\} \in \wp(r)$, $i_1 < \cdots < i_s$.

$\mathcal{D}\left(B^{|r}\right)$ is a graded *-algebra, and the graded involution

$$\bar{}\,: \mathcal{D}\left(B^{|r}\right) \to \mathcal{D}\left(B^{|r}\right)$$

is uniquely defined by the rules

{i} $\bar{\bar{f}} = f$ and $\overline{fh} = \overline{hf}$ for all $f, h \in \mathcal{D}\left(B^{|r}\right)$,
{ii} $\bar{}$ is $\mathbb{C}$-antilinear, and restricted to $\mathcal{C}^\infty(B)$ it is just the identity,

{iii} $\overline{\zeta_i}$ is the i-th standard generator of $\bigwedge(\mathbb{C}^r) \hookrightarrow \mathcal{D}\left(B^{|r}\right)$ embedded as the **third** factor, where ζ_i denotes the i-th odd holomorphic standard coordinate on $B^{|r}$, which is the i-th standard generator of $\bigwedge(\mathbb{C}^r) \hookrightarrow \mathcal{D}\left(B^{|r}\right)$ embedded as the **second** factor, $i = 1, \ldots, r$.

With the help of this graded involution we are able to decompose every $f \in \mathcal{D}\left(B^{|r}\right)$ uniquely as

$$f = \sum_{I,J \in \wp(r)} f_{IJ} \zeta^I \overline{\zeta}^J,$$

where $f_{IJ} \in \mathcal{C}^\infty(B)^{\mathbb{C}}$, $I, J \in \wp(r)$, and $\overline{\zeta}^J := \overline{\zeta_{i_1}} \ldots \overline{\zeta_{i_s}}$ for all $J = \{j_1, \ldots, j_s\} \in \wp(r)$, $j_1 < \cdots < j_s$.

For a discussion of super automorphic and super cusp forms we restrict ourselves to the case of the LIE group $G := sS\left(U(n,1) \times U(r)\right)$, $n \in \mathbb{N} \setminus \{0\}$, $r \in \mathbb{N}$, acting on the complex (n,r)-dimensional super unit ball $B^{|r}$. So far there seems to be no classification of super complex bounded symmetric domains although we know the basic examples, see for example Chapter IV of [2], which we follow here. The group G is the body of the super LIE group $SU(n,1|r)$ studied in [2] acting on $B^{|r}$. The fact that an ordinary discrete subgroup (which means a sub super LIE group of super dimension $(0,0)$) of a super LIE group is just an ordinary discrete subgroup of the body justifies our restriction to an ordinary LIE group acting on $B^{|r}$ since purpose of this article is to study automorphic and cusp forms with respect to a lattice. In any case one can see the odd directions of the complex super domain $B^{|r}$ already in G since it is an almost direct product of the semisimple LIE group $SU(n,1)$ acting on the body B and $U(r)$ acting on $\bigwedge(\mathbb{C}^r)$. Observe that if $r > 0$ the full automorphism group of $B^{|r}$, without any isometry condition, is never a super LIE group since one can show that otherwise its super LIE algebra would be the super LIE algebra of integrable super vector fields on $B^{|r}$, which has unfortunately infinite dimension.

Let us remark on two striking facts:

(i) the construction of our spanning set uses FOURIER decomposition exactly three times, which is not really surprising, since this corresponds to the three factors in the IWASAWA decomposition $G = KAN$.

(ii) super automorphic resp. cusp forms introduced this way are equivalent (but not one-to-one) to the notion of 'twisted' vector-valued automorphic resp. cusp forms.

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2. THE SPACE OF SUPER CUSP FORMS

Let $n \in \mathbb{N} \setminus \{0\}$, $r \in \mathbb{N}$ and

$$\begin{aligned} G &:= sS\left(U(n,1) \times U(r)\right) \\ &:= \left\{ \left(\begin{array}{c|c} g' & 0 \\ \hline 0 & E \end{array} \right) \in U(n,1) \times U(r) \,\middle|\, \det g' = \det E \right\}, \end{aligned}$$

which is a real $((n+1)^2 + r^2 - 1)$-dimensional LIE group. Let $\mathcal{B} := B^{|r}$, where

$$B := \left\{ \mathbf{z} \in \mathbb{C}^n \middle| \, \mathbf{z}^* \mathbf{z} < 1 \right\} \subset \mathbb{C}^n$$

denotes the usual unit ball, with even coordinate functions $z_1, \ldots, z_n$ and odd coordinate functions $\zeta_1, \ldots, \zeta_r$. Then we have a holomorphic action of G on $\mathcal{B}$ given by super fractional linear

(MÖBIUS) transformations

$$g\left(\frac{\mathbf{z}}{\zeta}\right) := \left(\frac{(A\mathbf{z}+\mathbf{b})(\mathbf{c}\mathbf{z}+d)^{-1}}{E\zeta(\mathbf{c}\mathbf{z}+d)^{-1}}\right),$$

where we split

$$g := \left(\begin{array}{c|c|c} A & \mathbf{b} & 0 \\ \hline \mathbf{c} & d & \\ \hline 0 & & E \end{array}\right) \begin{array}{l} \}n \\ \leftarrow n+1 \\ \}r \end{array}.$$

The stabilizer of $\mathbf{0} \hookrightarrow \mathcal{B}$ is

$$\begin{aligned} K &:= sS((U(n)\times U(1))\times U(r)) \\ &= \left\{ \left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & d & \\ \hline 0 & & E \end{array}\right) \in U(n)\times U(1)\times U(r) \,\middle|\, d\det A = \det E \right\}. \end{aligned}$$

On $G\times B$ we define the cocycle $j \in \mathcal{C}^\infty(G)^{\mathbb{C}}\hat{\otimes}\mathcal{O}(B)$ as $j(g,\mathbf{z}) := (\mathbf{c}\mathbf{z}+d)^{-1}$ for all $g\in G$ and $\mathbf{z}\in B$. Observe that $j(w) := j(w,\mathbf{z}) \in U(1)$ is independent of $\mathbf{z}\in B$ for all $w\in K$ and therefore defines a character on the group K.

Let $k\in\mathbb{Z}$ be fixed. Then we have a right-representation of G

$$|_g : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{B}), f \mapsto f|_g := f\left(g\left(\frac{\mathbf{z}}{\zeta}\right)\right) j(g,\mathbf{z})^k,$$

for all $g\in G$, which fixes $\mathcal{O}(\mathcal{B})$. Finally let Γ be a discrete subgroup of G.

Definition 2.1 (Super Automorphic Forms)**.** Let $f\in\mathcal{O}(\mathcal{B})$. Then f is called a super automorphic form for Γ of weight k if and only if $f|_\gamma = f$ for all $\gamma\in\Gamma$. We denote the space of super automorphic forms for Γ of weight k by $sM_k(\Gamma)$.

Let us define a lift:

$$\begin{aligned} \widetilde{\ } : \mathcal{D}(\mathcal{B}) &\to \mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \mathcal{D}\left(\mathbb{C}^{0|r}\right) \simeq \mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \bigwedge(\mathbb{C}^r) \boxtimes \bigwedge(\mathbb{C}^r), \\ f &\mapsto \widetilde{f}, \end{aligned}$$

where

$$\begin{aligned} \widetilde{f}(g) &:= f|_g\left(\frac{\mathbf{0}}{\eta}\right) \\ &= f\left(g\left(\frac{\mathbf{0}}{\eta}\right)\right) j(g,\mathbf{0})^k \end{aligned}$$

for all $f\in\mathcal{D}(\mathcal{B})$ and $g\in G$ and we use the odd coordinate functions $\eta_1,\ldots,\eta_r$ on $\mathbb{C}^{0|r}$. Let $f\in\mathcal{O}(\mathcal{B})$. Then clearly $\widetilde{f}\in\mathcal{C}^\infty(G)^{\mathbb{C}}\otimes\mathcal{O}\left(\mathbb{C}^{0|r}\right)$ and $f\in sM_k(\Gamma) \Leftrightarrow \widetilde{f}\in\mathcal{C}^\infty\left(\Gamma\backslash G\right)^{\mathbb{C}}\otimes\mathcal{O}\left(\mathbb{C}^{0|r}\right)$ since for all $g\in G$

$$\begin{array}{ccc} \mathcal{C}^\infty(G)^{\mathbb{C}}\otimes\mathcal{D}\left(\mathbb{C}^{0|r}\right) & \xrightarrow{l_g} & \mathcal{C}^\infty(G)^{\mathbb{C}}\otimes\mathcal{D}\left(\mathbb{C}^{0|r}\right) \\ \uparrow\sim & & \uparrow\sim \\ \mathcal{D}(\mathcal{B}) & \xrightarrow[|_g]{} & \mathcal{D}(\mathcal{B}) \end{array}$$

commutes, where $l_g : \mathcal{C}^\infty(G)\to\mathcal{C}^\infty(G)$ denotes the left translation with $g\in G$, $l_g(f)(x) := f(gx)$ for all $x\in G$. Let $\langle\ ,\ \rangle$ be the canonical scalar product on $\mathcal{D}\left(\mathbb{C}^{0|r}\right)\simeq\bigwedge(\mathbb{C}^{2r})$ (semilinear in the second entry). Then for all $a\in\mathcal{D}\left(\mathbb{C}^{0|r}\right)$ we write $|a| := \sqrt{\langle a,a\rangle}$, and $\langle\ ,\ \rangle$ induces

a 'scalar product'

$$(f,h)_\Gamma := \int_{\Gamma\backslash G} \left\langle \widetilde{h}, \widetilde{f} \right\rangle$$

for all $f,h \in \mathcal{D}(\mathcal{B})$ such that $\left\langle \widetilde{h}, \widetilde{f} \right\rangle \in L^1(\Gamma\backslash G)$, and for all $s \in]0,\infty]$ a 'norm'

$$||f||_{s,\Gamma}^{(k)} := \left|\left|\quad \left|\widetilde{f}\right| \quad\right|\right|_{s,\Gamma\backslash G}$$

for all $f \in \mathcal{D}(\mathcal{B})$ such that $\left|\widetilde{f}\right| \in \mathcal{C}^\infty(\Gamma\backslash G)$. On G we always use the (left and right) HAAR measure. Let us define

$$L_k^s(\Gamma\backslash\mathcal{B}) := \left\{ f \in \mathcal{D}(\mathcal{B}) \,\middle|\, \widetilde{f} \in \mathcal{C}^\infty(\Gamma\backslash G)^{\mathbb{C}} \otimes \mathcal{D}\left(\mathbb{C}^{0|r}\right), ||f||_{s,\Gamma}^{(k)} < \infty \right\}.$$

Definition 2.2 (Super Cusp Forms)**.** Let $f \in sM_k(\Gamma)$. f is called a super cusp form for Γ of weight k if and only if $f \in L_k^2(\Gamma\backslash\mathcal{B})$. The $\mathbb{C}$- vector space of all super cusp forms for Γ of weight k is denoted by $sS_k(\Gamma)$. It is a HILBERT space with inner product $(\ ,\ )_\Gamma$.

Observe that $|_g$ respects the splitting

$$\mathcal{O}(\mathcal{B}) = \bigoplus_{\rho=0}^{r} \mathcal{O}^{(\rho)}(\mathcal{B})$$

for all $g \in G$, where $\mathcal{O}^{(\rho)}(\mathcal{B})$ is the space of all $f = \sum_{I\in\wp(r),|I|=\rho} f_I$, all $f_I \in \mathcal{O}(\mathcal{B})$, $I \in \wp(r)$, $|I| = \rho$, $\rho = 0,\dots,r$, and $\widetilde{\ }$ maps the space $\mathcal{O}^{(\rho)}(\mathcal{B})$ into $\mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \mathcal{O}^{(\rho)}\left(\mathbb{C}^{0|r}\right)$. Therefore we have splittings

$$sM_k(\Gamma) = \bigoplus_{\rho=0}^{r} sM_k^{(\rho)}(\Gamma) \quad \text{and} \quad sS_k(\Gamma) = \bigoplus_{\rho=0}^{r} sS_k^{(\rho)}(\Gamma),$$

where $sM_k^{(\rho)}(\Gamma) := sM_k(\Gamma) \cap \mathcal{O}^{(\rho)}(\mathcal{B})$, $sS_k^{(\rho)}(\Gamma) := sS_k(\Gamma) \cap \mathcal{O}^{(\rho)}(\mathcal{B})$, $\rho = 0,\dots,r$, and the last sum is orthogonal.

As shown in [10] and in Section 3.2 of [11] there is an analogon to SATAKE's theorem in the super case:

Theorem 2.1. *Let $\rho \in \{0,\dots,r\}$. Assume $\Gamma\backslash G$ is compact or $n \geq 2$ and $\Gamma \sqsubset G$ is a lattice (discrete such that $\operatorname{vol}\Gamma\backslash G < \infty$, $\Gamma\backslash G$ not necessarily compact). If $k \geq 2n - \rho$ then*

$$sS_k^{(\rho)}(\Gamma) = sM_k^{(\rho)}(\Gamma) \cap L_k^s(\Gamma\backslash\mathcal{B})$$

for all $s \in [1,\infty]$.

As in the classical case this theorem implies that if $\Gamma\backslash G$ is compact or $n \geq 2$, $\Gamma \sqsubset G$ is a lattice and $k \geq 2n - \rho$, then the HILBERT space $sS_k^{(\rho)}(\Gamma)$ is finite dimensional.

We will use the JORDAN triple determinant $\Delta : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given by

$$\Delta(\mathbf{z},\mathbf{w}) := 1 - \mathbf{w}^*\mathbf{z}$$

for all $\mathbf{z},\mathbf{w} \in \mathbb{C}^n$. Let us recall the basic properties:

(i) $|j(g,\mathbf{0})| = \Delta(g\mathbf{0},g\mathbf{0})^{\frac{1}{2}}$ for all $g \in G$,
(ii) $\Delta(g\mathbf{z},g\mathbf{w}) = \Delta(\mathbf{z},\mathbf{w})\, j(g,\mathbf{z})\, \overline{j(g,\mathbf{w})}$ for all $g \in G$ and $\mathbf{z},\mathbf{w} \in B$, and
(iii) $\int_B \Delta(\mathbf{z},\mathbf{z})^\lambda\, dV_{\mathrm{Leb}} < \infty$ if and only if $\lambda > -1$.

We have the G-invariant volume element $\Delta(\mathbf{z},\mathbf{z})^{-(n+1)}dV_{\mathrm{Leb}}$ on B.
For all $I\in\wp(r)$, $h\in\mathcal{O}(B)$, $\mathbf{z}\in B$ and

$$g=\left(\begin{array}{c|c}*&0\\\hline 0&E\end{array}\right)\in G$$

we have

$$h\zeta^I\big|_g(\mathbf{z})=h(g\mathbf{z})(E\eta)^I j(g,\mathbf{z})^{k+|I|},$$

where $E\in U(r)$. So for all $s\in]0,\infty]$, $f=\sum_{I\in\wp(r)}f_I\zeta^I$ and $h=\sum_{I\in\wp(r)}h_I\zeta^I\in\mathcal{O}(\mathcal{B})$ we have

$$||f||^{(k)}_{s,\Gamma}\equiv\left|\left|\sqrt{\sum_{I\in\wp(r)}f_I^2\Delta(\mathbf{z},\mathbf{z})^{k+|I|}}\right|\right|_{s,\Gamma\backslash B,\Delta(\mathbf{z},\mathbf{z})^{-(n+1)}dV_{\mathrm{Leb}}}$$

if $\widetilde{f}\in\mathcal{C}^\infty(G)\otimes\mathcal{O}\left(\mathbb{C}^{0|r}\right)$ and

$$(f,h)_\Gamma\equiv\sum_{I\in\wp(r)}\int_{\Gamma\backslash B}\overline{f_I}h_I\Delta(\mathbf{z},\mathbf{z})^{k+|I|-(n+1)}\,dV_{\mathrm{Leb}}$$

if $\left\langle\widetilde{h},\widetilde{f}\right\rangle\in L^1(\Gamma\backslash G)$, where '$\equiv$' means equality up to a constant $\neq 0$ depending on Γ.

For the explicit computation of the elements of our spanning set in Theorem 4.3 we need the following lemmas:

Lemma 2.2 (Convergence of relative POINCARÉ series)**.** *Let $\Gamma_0\sqsubset\Gamma$ be a subgroup and*

$$f\in sM_k(\Gamma_0)\cap L^1_k(\Gamma_0\backslash\mathcal{B}).$$

Then

$$\Phi:=\sum_{\gamma\in\Gamma_0\backslash\Gamma}f|_\gamma\text{ and }\Phi':=\sum_{\gamma\in\Gamma_0\backslash\Gamma}\widetilde{f}(\gamma\diamondsuit)$$

converge absolutely and uniformly on compact subsets of B resp. G,

$$\Phi\in sM_k(\Gamma)\cap L^1_k(\Gamma\backslash\mathcal{B}),$$

$\widetilde{\Phi}=\Phi'$, and for all $\varphi\in sM_k(\Gamma)\cap L^\infty_k(\Gamma\backslash\mathcal{B})$ we have

$$(\Phi,\varphi)_\Gamma=(f,\varphi)_{\Gamma_0}.$$

The symbol '$\diamondsuit$' here and also later simply stands for the argument of the function. So $\widetilde{f}(\gamma\diamondsuit)\in\mathcal{C}^\infty(G)^{\mathbb{C}}\otimes\bigwedge(\mathbb{C}^r)$ is a short notation for the smooth map

$$G\to\bigwedge(\mathbb{C}^r),g\mapsto\widetilde{f}(\gamma g).$$

Proof. Standard, on using the mean value property of holomorphic functions for all $k\in\mathbb{Z}$ without any further assumption on k. □

Lemma 2.3. *Let $I\in\wp(r)$ and $k\geq 2n+1-|I|$. Then for all $\mathbf{w}\in B$*

$$\Delta(\diamondsuit,\mathbf{w})^{-k-|I|}\zeta^I\in\mathcal{O}^{|I|}(\mathcal{B})\cap L^1_k(\mathcal{B}),$$

and for all $f=\sum_{J\in\wp(r)}f_J\zeta^J\in\mathcal{O}(\mathcal{B})\cap L^\infty_k(\mathcal{B})$ we have

$$\left(\Delta(\diamondsuit,\mathbf{w})^{-k-|I|}\zeta^I,f\right)\equiv f_I(\mathbf{w}),$$

where $(\ ,\ ):=(\ ,\ )_{\{1\}}$.

Since the proof is also standard, we will omit it here. It can be found in [11].

3. THE STRUCTURE OF THE GROUP G

We have a canonical embedding

$$G' := SU(p,q) \hookrightarrow G, g' \mapsto \left(\begin{array}{c|c} g' & 0 \\ \hline 0 & 1 \end{array}\right),$$

and the canonical projection

$$G \to U(r), g := \left(\begin{array}{c|c} g' & 0 \\ \hline 0 & E \end{array}\right) \mapsto E_g := E$$

induces a group isomorphism

$$G/G' \simeq U(r).$$

Obviously $K' = K \cap G' = S(U(n) \times U(1))$ is the stabilizer of $\mathbf{0}$ in G'. Let A denote the common standard maximal split abelian subgroup of G and G' given by the image of the LIE group embedding

$$\mathbb{R} \hookrightarrow G', t \mapsto a_t := \left(\begin{array}{c|c|c} \cosh t & 0 & \sinh t_1 \\ \hline 0 & 1 & 0 \\ \hline \sinh t & 0 & \cosh t \end{array}\right).$$

Then the centralizer M of A in K is the group of all

$$\left(\begin{array}{c|c|c} \begin{array}{c|c} \varepsilon & 0 \\ \hline 0 & u \end{array} & 0 & 0 \\ \hline 0 & \varepsilon & \\ \hline 0 & & E \end{array}\right),$$

where $\varepsilon \in U(1)$, $u \in U(p-1)$ and $E \in U(r)$ such that $\varepsilon^2 \det u = \det E$. Let $M' = K' \cap M = G' \cap M$ be the centralizer of A in K'. The centralizer of G' in G is precisely

$$Z_G(G') := \left\{ \left(\begin{array}{c|c} \varepsilon 1 & 0 \\ \hline 0 & E \end{array}\right) \middle| \varepsilon \in U(1), E \in U(r), \varepsilon^{p+1} = \det E \right\} \sqsubset M,$$

and $G' \cap Z_G(G') = Z(G')$. An easy calculation shows that $G = G'Z_G(G')$. So $K = K'Z_G(G')$ and $M = M'Z(G')$. Therefore if we decompose the adjoint representation of A as

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha,$$

where for all $\alpha \in \mathbb{R}$

$$\mathfrak{g}^\alpha := \left\{ \xi \in \mathfrak{g} \middle| \mathrm{Ad}_{a_t}(\xi) = e^{\alpha t} \right\}$$

is the corresponding root space and

$$\Phi := \{ \alpha \in \mathbb{R} | \, g^\alpha \neq 0 \}$$

is the root system, then we see that Φ is at the same time the root system of G', so $\Phi = \{0, \pm 2\}$ if $n = 1$ and $\Phi = \{0, \pm 1, \pm 2\}$ if $n \geq 2$. Furthermore, if $\alpha \neq 0$ then $\mathfrak{g}^\alpha \sqsubset \mathfrak{g}'$ is at the same time the corresponding root space of $\mathfrak{g}'$, and finally $\mathfrak{g}^0 = \mathfrak{a} \oplus \mathfrak{m} = \mathfrak{a} \oplus \mathfrak{m}' \oplus \mathfrak{z}_\mathfrak{g}(\mathfrak{g}')$.

Lemma 3.1.

$$N(A) = AN_K(A) = N(AM) \sqsubset N(M).$$

Proof. Simple calculation. □

In particular we have the WEYL group

$$W := M \backslash N_K(A) \simeq M' \backslash N_{K'}(A) \simeq \{\pm 1\}$$

acting on $A \simeq \mathbb{R}$ via sign change. For the main result, Theorem 4.3, of this article the following definition is crucial:

Definition 3.1. Let $g_0 \in G$.
(i) g_0 is called loxodromic if and only if there exists $g \in G$ such that $g_0 \in gAMg^{-1}$.
(ii) If g_0 is loxodromic, it is called regular if and only if $g_0 = ga_t w g^{-1}$ with $t \in \mathbb{R} \setminus \{0\}$ and $w \in M$.
(iii) If $\gamma \in \Gamma$ is regular loxodromic then it is called primitive in Γ if and only if $\gamma = \gamma'^{\nu}$ implies $\nu \in \{\pm 1\}$ for all loxodromic $\gamma' \in \Gamma$ and $\nu \in \mathbb{Z}$.

Clearly for all $\gamma \in \Gamma$ regular loxodromic there exists $\gamma' \in \Gamma$ primitive regular loxodromic and $\nu \in \mathbb{N} \setminus \{0\}$ such that $\gamma = \gamma'^{\nu}$.

Lemma 3.2. *Let $g_0 \in G$ be regular loxodromic, $g \in G$, $w \in M$ and $t \in \mathbb{R} \setminus \{0\}$ such that $g_0 = ga_t w g^{-1}$. Then g is uniquely determined up to right translation by elements of $AN_K(A)$, and t is uniquely determined up to sign.*

Proof. By straight forward computation or using the following strategy: Let $g' \in G$, $w' \in M$ and $t' \in \mathbb{R}$ such that $g_0 = g' a_{t'} w' g'^{-1}$ also. Then $a_t w = (g^{-1}g')\, a_{t'} w' \,(g^{-1}g')^{-1}$. Since $t \in \mathbb{R} \setminus \{0\}$ and because of the root space decomposition, $\mathfrak{a} + \mathfrak{m}$ must be the largest subspace of $\mathfrak{g}$ on which $\mathrm{Ad}_{a_t w}$ is orthogonal with respect to an appropiate scalar product. So $\mathrm{Ad}_{g^{-1}g'}$ maps $\mathfrak{a} + \mathfrak{m}$ into itself. This implies $g^{-1}g' \in N(AM) = AN_K(A)$ by Lemma 3.1. □

4. THE MAIN RESULT

Let $\rho \in \{0, \ldots, r\}$. Assume $\Gamma\backslash G$ compact **or** $n \geq 2$, $\mathrm{vol}\, \Gamma\backslash G < \infty$ and $k \geq 2n - \rho$. Let $C > 0$ be given. Let us consider a regular loxodromic $\gamma_0 \in \Gamma$. Let $g \in G$, $w_0 \in M$ and $t_0 > 0$ such that $\gamma_0 = ga_{t_0} w_0 g^{-1}$.

There exists a torus $\mathbb{T} := \langle \gamma_0 \rangle \backslash\, gAM$ belonging to γ_0. From Lemma 3.2 it follows that $\mathbb{T}$ is independent of g up to right translation with an element of the WEYL group $W = M\backslash N_K(A)$.

Let $f \in sS_k(\Gamma)$. Then $\widetilde{f} \in \mathcal{C}^\infty(\Gamma\backslash G)^{\mathbb{C}} \otimes \mathcal{O}\left(\mathbb{C}^{0|r}\right)$. Define $h \in \mathcal{C}^\infty(\mathbb{R} \times M)^{\mathbb{C}} \otimes \mathcal{O}\left(\mathbb{C}^{0|r}\right)$ as

$$h(t, w) := \widetilde{f}(ga_t w)$$

for all $(t, w) \in \mathbb{R} \times M$ 'screening' the values of $\widetilde{f}$ on $\mathbb{T}$. Then clearly $h(t, w) = h(t, 1, E_w \eta j(w)) j(w)^k$, and so $h(t, w) = h(t, 1, E_w \eta) j(w)^{k+\rho}$ if $f \in sS_k^{(\rho)}(\Gamma)$, for all $(t, w) \in \mathbb{R} \times M$. Clearly $E_0 := E_{w_0} \in U(r)$. So we can choose $g \in G$ such that E_0 is diagonal without changing $\mathbb{T}$. Choose $D \in \mathbb{R}^{r \times r}$ diagonal such that $\exp(2\pi i D) = E_0$ and $\chi \in \mathbb{R}$ such that $j(w_0) = e^{2\pi i \chi}$. D and χ are uniquely determined by w_0 up to $\mathbb{Z}$. If

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix}$$

with $d_1, \ldots, d_r \in \mathbb{R}$ and $I \in \wp(r)$, then we define $\mathrm{tr}_I D := \sum_{j \in I} d_j$.

Theorem 4.1 (FOURIER expansion of h)**.**
(i) *$h(t+t_0,w) = h\left(t, w_0^{-1}w\right)$ for all $(t,w) \in \mathbb{R}\times M$, and there exist unique $b_{I,m} \in \mathbb{C}$, $I \in \wp(r)$, $m \in \frac{1}{t_0}\left(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D\right)$, such that*

$$h(t,w) = \sum_{I\in\wp(r)} j(w)^{k+|I|} \sum_{m\in\frac{1}{t_0}(\mathbb{Z}-(k+|I|)\chi-\mathrm{tr}_I D)} b_{I,m} e^{2\pi i m t} \left(E_w \eta\right)^I$$

for all $(t,w) \in \mathbb{R}\times M$, where the sum converges uniformly in all derivatives.
(ii) *If $f \in sS_k^{(\rho)}(\Gamma)$, $b_{I,m} = 0$ for all $I \in \wp(r)$, $|I| = \rho$, and $m \in \frac{1}{t_0}(\mathbb{Z} - (k+\rho)\chi - \mathrm{tr}_I D) \cap]-C, C[$ then there exists $H \in \mathcal{C}^\infty(\mathbb{R}\times M)^{\mathbb{C}} \otimes \bigwedge(\mathbb{C}^r)$ uniformly* LIPSCHITZ *continuous with a* LIPSCHITZ *constant $C_2 \geq 0$ independent of γ_0 such that*

$$h = \partial_t H,$$

$$H(t,w) = j(w)^k H\left(t, 1, E_w \eta j(w)\right)$$

and

$$H(t+t_0,w) = H\left(t, w_0^{-1}w\right)$$

for all $(t,w) \in \mathbb{R}\times M$.

Proof. (i) Let $t \in \mathbb{R}$ and $w \in M$. Then

$$\begin{aligned} h(t+t_0,w) &= \widetilde{f}\left(g a_{t_0} a_t w\right) = \widetilde{f}\left(\gamma_0 g w_0^{-1} a_t w\right) = \widetilde{f}\left(g a_t w_0^{-1} w\right) \\ &= h\left(t, w_0^{-1}w\right), \end{aligned}$$

and so

$$\begin{aligned} h(t+t_0,1) &= h\left(t, w_0^{-1}\right) \\ &= j(w_0)^{-k} h\left(t, 1, E_0^{-1}\eta j(w_0)^{-1}\right) \\ &= j(w_0)^{-k} \sum_{I\in\wp(r)} h(t,1)\, e^{-2\pi i \mathrm{tr}_I D} \eta^I j(w_0)^{-|I|} \\ &= \sum_{I\in\wp(r)} e^{-2\pi i((k+|I|)\chi+\mathrm{tr}_I D)} h_I(t,1)\eta^I. \end{aligned}$$

Therefore $h_I(t+t_0,1) = e^{-2\pi i((k+|I|)\chi+\mathrm{tr}_I D)} h_I(t,1)$ for all $I \in \wp(r)$, and the rest follows by a standard FOURIER expansion. □

To prove (ii) we need the following lemma:

Lemma 4.2 (Generalization of the reverse BERNSTEIN inequality)**.** *Let $t_0 \in \mathbb{R}\setminus\{0\}$, $\nu \in \mathbb{R}$ and $C > 0$. Let $\mathcal{S}$ be the space of all convergent* FOURIER *series*

$$s = \sum_{m\in\frac{1}{t_0}(\mathbb{Z}-\nu), |m|\geq C} s_l e^{2\pi i m \diamondsuit} \in \mathcal{C}^\infty(\mathbb{R})^{\mathbb{C}},$$

for all $s_m \in \mathbb{C}$. Then

$$\widehat{\ } : \mathcal{S} \to \mathcal{S}, s = \sum_{m\in\frac{1}{t_0}(\mathbb{Z}-\nu), |m|\geq C} s_m e^{2\pi i m\diamondsuit} \mapsto \widehat{s} := \sum_{m\in\frac{1}{t_0}(\mathbb{Z}-\nu), |m|\geq C} \frac{s_m}{2\pi i m} e^{2\pi i m\diamondsuit}$$

is a well-defined linear map, and $||\widehat{s}||_\infty \leq \frac{6}{\pi C}||s||_\infty$ for all $s \in \mathcal{S}$.

Proof. This can be deduced from the ordinary reverse BERNSTEIN inequality, see for example Theorem 8.4 in Chapter I of [9].

Now we prove Theorem 4.1 (ii). Fix some $I \in \wp(r)$ such that $|I| = \rho$ and $b_{I,m} = 0$ for all $m \in \frac{1}{t_0}(\mathbb{Z} - (k+\rho)\chi - \mathrm{tr}_I D) \cap]-C, C[$. Then if we define $\nu := (k+\rho)\chi + \mathrm{tr}_I D \in \mathbb{R}$ we have

$$h_I(\diamondsuit, 1) = \sum_{m \in \frac{1}{t_0}(\mathbb{Z}-\nu), |m| \geq C} b_{I,m} e^{2\pi i m \diamondsuit},$$

and so we can apply the generalized reverse BERNSTEIN inequality, Lemma 4.2, to h_I. Therefore we can define

$$H'_I := \widehat{h_I(\diamondsuit, 1)} = \sum_{m \in \frac{1}{t_0}(\mathbb{Z}-\nu), |m| \geq C} \frac{b_{I,m}}{2\pi i m} e^{2\pi i m \diamondsuit} \in \mathcal{C}^\infty(\mathbb{R})^{\mathbb{C}}.$$

$\left|\widetilde{f}\right| \in L^\infty(G)$ by SATAKE's theorem, Theorem 2.1, and so there exists a constant $C' > 0$ independent of γ_0 and I such that $||h_I||_\infty < C'$, and now Lemma 4.2 tells us that

$$||H'_I||_\infty \leq \frac{6}{\pi C} ||h(\diamondsuit, 1)||_\infty \leq \frac{6C'}{\pi C}.$$

Clearly $h_I(\diamondsuit, 1) = \partial_t H'_I$.

Since j is smooth on the compact set M, $j^{k+\rho}(E_w\eta)^I$ is uniformly LIPSCHITZ continuous on M with a common LIPSCHITZ constant C'' independent of γ_0 and I. So we see that $H \in \mathcal{C}^\infty(\mathbb{R}, M)^{\mathbb{C}} \otimes \bigwedge(\mathbb{C}^r)$ defined as

$$H(t, w) := \sum_{I \in \wp(r)} j(w)^{k+\rho} H'_I(t) (E_w\eta)^I$$

for all $(t, w) \in \mathbb{R} \times M$ is uniformly LIPSCHITZ continuous with LIPSCHITZ constant $C_2 := \left(\frac{6C''}{\pi C} + 1\right) C'$ independent of γ_0, and the rest is trivial. □

Let $I \in \wp(r)$ and $m \in \frac{1}{t_0}(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D)$. Since $sS_k(\Gamma)$ is a HILBERT space and $sS_k(\Gamma) \to \mathbb{C}, f \mapsto b_{I,m}$ is linear and continuous, there exists exactly one $\varphi_{\gamma_0,I,m} \in sS_k(\Gamma)$ such that $b_{I,m} = (\varphi_{\gamma_0,I,m}, f)$ for all $f \in sS_k(\Gamma)$. Clearly $\varphi_{\gamma_0,I,m} \in sS_k^{(|I|)}(\Gamma)$.

For the remainder of the article for simplicity we write $m \in]-C, C[$ instead of $m \in \frac{1}{t_0}(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D) \cap]-C, C[$. In the last section we will compute $\varphi_{\gamma_0,I,m}$ as a relative POINCARÉ series. One can check that the family

$$\left\{\varphi_{\gamma_0,I,m}\right\}_{I \in \wp(r), |I|=\rho, m \in]-C,C[}$$

is independent of the choice of g, D and χ up to multiplication with a unitary matrix with entries in $\mathbb{C}$ and invariant under conjugating γ_0 with elements of Γ.

Now we can state our main theorem: Let Ω be a fundamental set for all primitive regular loxodromic $\gamma_0 \in \Gamma$ modulo conjugation by elements of Γ and

$$\widetilde{Z} := \overline{\left\{ m \in Z_G(G') \;\middle|\; \exists g' \in G' : mg' \in \Gamma \right\}} \sqsubset Z_G(G').$$

Then clearly $\Gamma \sqsubset G'\widetilde{Z}$. Recall that we still assume

- $\Gamma\backslash G$ **compact** or
- $n \geq 2$, $\mathrm{vol}\,\Gamma\backslash G < \infty$ and $k \geq 2n - \rho$.

Theorem 4.3 (Spanning set for $sS_k(\Gamma)$)**.** *Assume that the right translation of A on $\Gamma\backslash G'\widetilde{Z}$ is topologically transitive. Then*

$$\{\varphi_{\gamma_0,I,m}|\,\gamma_0\in\Omega, I\in\wp(r), |I|=\rho, m\in]-C,C[\}$$

is a spanning set for $sS_k^{(\rho)}(\Gamma)$.

For proving this result we need an ANOSOV type theorem for G and the unbounded realization of $\mathcal{B}$, which we will discuss in the following two sections.

Remark 1.

(i) If there is some subgroup $\widetilde{M}\sqsubset Z_G(G')$ such that $\Gamma\sqsubset G'\widetilde{M}$ and the right translation of A on $\Gamma\backslash G'\widetilde{M}$ is topologically transitive then necessarily $\widetilde{M}Z(G')=\widetilde{Z}$ and there exists $g_0\in G'$ such that $G'\widetilde{Z}=\overline{\Gamma g_0 A}$. The latter statement is a trivial consequence of the fact that $\widetilde{Z}\sqsubset M$.

(ii) In the case where $\Gamma\cap G'\sqsubset\Gamma$ is of finite index or equivalently $\widetilde{Z}$ is finite then we know that the right translation of A on $\Gamma\backslash G'\widetilde{Z}$ is topologically transitive because of MOORE's ergodicity theorem, see [13] Theorem 2.2.6, and since then $\Gamma\cap G'\sqsubset G'$ is a lattice.

(iii) There is a finite-to-one correspondence between Ω and the set of closed geodesics of $\Gamma\backslash B$ assigning to each primitive loxodromic element $\gamma_0=ga_{t_0}w_0g^{-1}\in\Gamma$, $g\in G$, $t_0>0$ and $w_0\in M$, the image of the unique geodesic $gA\mathbf{0}$ of B normalized by γ_0 under the canonical projection $B\to\Gamma\backslash B$. It is of length t_0 if there is no irregular point of $\Gamma\backslash B$ on $gA\mathbf{0}$.

5. AN ANOSOV TYPE RESULT FOR THE GROUP G

On the LIE group G we have a smooth flow $(\varphi_t)_{t\in\mathbb{R}}$ given by the right translation by elements of A:

$$\varphi_t: G\to G, g\mapsto ga_t.$$

This turns out to be partially hyperbolic, and so we can apply a partial ANOSOV closing lemma. Let me mention that the flow $(\varphi_t)_{t\in\mathbb{R}}$ descends to the ordinary geodesic flow on the unit tangent bundle $SB\simeq M\backslash G$. Let us first have a look at the general theory of partial hyperbolicity: Let W be, for the moment, a smooth Riemannian manifold.

Definition 5.1 (Partially Hyperbolic Diffeomorphism and Flow)**.** Let $C>1$.

(i) Let φ be a $\mathcal{C}^\infty$-diffeomorphism of W. Then φ is called partially hyperbolic with constant C if and only if there exists an orthogonal $D\varphi$ (and therefore $D\varphi^{-1}$) -invariant $\mathcal{C}^\infty$-splitting

$$TW=T^0\oplus T^+\oplus T^- \tag{5.1}$$

of the tangent bundle TW such that $T^0\oplus T^+$, $T^0\oplus T^-$, T^0, T^+ and T^- are closed under the commutator, $D\varphi|_{T^0}$ is an isometry, $||D\varphi|_{T^-}||\le\frac{1}{C}$ and $||D\varphi^{-1}|_{T^+}||\le\frac{1}{C}$.

(ii) Let $(\varphi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^\infty$-flow on W. Then $(\varphi_t)_{t\in\mathbb{R}}$ is called partially hyperbolic with constant C if and only if all φ_t, $t>0$ are partially hyperbolic diffeomorphisms with a common splitting (5.1) and constants e^{Ct} resp. and T^0 contains the generator of the flow.

A partially hyperbolic diffeomorphism φ gives rise to $\mathcal{C}^\infty$-foliations on W corresponding to the splitting $TW=T^0\oplus T^+\oplus T^-$. Let us denote the distances along the $T^0\oplus T^+$-, T^0-, T^+- respectively T^--leaves by $d^{0,+}$, d^0, d^+ and d^-.

Definition 5.2. Let $TW=T^0\oplus T^+\oplus T^-$ be an orthogonal $\mathcal{C}^\infty$-splitting of the tangent bundle TW of W such that $T^0\oplus T^+$, T^0, T^+ and T^- are closed under the commutator, $C'\ge 1$ and $U\subset W$. U is called C'-rectangular (with respect to the splitting $TW=T^0\oplus T^+\oplus T^-$) if and only if for all $y,z\in U$

{i} there exists a unique intersection point $a \in U$ of the $T^0 \oplus T^+$-leaf containing y and the T^--leaf containing z and a unique intersection point $b \in U$ of the $T^0 \oplus T^+$-leaf containing z and the T^--leaf containing y,

$$d^{0,+}(y,a), d^-(y,b), d^-(z,a), d^{0,+}(z,b) \leq C' d(y,z),$$

and

$$\frac{1}{C'} d^{0,+}(z,b) \leq d^{0,+}(y,a) \leq C' d^{0,+}(z,b),$$
$$\frac{1}{C'} d^-(z,a) \leq d^-(y,b) \leq C' d^-(z,a).$$

{ii} if y and z belong to the same $T^0 \oplus T^+$-leaf there exists a unique intersection point $c \in U$ of the T^0-leaf containing y and the T^+-leaf containing z and a unique intersection point $d \in U$ of the T^0-leaf containing z and the T^+-leaf containing y,

$$d^0(y,c), d^+(y,d), d^+(z,c), d^0(z,d) \leq C' d^{0,+}(y,z),$$

and

$$\frac{1}{C'} d^0(z,d) \leq d^0(y,c) \leq C' d^0(z,d),$$
$$\frac{1}{C'} d^+(z,c) \leq d^+(y,d) \leq C' d^+(z,c).$$

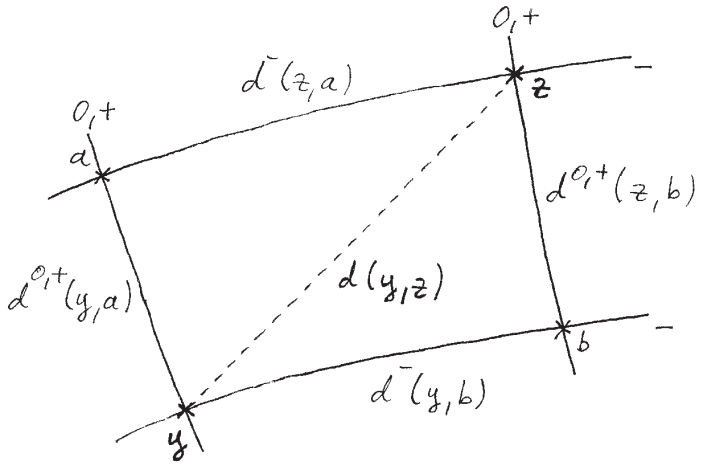

Figure 5.1: Intersection points in {i}.

Since the splitting $TW = T^0 \oplus T^+ \oplus T^-$ is orthogonal and smooth we see that for all $x \in W$ and $C' > 1$ there exists a C'-rectangular neighbourhood of x.

Theorem 5.1 (Partial ANOSOV closing lemma)**.** *Let φ be a partially hyperbolic diffeomorphism with constant C, let $x \in W$, $C' \in]1, C[$ and $\delta > 0$ such that $\overline{U_\delta(x)}$ is contained in a C'-rectangular subset $U \subset W$.*

If $d(x, \varphi(x)) \leq \delta \frac{1-\frac{C'}{C}}{C'^2+1}$ then there exist $y, z \in U$ such that

(i) *x and y belong to the same T^--leaf and*

$$d^-(x,y) \leq \frac{C'}{1-\frac{C'}{C}} d(x,\varphi(x)),$$

(ii) *y and $\varphi(y)$ belong to the same $T^0 \oplus T^+$-leaf and*

$$d^{0,+}(y,\varphi(y)) \leq C'^2 d(x,\varphi(x)),$$

(iii) *y and z belong to the same T^+-leaf and*

$$d^+(\varphi(y),\varphi(z)) \leq \frac{C'^3}{1-\frac{C'}{C}} d(x,\varphi(x)),$$

(iv) *z and $\varphi(z)$ belong to the same T^0-leaf and*

$$d^0(z,\varphi(z)) \leq \mathcal{C}'^4 d(x,\varphi(x)).$$

The proof, which will not be given here, uses a standard argument obtaining the points y and $\varphi(z)$ as limits of certain CAUCHY sequences. The interested reader will find it in [11].

Now let us return to the flow $(\varphi_t)_{t\in\mathbb{R}}$ on G and choose a left invariant metric on G such that $\mathfrak{g}^\alpha$, $\alpha \in \Phi \setminus \{0\}$, $\mathfrak{a}$ and $\mathfrak{m}$ are pairwise orthogonal and the isomorphism $\mathbb{R} \simeq A \subset G$ is isometric. Then since the flow $(\varphi_t)_{t\in\mathbb{R}}$ commutes with left translations it is indeed partially hyperbolic with constant 1 and the unique left invariant splitting of TG given by

$$T_1G = \mathfrak{g} = \underbrace{\mathfrak{a}\oplus\mathfrak{m}}_{T_1^0:=} \oplus \underbrace{\bigoplus_{\alpha\in\Phi,\alpha>0} \mathfrak{g}^\alpha}_{T_1^-:=} \oplus \underbrace{\bigoplus_{\alpha\in\Phi,\alpha<0} \mathfrak{g}^\alpha}_{T_1^+:=}.$$

For all $L \subset G$ compact, $T, \varepsilon > 0$ define

$$M_{L,T} := \left\{ g a_t g^{-1} \middle| \, g \in L, t \in [-T,T] \right\}$$

and

$$N_{L,T,\varepsilon} := \{ g \in G \,|\mathrm{dist}\,(g, M_{L,T}) \leq \varepsilon \}.$$

Lemma 5.2. *For all $L \subset G$ compact there exist $T_0, \varepsilon_0 > 0$ such that $\Gamma \cap N_{L,T_0,\varepsilon_0} = \{1\}$.*

Proof. Let $L \subset G$ be compact and $T > 0$. Then $M_{L,T}$ is compact, and so there exists $\varepsilon > 0$ such that $N_{L,T,\varepsilon}$ is again compact. Since Γ is discrete, $\Gamma \cap N_{L,T,\varepsilon}$ is finite. Clearly for all T, T', ε and $\varepsilon' > 0$ if $T \leq T'$ and $\varepsilon \leq \varepsilon'$ then $N_{L,T,\varepsilon} \subset N_{L,T',\varepsilon'}$, and finally

$$\bigcap_{T,\varepsilon>0} N_{T,\varepsilon} = \{1\}.$$

□

Here now is the quintessence of this section:

Theorem 5.3.

(i) *For all $T_1 > 0$ there exist $C_1 \geq 1$ and $\varepsilon_1 > 0$ such that for all $x \in G$, $\gamma \in \Gamma$ and $T \geq T_1$ if*

$$\varepsilon := d(\gamma x, x a_T) \leq \varepsilon_1$$

then there exist $z \in G$, $w \in M$ and $t_0 > 0$ such that $\gamma z = z a_{t_0} w$ (and so γ is regular loxodromic), $d((t_0,w),(T,1)) \leq C_1\varepsilon$ and for all $\tau \in [0,T]$

$$d(x a_\tau, z a_\tau) \leq C_1 \varepsilon \left(e^{-\tau} + e^{-(T-\tau)} \right).$$

(ii) *For all $L \subset G$ compact there exists $\varepsilon_2 > 0$ such that for all $x \in L$, $\gamma \in \Gamma$ and $T \in [0, T_0]$, $T_0 > 0$ given by Lemma 5.2, if*

$$\varepsilon := d(\gamma x, x a_T) \leq \varepsilon_2$$

then $\gamma = 1$ and $T \leq \varepsilon$.

Proof. (i) Let $T_1 > 0$ and define

$$C_1 := \max\left(\frac{e^{\frac{3}{2}T_1}}{1 - e^{-\frac{T_1}{2}}}, e^{2T_1}\right) \geq 1.$$

Define $C' := e^{\frac{T_1}{2}}$, let U be a C'-rectangular neighbourhood of $1 \in G$ and let $\delta > 0$ such that $\overline{U_\delta(1)} \subset U$. Then by the left invariance of the splitting and the metric on G we see that gU is a C'-rectangular neighbourhood of g and $\overline{U_\delta(g)} = g\overline{U_\delta(1)} \subset gU$ for all $g \in G$. Define

$$\varepsilon_1 := \min\left(\delta \frac{1 - e^{-\frac{T_1}{2}}}{e^{T_1} + 1}, \frac{T_1}{C_1}\right) > 0.$$

Now assume $\gamma \in \Gamma$ and $T \geq T_1$ such that

$$\varepsilon := d(\gamma x, x a_{T\mathbf{v}}) \leq \varepsilon_1.$$

Then $\varphi : G \to G, g \mapsto \gamma^{-1} g a_T$ is a partially hyperbolic diffeomorphism with constant $e^{T_1} > 1$ and the corresponding splitting $TG = T^0 \oplus T^+ \oplus T^-$. Then since

$$\varepsilon \leq \delta \frac{1 - e^{-\frac{T_1}{2}}}{e^{T_1} + 1} = \delta \frac{1 - C' e^{-T_1}}{C'^2 + 1}$$

the partial ANOSOV closing lemma, Theorem 5.1, tells us that there exist $y, z \in G$ such that

(i) x and y belong to the same T^--leaf and

$$d^-(x, y) \leq \varepsilon \frac{C'}{1 - \frac{C'}{C}},$$

(iii) y and z belong to the same T^+-leaf and

$$d^+(y a_{T\mathbf{v}}, z a_{T\mathbf{v}}) \leq \varepsilon \frac{C'^3}{1 - \frac{C'}{C}},$$

(iv) γz and $z a_{T\mathbf{v}}$ belong to the same T^0-leaf and

$$d^0(\gamma z, z a_{T\mathbf{v}}) \leq \varepsilon C'^4.$$

In (iii) and (iv) we already used that the metric and the flow are left invariant. So by (iv) and since the T^0-leaf containing $z a_T$ is zAM, there exist $w \in M$ and $t_0 \in \mathbb{R}$ such that $\gamma z = z a_{t_0} w$. So

$$d^0(a_{t_0 - T} w, 1) \leq \varepsilon C'^4,$$

and so, since $AM \simeq \mathbb{R} \times M$ isometrically, we see that

$$d((t_0, w), (T, 1)) \leq \varepsilon C'^4 = \varepsilon e^{2T_1} \leq \varepsilon C_1.$$

In particular, $|t_0 - T| \leq T_1$, and so $t_0 > 0$.

Now let $\tau \in [0, T]$. Then since x and y belong to the same T^--leaf, the same is true for $x a_\tau$ and $y a_\tau$, and

$$d^-(x a_\tau, y a_\tau) \leq d^-(x, y) e^{-\tau} \leq \varepsilon \frac{C'}{1 - \frac{C'}{C}} e^{-\tau} \leq \varepsilon C_1 e^{-\tau}.$$

Since y and z belong to the same T^+-leaf, the same is true for ya_τ and za_τ, and

$$\begin{aligned} d^+\left(ya_\tau, za_\tau\right) &\leq d^+\left(ya_T, za_T\right) e^{-(T-\tau)} \\ &\leq \varepsilon \frac{C'^3}{1-\frac{C'}{C}} e^{-(T-\tau)} \leq \varepsilon C_1 e^{-(T-\tau)}. \end{aligned}$$

Combining these two inequalities we obtain

$$d\left(xa_\tau, za_\tau\right) \leq \varepsilon C_1 \left(e^{-\tau} + e^{-(T-\tau)}\right).$$

(ii) Let $L \subset G$ be compact and let $c \geq 1$ be given such that $||\mathrm{Ad}_g||, \left|\left|\mathrm{Ad}_g^{-1}\right|\right| \leq c$ and therefore

$$\frac{1}{c} d(ag, bg) \leq d(a, b) \leq c d(ag, bg)$$

for all $g \in L$ and $a, b \in G$. Let $\varepsilon_0 > 0$ be given by Lemma 5.2 and define

$$\varepsilon_2 := \frac{\varepsilon_0}{c} > 0.$$

Let $x \in L$, $\gamma \in \Gamma$ and $T \in [0, T_0]$ such that

$$\varepsilon := d\left(\gamma x, xa_T\right) \leq \varepsilon_2.$$

Then since $x \in L$, we get

$$d\left(\gamma, xa_T x^{-1}\right) \leq c\varepsilon \leq \varepsilon_0$$

and so $\gamma \in \Gamma \cap N_{L,T_0,\varepsilon_0}$. This implies $\gamma = 1$ and so $d\left(1, a_T\right) = \varepsilon$ and therefore $T \leq \varepsilon$. □

6. THE UNBOUNDED REALIZATION

Let $\mathfrak{n} \sqsubset \mathfrak{g}'$ be the standard maximal nilpotent sub LIE algebra, which is at the same time the direct sum of all root spaces of $\mathfrak{g}'$ of positive roots with respect to $\mathfrak{a}$. Let $N := \exp \mathfrak{n}$. Then we have an IWASAWA decomposition

$$G = NAK,$$

N is 2-step nilpotent, and so $N' := [N, N]$ is at the same time the center of N.

Now we transform the whole problem to the unbounded realization via the partial CAYLEY transformation

$$R := \left(\begin{array}{c|c|c} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \hline 0 & 1 & 0 \\ \hline -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{array}\right) \begin{array}{l} \leftarrow 1 \\ \}n-1 \\ \leftarrow n+1 \end{array} \in G'^{\mathbb{C}} = SL(n+1, \mathbb{C})$$

mapping B biholomorphically onto the unbounded domain

$$H := \left\{ \mathbf{w} = \left(\begin{array}{c} w_1 \\ \hline \mathbf{w}_2 \end{array}\right) \begin{array}{l} \leftarrow 1 \\ \}n-1 \end{array} \in \mathbb{C}^n \,\middle|\, \mathrm{Re}\, w_1 > \frac{1}{2} \mathbf{w}_2^* \mathbf{w}_2 \right\}.$$

We see that

$$RG'R^{-1} \sqsubset G'^{\mathbb{C}} = SL(n+1, \mathbb{C}) \hookrightarrow GL(n+1, \mathbb{C}) \times GL(r, \mathbb{C})$$

acts holomorphically and transitively on H via fractional linear transformations, and explicit calculations show that

$$a'_t := Ra_t R^{-1} = \left(\begin{array}{c|c|c} e^t & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & e^{-t} \end{array}\right) \begin{array}{l} \leftarrow 1 \\ \}n-1 \\ \leftarrow n+1 \end{array}$$

for all $t \in \mathbb{R}$, and RNR^{-1} is the image of

$$\mathbb{R} \times \mathbb{C}^{n-1} \to RG'R^{-1}, (\lambda, \mathbf{u}) \mapsto n'_{\lambda,\mathbf{u}} := \left(\begin{array}{c|c|c} 1 & \mathbf{u}^* & i\lambda + \frac{1}{2}\mathbf{u}^*\mathbf{u} \\ \hline 0 & 1 & \mathbf{u} \\ \hline 0 & 0 & 1 \end{array}\right),$$

which is a $\mathcal{C}^\infty$-diffeomorphism onto its image, with the multiplication rule

$$n'_{\lambda,\mathbf{u}} n'_{\mu,\mathbf{v}} = n'_{\lambda+\mu+\mathrm{Im}\ (\mathbf{u}^*\mathbf{v}),\mathbf{u}+\mathbf{v}}$$

for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{n-1}$, so N is exactly the HEISENBERG group H_n acting on H as pseudo translations

$$\mathbf{w} \mapsto \left(\frac{w_1 + \mathbf{u}^*\mathbf{w}_2 + i\lambda + \frac{1}{2}\mathbf{u}^*\mathbf{u}}{\mathbf{w}_2 + \mathbf{u}}\right).$$

Define $j(R, \mathbf{z}) = \frac{\sqrt{2}}{1-z_1} \in \mathcal{O}(B)$, $j(R^{-1}, \mathbf{w}) := j(R, R^{-1}\mathbf{w})^{-1} = \frac{\sqrt{2}}{1+w_1} \in \mathcal{O}(H)$, and for all

$$g \in RGR^{-1} = \left(\begin{array}{c|c|c} A & \mathbf{b} & \\ \hline \mathbf{c} & d & 0 \\ \hline 0 & & E \end{array}\right) \in RGR^{-1}$$

define

$$j(g, \mathbf{w}) = j\left(R, R^{-1}g\mathbf{w}\right) j\left(R^{-1}gR, R^{-1}\mathbf{w}\right) j\left(R^{-1}, \mathbf{w}\right) = \frac{1}{\mathbf{c}\mathbf{w} + d}.$$

Let $\mathcal{H} := H^{|r}$ with even coordinate functions $w_1, \ldots, w_n$ and odd coordinate functions $\vartheta_1, \ldots, \vartheta_r$. R commutes with all $g \in Z_G(G')$, and we have a right-representation of the group RGR^{-1} on $\mathcal{D}(\mathcal{H})$ given by

$$|_g : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}), f \mapsto f\left(g\left(\frac{\diamondsuit}{\vartheta}\right)\right) j(g, \diamondsuit)^k$$

for all $g \in RGR^{-1}$. If we define

$$|_R : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{B}), f \mapsto f\left(R\left(\frac{\diamondsuit}{\zeta}\right)\right) j(R, \diamondsuit)^k$$

and

$$|_{R^{-1}} : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{H}), f \mapsto f\left(R^{-1}\left(\frac{\diamondsuit}{\vartheta}\right)\right) j\left(R^{-1}, \diamondsuit\right)^k,$$

then we see that we get a commuting diagram

$$\begin{array}{ccc} \mathcal{D}(\mathcal{H}) & \overset{|_{RgR^{-1}}}{\longrightarrow} & \mathcal{D}(\mathcal{H}) \\ {\scriptstyle |_R}\ \downarrow & & \downarrow\ {\scriptstyle |_R} \\ \mathcal{D}(\mathcal{B}) & \underset{|_g}{\longrightarrow} & \mathcal{D}(\mathcal{B}) \end{array}.$$

Now define the sesqui polynomial Δ' on $H \times H$, holomorphic in the first and antiholomorphic in the second variable, as

$$\Delta'(\mathbf{z}, \mathbf{w}) := \Delta\left(R^{-1}\mathbf{z}, R^{-1}\mathbf{w}\right) j\left(R^{-1}, \mathbf{z}\right)^{-1} \overline{j\left(R^{-1}, \mathbf{w}\right)}^{-1} = z_1 + \overline{w_1} - \mathbf{w}_2^*\mathbf{z}_2$$

for all $\mathbf{z}, \mathbf{w} \in H$. Clearly $\left|\det(\mathbf{z} \mapsto R\mathbf{z})'\right| = |j(R, \mathbf{z})|^{n+1}$ for all $\mathbf{z} \in B$. So

$$\left|\det(\mathbf{w} \mapsto g\mathbf{w})'\right| = |j(g, \mathbf{w})|^{n+1},$$

$$|j(g, \mathbf{e}_1)| = \Delta'(g\mathbf{e}_1, g\mathbf{e}_1)^{\frac{1}{2}}$$

for all $g \in RGR^{-1}$ and $\Delta'(\mathbf{w},\mathbf{w})^{-(n+1)}\, dV_{\mathrm{Leb}}$ is the RGR^{-1} -invariant volume element on H. If $f = \sum_{I\in\wp(r)} f_I \zeta^I \in \mathcal{O}(\mathcal{B})$, all $f_I \in \mathcal{O}(B)^{\mathbb{C}}$, $I \in \wp(r)$, then

$$f|_{R^{-1}} = \sum_{I\in\wp(r)} f_I\left(R^{-1}\diamondsuit\right) j\left(R^{-1},\diamondsuit\right)^{k+|I|}\vartheta^I \in \mathcal{O}(\mathcal{H}),$$

and if $f = \sum_{I\in\wp(r)} f_I\vartheta^I \in \mathcal{O}(\mathcal{H})$, all $f_I \in \mathcal{C}^\infty(H)^{\mathbb{C}}$, $I \in \wp(r)$, and $g \in RGR^{-1}$, then

$$f|_g = \sum_{I\in\wp(r)} f_I(g\diamondsuit)\, j(g,\diamondsuit)^{k+|I|}\,(E_g\vartheta)^I \in \mathcal{O}(\mathcal{H}).$$

Let $\partial H = \left\{\mathbf{w}\in\mathbb{C}^n \,\middle|\, \mathrm{Re}\, w_1 = \frac{1}{2}\mathbf{w}_2^*\mathbf{w}\right\}$ be the boundary of H in $\mathbb{C}^n$. Then Δ' and ∂H are RNR^{-1}-invariant, and RNR^{-1} acts transitively on ∂H and on each

$$\left\{\mathbf{w}\in H \,\middle|\, \Delta'(\mathbf{w},\mathbf{w}) = e^{2t}\right\} = RNa_t\mathbf{0},$$

$t \in \mathbb{R}$.

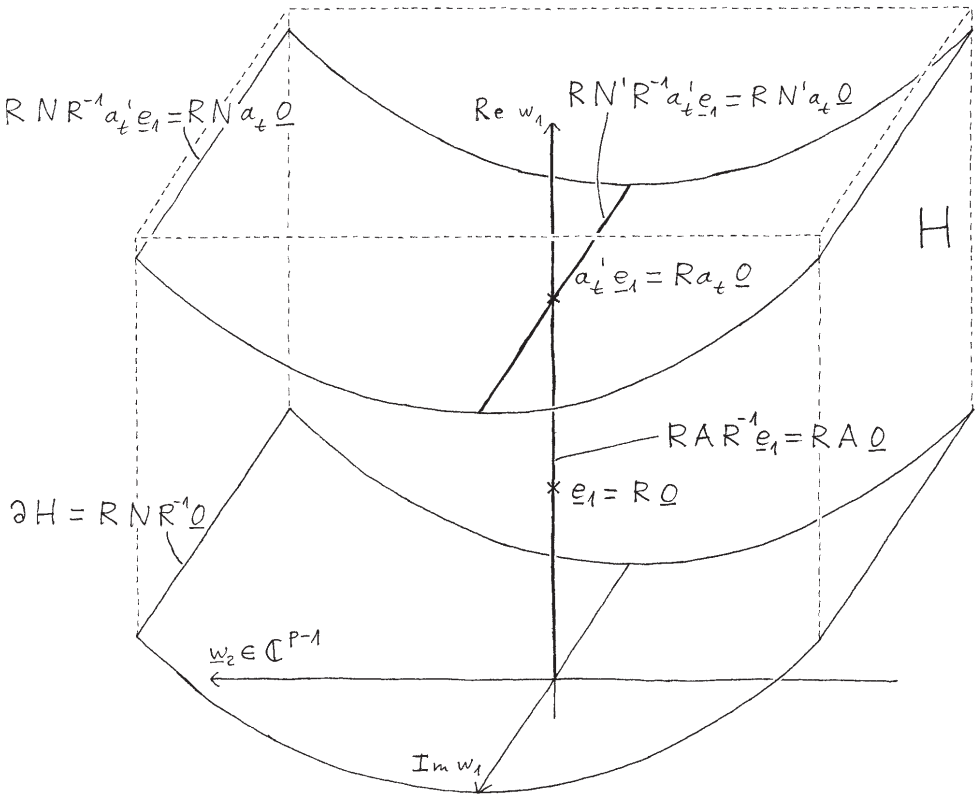

Figure 6.1: The geometry of H.

All geodesics in H can be written in the form

$$\mathbb{R} \to H, t \mapsto \mathbf{w}_t := Rga_t\mathbf{0} = RgR^{-1}a_t'\mathbf{e}_1$$

with some $g \in G$, and conversely all these curves are geodesics in H. We have to distinguish two cases: Either the geodesic connects ∞ with a point in ∂H, or it connects two points in ∂H. In the second case we have

$$\lim_{t\to\pm\infty} \Delta'(\mathbf{w}_t,\mathbf{w}_t) = 0,$$

so we may assume without loss of generality that $\Delta'(\mathbf{w}_t, \mathbf{w}_t)$ is maximal for $t = 0$, otherwise we have to reparametrize the geodesic using ga_T, $T \in \mathbb{R}$ appropriately chosen, instead of g.

Lemma 6.1.
(i) *Let*

$$\mathbb{R} \to H, t \mapsto \mathbf{w}_t := Rga_t\mathbf{0} = RgR^{-1}a'_t\mathbf{e}_1$$

be a geodesic in H such that $\lim_{t\to\infty} \mathbf{w}_t = \infty$ and $\lim_{t\to-\infty} \mathbf{w}_t \in \partial H$ with respect to the euclidian metric on $\mathbb{C}^p$. Then for all $t \in \mathbb{R}$

$$\Delta'(\mathbf{w}_t, \mathbf{w}_t) = e^{2t}\Delta'(\mathbf{w}_0, \mathbf{w}_0),$$

and if instead $\lim_{t\to-\infty} \mathbf{w}_t = \infty$ and $\lim_{t\to\infty} \mathbf{w}_t \in \partial H$, then

$$\Delta'(\mathbf{w}_t, \mathbf{w}_t) = e^{-2t}\Delta'(\mathbf{w}_0, \mathbf{w}_0).$$

(ii) *Let*

$$\mathbb{R} \to H, t \mapsto \mathbf{w}_t := Rga_t\mathbf{0} = RgR^{-1}a'_t\mathbf{e}_1$$

be a geodesic in H connecting two points in ∂H such that $\Delta'(\mathbf{w}_t, \mathbf{w}_t)$ is maximal for $t = 0$. Then

$$\mathbb{R} \to \mathbb{R}_{>0}, t \mapsto \Delta'(\mathbf{w}_t, \mathbf{w}_t)$$

is strictly increasing on $\mathbb{R}_{\le 0}$ and strictly decreasing on $\mathbb{R}_{\ge 0}$, and for all $t \in \mathbb{R}$

$$\Delta'(\mathbf{w}_{-t}, \mathbf{w}_{-t}) = \Delta'(\mathbf{w}_t, \mathbf{w}_t)$$

and

$$e^{-2|t|}\Delta'(\mathbf{w}_0, \mathbf{w}_0) \le \Delta'(\mathbf{w}_t, \mathbf{w}_t) \le 4e^{-2|t|}\Delta'(\mathbf{w}_0, \mathbf{w}_0).$$

Proof. (i) Since RNR^{-1} acts transitively on ∂H and Δ' is RNR^{-1}-invariant we can assume without loss of generality that the geodesic connects $\mathbf{0}$ and ∞. But in H a geodesic is uniquely determined up to reparametrization by its endpoints. So we see that in the first case

$$w_t = a'_t x\mathbf{e}_1 = e^{2t}x\mathbf{e}_1$$

and in the second case

$$w_t = a'_{-t} x\mathbf{e}_1 = e^{-2t}x\mathbf{e}_1$$

both with an appropriately chosen $x > 0$.

(ii) Let $u, y \in \mathbb{R}$ and $\mathbf{s} \in \mathbb{C}^{p-1}$ such that $y^2 + \mathbf{s}^*\mathbf{s} = 1$. Then

$$\mathbb{R} \to H, t \mapsto \mathbf{w}_t^{(u,y,\mathbf{s})} := \frac{e^u}{1 + y^2\tanh^2 t}\left(\frac{e^u\left(1 - y^2\tanh^2 t + 2iy\tanh t\right)}{\sqrt{2}\tanh t\,(1 + iy\tanh t)\,\mathbf{s}}\right)$$

is a geodesic through $e^{2u}\mathbf{e}_1$ in H since it is the image of the standard geodesic

$$\mathbb{R} \to B, t \mapsto a_t\mathbf{0} = \begin{pmatrix}\tanh t\\ \hline \mathbf{0}\end{pmatrix}$$

in B under the transformation

$$\underbrace{a'_u}_{\in RAR^{-1}\sqsubset RG'R^{-1}} \qquad \underbrace{R\left(\begin{array}{cc|c} iy & -\mathbf{s}^* & 0\\ \mathbf{s} & -iy & 0\\ \hline 0 & 0 & 1\end{array}\right)}_{\in K'\sqsubset G'}.$$

So we see that

$$\partial_t \mathbf{w}_t^{(u,y,\mathbf{s})}\Big|_{t=0} = \begin{pmatrix}2ie^{2u}y\\ \hline \sqrt{2}e^u\mathbf{s}\end{pmatrix} \in T_{e^{2u}\mathbf{e}_1}H$$

is a unit vector with respect to the RGR^{-1}-invariant metric on H coming from B via R. Now since RNR^{-1} acts transitively on each

$$\left\{\mathbf{w} \in H \left| \Delta'(\mathbf{w},\mathbf{w}) = e^{2t} \right.\right\} = RNa_t\mathbf{0},$$

$t \in \mathbb{R}$, and Δ' is invariant under RNR^{-1} we may assume without loss of generality that $\mathbf{w}_0 = e^{2u}\mathbf{e}_1$ with an appropriate $u \in \mathbb{R}$. Since $\Delta'(\mathbf{w}_t,\mathbf{w}_t)$ is maximal for $t=0$ we know that $\partial_t \mathbf{w}_t|_{t=0}$ is a unit vector in $i\mathbb{R} \oplus \mathbb{C}^{p-1} \sqsubset T_{\mathbf{e}_1}H$, and therefore there exist $y \in \mathbb{R}$ and $\mathbf{s} \in \mathbb{C}^{p-1}$ such that $y^2 + \mathbf{s}^*\mathbf{s} = 1$ and

$$\partial_t \mathbf{w}_t|_{t=0} = \begin{pmatrix} 2ie^{2u}y \\ \sqrt{2}e^u\mathbf{s} \end{pmatrix}.$$

Since the geodesic is uniquely determined by $\mathbf{w}_0$ and $\partial_t\mathbf{w}_t|_{t=0}$ we see that $\mathbf{w}_t = \mathbf{w}_t^{(u,y,\mathbf{s})}$ for all $t \in \mathbb{R}$, and so a straight forward calculation shows that

$$\begin{aligned}\Delta'(\mathbf{w}_t,\mathbf{w}_t) &= 2e^{2u}\frac{1-\tanh^2 t}{1+y^2\tanh^2 t} \\ &= \frac{8e^{2u}}{(1+y^2)(e^{2t}+e^{-2t})+2\mathbf{s}^*\mathbf{s}}.\end{aligned}$$

The rest is an easy exercise using $y^2 + \mathbf{s}^*\mathbf{s} = 1$. □

For all $t \in \mathbb{R}$ define $A_{>t} := \{a_\tau | \tau > t\} \subset A$.

Theorem 6.2 (A 'fundamental domain' for $\Gamma\backslash G$)**.** *There exist $\eta \subset N$ open and relatively compact, $t_0 \in \mathbb{R}$ and $\Xi \subset G'$ finite such that if we define*

$$\Omega := \bigcup_{g\in\Xi} g\eta A_{>t_0}K$$

then

(i) *$g^{-1}\Gamma g \cap NZ_G(G') \sqsubset NZ_G(G')$ and $g^{-1}\Gamma g \cap N'Z_G(G') \sqsubset N'Z_G(G')$ are lattices, and*

$$NZ_G(G') = \left(g^{-1}\Gamma g \cap NZ_G(G')\right)\eta Z_G(G')$$

for all $g \in \Xi$,

(ii) *$G = \Gamma\Omega$,*

(iii) *the set $\{\gamma \in \Gamma | \gamma\Omega \cap \Omega \neq \emptyset\}$ is finite.*

Proof. The theorem is a direct consequence of Theorem 0.6 (i) - (iii), Theorem 0.7, Lemma 3.16 and Lemma 3.18 of [5]. For a detailed derivation see [10] or Section 3.2 of [11]. □

Now clearly the set of cusps of $\Gamma\backslash B$ in $\Gamma\backslash\partial B$ is contained in the set

$$\left\{\lim_{t\to+\infty} \Gamma g a_t\mathbf{0} | g \in \Xi\right\},$$

and is therefore finite as expected, where the limits are taken with respect to the Euclidian metric on B.

Corollary 6.3. *Let $t_0 \in \mathbb{R}$, $\eta \subset N$ and $\Xi \subset G$ be given by Theorem 6.2. Let $h \in \mathcal{C}(\Gamma\backslash G)^{\mathbb{C}}$ and $s \in]0,\infty]$. Then $h \in L^s(\Gamma\backslash G)$ if and only if $h(g\diamondsuit) \in L^s(\eta A_{>t_0}K)$ for all $g \in \Xi$.*

Let $f \in sM_k(\Gamma)$ and $g \in \Xi$. Then we can decompose $f|_g|_{R^{-1}} = \sum_{I\in\wp(r)} q_I\vartheta^I \in \mathcal{O}(\mathcal{H})$, all $q_I \in \mathcal{O}(H)$, $I \in \wp(r)$, and by Theorem 6.2 (i) we know that $g^{-1}\Gamma g \cap N'Z_G(G') \not\subset Z_G(G')$. So let $n \in g^{-1}\Gamma g \cap N'Z_G(G') \setminus Z_G(G')$,

$$RnR^{-1} = n'_{\lambda_0,\mathbf{0}}\left(\begin{array}{c|c} \varepsilon 1 & 0 \\ \hline 0 & E_0 \end{array}\right),$$

$\lambda_0 \in \mathbb{R} \setminus \{0\}$, $\varepsilon \in U(1)$, $E_0 \in U(r)$, $\varepsilon^{n+1} = \det E$.

$j(RnR^{-1}) := j(RnR^{-1}, \mathbf{w}) = \varepsilon^{-1} \in U(1)$ is independent of $\mathbf{w} \in H$. So there exists $\chi \in \mathbb{R}$ such that $j(RnR^{-1}) = e^{2\pi i\chi}$. Without loss of generality we can assume that E_0 is diagonal, otherwise conjugate n with an appropriate element of $Z_G(G')$. So there exists $D \in \mathbb{R}^{r\times r}$ diagonal such that $E_0 = \exp(2\pi iD)$.

Theorem 6.4 (FOURIER expansion of $f|_g|_{R^{-1}}$ **).**

(i) *There exist unique* $c_{I,m} \in \mathcal{O}(\mathbb{C}^{n-1})$, $I \in \wp(r)$, $m \in \frac{1}{\lambda_0}(\mathbb{Z} - \operatorname{tr}_I D - (k+|I|)\chi)$, *such that*

$$q_I(\mathbf{w}) = \sum_{m\in\frac{1}{\lambda_0}(\mathbb{Z}-\operatorname{tr}_I D-(k+|I|)\chi)} c_{I,m}(\mathbf{w}_2)\, e^{2\pi m w_1}$$

for all $\mathbf{w} \in H$ *and* $I \in \wp(r)$, *and so*

$$f|_g|_{R^{-1}}(\mathbf{w}) = \sum_{I\in\wp(r)} \sum_{m\in\frac{1}{\lambda_0}(\mathbb{Z}-\operatorname{tr}_I D-(k+|I|)\chi)} c_{I,m}(\mathbf{w}_2)\, e^{2\pi m w_1}\vartheta^I$$

for all $\mathbf{w} = \begin{pmatrix} w_1 \\ \hline \mathbf{w}_2 \end{pmatrix} \begin{matrix} \leftarrow 1 \\ \} n-1 \end{matrix} \in H$, *where the convergence is absolute and compact.*

(ii) $c_{I,m} = 0$ *for all* $I \in \wp(r)$ *and* $m > 0$ *(this is a super analogon for* KOECHER*'s principle, see for example Section 11.5 of* [1] *), and if* $\operatorname{tr}_I D + (k+|I|)\chi \in \mathbb{Z}$, *then* $c_{I,0}$ *is a constant.*

(iii) *Let* $I \in \wp(r)$ *and* $s \in [1,\infty]$ *. If* $\operatorname{tr}_I D + (k+|I|)\chi \notin \mathbb{Z}$, *then*

$$q_I \Delta'(\mathbf{w},\mathbf{w})^{\frac{k+|I|}{2}} \in L^s(R\eta A_{>t_0}\mathbf{0})$$

with respect to the RGR^{-1} *-invariant measure* $\Delta'(\mathbf{w},\mathbf{w})^{-(n+1)}\, dV_{\mathrm{Leb}}$ *on* H. *If* $\operatorname{tr}_I D + (k+|I|)\chi \in \mathbb{Z}$ *and* $k \geq 2n - |I|$ *then*

$$q_I \Delta'(\mathbf{w},\mathbf{w})^{\frac{k+|I|}{2}} \in L^s(R\eta A_{>t_0}\mathbf{0})$$

with respect to the RGR^{-1} *-invariant measure on* H *if and only if* $c_{I,0} = 0$.

A proof can be found in [10] or [11] Section 3.2.

7. PROOF OF THE MAIN RESULT

We have a LIE algebra embedding

$$\rho : \mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{g}'^{\mathbb{C}} = \mathfrak{sl}(n+1,\mathbb{C}), \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \left(\begin{array}{c|c|c} a & 0 & b \\ \hline 0 & 0 & 0 \\ \hline c & 0 & -a \end{array}\right).$$

Obviously the preimage of $\mathfrak{g}'$ under ρ is $\mathfrak{su}(1,1)$, the preimage of $\mathfrak{k}'$ under ρ is $\mathfrak{s}(\mathfrak{u}(1)\oplus\mathfrak{u}(1)) \simeq \mathfrak{u}(1)$ and ρ lifts to a LIE group homomorphism

$$\widetilde{\rho} : SL(2,\mathbb{C}) \to G'^{\mathbb{C}} = SL(n+1,\mathbb{C}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{array}{c|c|c} a & 0 & b \\ \hline 0 & 0 & 0 \\ \hline c & 0 & d \end{array}\right)$$

such that $\widetilde{\rho}(SU(1,1)) \sqsubset G'$.

Let us now identify the elements of $\mathfrak{g}$ with the corresponding left invariant differential operators. They are defined on a dense subset of $L^2(\Gamma\backslash G)$, and define

$$\mathcal{D} := \rho\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{a} \ , \ \mathcal{D}' := \rho\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \in \mathfrak{g}' \text{ and}$$
$$\phi := \rho\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{k}'.$$

The $\mathbb{R}$-linear span of $\mathcal{D}$, $\mathcal{D}'$ and ϕ is the 3-dimensional sub LIE algebra $\rho(\mathfrak{su}(1,1))$ of $\mathfrak{g}' \sqsubset \mathfrak{g}$, and $\mathcal{D}$ generates the flow φ_t. ϕ generates a subgroup of K', being the image of the LIE group embedding

$$\mathbb{R}/2\pi\mathbb{Z} \hookrightarrow K, t \mapsto \exp(t\phi) = \widetilde{\rho}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

Now define

$$\mathcal{D}^+ := \frac{1}{2}\left(\mathcal{D} - i\mathcal{D}'\right), \mathcal{D}^- := \frac{1}{2}\left(\mathcal{D} + i\mathcal{D}'\right) \text{ and } \Psi := -i\phi$$

as left invariant differential operators on G. Then we get the commutation relations

$$\left[\Psi, \mathcal{D}^+\right] = 2\mathcal{D}^+, \left[\Psi, \mathcal{D}^-\right] = -2\mathcal{D}^- \text{ and } \left[\mathcal{D}^+, \mathcal{D}^-\right] = \Psi,$$

and since G is unimodular

$$\left(\mathcal{D}^+\right)^* = -\mathcal{D}^-, \left(\mathcal{D}^-\right)^* = -\mathcal{D}^+ \text{ and } \Psi^* = \Psi.$$

So by standard FOURIER analysis

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_{\nu\in\mathbb{Z}}} H_\nu$$

as an orthogonal sum, where

$$H_\nu := \left\{ F \in L^2(\Gamma\backslash G) \cap \text{ domain } \Psi \middle| \ \Psi F = \nu F \right\}$$

for all $\nu \in \mathbb{Z}$. By a simple calculation we obtain

$$\mathcal{D}^+\left(H_\nu \cap \text{ domain } \mathcal{D}^+\right) \subset H_{\nu+2} \text{ and } \mathcal{D}^-\left(H_\nu \cap \text{ domain } \mathcal{D}^-\right) \subset H_{\nu-2}$$

for all $\nu \in \mathbb{Z}$.

Lemma 7.1. *$\mathcal{D}^-\widetilde{h} = 0$ for all $h \in \mathcal{O}(\mathcal{B})$.*

Proof. Let $g \in G$. Then again $h|_g \in \mathcal{O}(\mathcal{B})$, and $\widetilde{h}(g\diamondsuit) = \widetilde{h|_g}$. So

$$\mathcal{D}^-\widetilde{h}(g) = \mathcal{D}^-\left(\widetilde{h}(g\diamondsuit)\right)(1) = \overline{\partial}_1 h|_g = 0.$$

□

Lemma 7.2. *Let $f \in sS_k^{(\rho)}(\Gamma)$. Then $\widetilde{f}$ is uniformly* LIPSCHITZ *continuous.*

Proof. Since on G we use a left invariant metric it suffices to show that there exists a constant $c \geq 0$ such that for all $g \in G$ and $\xi \in \mathfrak{g}$ with $||\xi||_2 \leq 1$

$$\left|\xi\widetilde{f}(g)\right| \leq c.$$

Then c is a LIPSCHITZ constant for $\widetilde{f}$. So choose an orthonormal basis $(\xi_1,\dots,\xi_N)$ of $\mathfrak{g}$ and a compact neighbourhood L of $\mathbf{0}$ in B. Then by CAUCHY's integral formula there exist $C',C''\geq 0$ such that for all $h\in\mathcal{O}(\mathcal{B})\cap L_k^\infty(\mathcal{B})$ and $n\in\{1,\dots,N\}$

$$\left|\left(\xi_n\widetilde{h}\right)(1)\right|\leq C'\int_L|h|\leq C'\mathrm{vol}\,L\,||h||_{\infty,L}\leq C''\mathrm{vol}\,L\left|\left|\widetilde{h}\right|\right|_\infty,$$

and since $\mathfrak{g}\to\mathbb{C},\xi\mapsto\left(\xi\widetilde{h}\right)(1)$ is linear we obtain

$$\left|\left(\xi\widetilde{h}\right)(1)\right|\leq NC''\mathrm{vol}\,L\left|\left|\widetilde{h}\right|\right|_\infty$$

for general $\xi\in\mathfrak{g}$ with $||\xi||_2\leq 1$. Now let $g\in G$. Then again $f|_g\in\mathcal{O}(\mathcal{B})$, $\widetilde{f}\,(g\diamondsuit)=\widetilde{f|_g}$, and by SATAKE's theorem, Theorem 2.1, f and so $f|_g\in L_k^\infty(\mathcal{B})$. So

$$\left|\xi\widetilde{f}(g)\right|=\left|\left(\xi\widetilde{f}\,(g\diamondsuit)\right)(1)\right|\leq NC''\mathrm{vol}\,L\left|\left|\widetilde{f}\,(g\diamondsuit)\right|\right|_\infty\leq NC''\mathrm{vol}\,L\left|\left|\widetilde{f}\right|\right|_\infty,$$

and we can define $c:=NC''\mathrm{vol}\,L\left|\left|\widetilde{f}\right|\right|_\infty$. □

Now let $f\in sS_k^{(\rho)}(\Gamma)$ such that $(\varphi_{\gamma_0,I,m},f)_\Gamma=0$ for all $\varphi_{\gamma_0,I,m}$, $\gamma_0\in\Gamma$ primitive loxodromic, $I\in\wp(r)$, $|I|=\rho$, $m\in]-C,C[$. We will show that $f=0$ in several steps.

Lemma 7.3. *There exists $F\in\mathcal{C}\left(\Gamma\backslash G\right)^{\mathbb{C}}\otimes\bigwedge\left(\mathbb{C}^r\right)$ uniformly* LIPSCHITZ *continuous on compact sets and differentiable along the flow φ_t such that*

$$f=\partial_\tau F\left(\diamondsuit a_\tau\right)|_{\tau=0}=\mathcal{D}F.$$

Proof. Here we use that the right translation with A on $\Gamma\backslash G'\widetilde{Z}$ is topologically transitive. So let $g_0\in G'$ such that $\overline{\Gamma g_0A}=G'\widetilde{Z}$ and define $s\in\mathcal{C}^\infty\left(\mathbb{R}\right)^{\mathbb{C}}\otimes\bigwedge\left(\mathbb{C}^r\right)$ by

$$s(t):=\int_0^t\widetilde{f}\,(g_0a_\tau)\,d\tau$$

for all $t\in\mathbb{R}$.

Step I. **Show that for all $L\subset G'\widetilde{Z}$ compact there exist constants $C_3\geq 0$ and $\varepsilon_3>0$ such that for all $t\in\mathbb{R}$, $T\geq 0$ and $\gamma\in\Gamma$ if $g_0a_t\in L$ and**

$$\varepsilon:=d\left(\gamma g_0a_t,g_0a_{t+T}\right)\leq\varepsilon_3$$

then $|s(t)-s(t+T)|\leq C_3\varepsilon$.

Let $L\subset G'\widetilde{Z}$ be compact, $T_0>0$ be given by Lemma 5.2 and $C_1\geq 1$ and ε_1 be given by Theorem 5.3 (i) with $T_1:=T_0$. Define $C_3:=\max\left(C_1\left(C_2+2c\right),\left|\left|\widetilde{f}\right|\right|_\infty\right)\geq 0$, where $C_2\geq 0$ is the LIPSCHITZ constant from Theorem 4.1 (ii) and $c\geq 0$ is the LIPSCHITZ constant of $\widetilde{f}$. Define $\varepsilon_3:=\min\left(\varepsilon_1,\varepsilon_2,\frac{T_0}{2C_1}\right)>0$, where $\varepsilon_2>0$ is given by Theorem 5.3 (ii).

Let $t\in\mathbb{R}$, $T\geq 0$ and $\gamma\in\Gamma$ such that $g_0a_t\in L$ and $\varepsilon:=d\left(\gamma g_0a_t,g_0a_{t+T}\right)\leq\varepsilon_3$.

First assume $T\geq T_0$. Then by Theorem 5.3 (i) since $\varepsilon\leq\varepsilon_1$ there exist $g\in G$, $w_0\in M$ and $t_0>0$ such that $\gamma g=ga_{t_0}w_0$, $d\left(\left(t_0,w_0\right),\left(T,1\right)\right)\leq C_1\varepsilon$, and for all $\tau\in[0,T]$

$$d\left(g_0a_{t+\tau},ga_\tau\right)\leq C_1\varepsilon\left(e^{-\tau}+e^{-(T-\tau)}\right).$$

We get

$$s(t+T)-s(t)=\underbrace{\int_0^T\widetilde{f}\,(ga_\tau)\,d\tau}_{I_1:=}+\underbrace{\int_0^T\left(\widetilde{f}\,(g_0a_{t+\tau})-\widetilde{f}\,(ga_\tau)\right)d\tau}_{I_2:=}$$

and

$$\begin{aligned}|I_2| &\leq \int_0^T \left|\widetilde{f}\left(g_0 a_{t+\tau}\right) - \widetilde{f}\left(g a_\tau\right)\right| d\tau \\ &\leq c \int_0^T d\left(g_0 a_{t+\tau}, g a_\tau\right) d\tau \\ &\leq c C_1 \varepsilon \int_0^T \left(e^{-\tau} + e^{-(T-\tau)}\right) d\tau \\ &\leq 2 c C_1 \varepsilon.\end{aligned}$$

Since $\gamma \in \Gamma$ is regular loxodromic, there exists $\gamma_0 \in \Gamma$ primitive loxodromic and $\nu \in \mathbb{N} \setminus \{0\}$ such that $\gamma = \gamma_0^\nu$. $\gamma_0 \in gAWg^{-1}$ since Lemma 3.2 tells us that $g \in G$ is already determined by γ up to right translation with elements of $AN_K(A)$. Choose $w' \in N_K(M)$, $t_0' > 0$ and $w_0' \in M$ such that $E_{w_0'}$ is diagonal and $\gamma = g w' a_{t_0'} w_0' \left(g w'\right)^{-1}$, and let $g' := gw'$. We define $h \in \mathcal{C}^\infty\left(\mathbb{R} \times M\right)^{\mathbb{C}} \otimes \bigwedge\left(\mathbb{C}^r\right)$ as

$$h(\tau, w) := \widetilde{f}\left(g' a_\tau w\right) = \widetilde{f}\left(g a_\tau w' w\right)$$

for all $\tau \in \mathbb{R}$ and $w \in M$. Then

$$I_1 = \int_0^T h\left(\tau, w'^{-1}\right) d\tau.$$

We can apply Theorem 4.1 (i) and, since f is perpendicular to all $\varphi_{\gamma_0, I, m}$, $I \in \wp(r)$, $m \in \left]-C, C\right[$, also Theorem 4.1 (ii) with $g' := gw'$ instead of g, and so

$$\begin{aligned}|I_1| &= \left|H\left(T, w'^{-1}\right) - H\left(0, w'^{-1}\right)\right| \\ &= \left|H\left(T, w'^{-1}\right) - H\left(t_0, w'^{-1} w_0\right)\right| \\ &\leq C_2 d\left((T, 1), (t_0, w_0)\right) \\ &\leq C_1 C_2 \varepsilon,\end{aligned}$$

where we used that $H\left(0, w'^{-1}\right) = H\left(t_0', w_0' w'^{-1}\right)$, choosing the left invariant metric on M, and the claim follows.

Now assume $T \leq T_0$. Then by Theorem 5.3 (ii), since $\varepsilon \leq \varepsilon_0$ we get $T \leq \varepsilon$ and so

$$|s(t+T) - s(t)| = \left|\int_0^T \widetilde{f}\left(g_0 a_{t+\tau}\right) d\tau\right| \leq \varepsilon \left|\left|\widetilde{f}\right|\right|_\infty.$$

Step II. **Show that there exists a unique** $F_1 \in \mathcal{C}\left(\Gamma \backslash G' \widetilde{Z}\right)^{\mathbb{C}} \otimes \bigwedge\left(\mathbb{C}^r\right)$ **uniformly** LIPSCHITZ **continuous on compact sets such that for all** $t \in \mathbb{R}$

$$s(t) = F_1\left(g_0 a_t\right).$$

By Step I for all $L \subset \Gamma \backslash G' \widetilde{Z}$ compact with $L^\circ \underset{\text{dense}}{\subset} L$ there exists a unique $F_L \in \mathcal{C}\left(\Gamma \backslash G' \widetilde{Z}\right)^{\mathbb{C}}$ uniformly LIPSCHITZ continuous such that for all $t \in \mathbb{R}$ if $\Gamma g_0 a_t \in L$ then $s(t) = F_L\left(\Gamma g_0 a_t\right)$. So we see that there exists a unique $F_1 \in \mathcal{C}\left(\Gamma \backslash G' \widetilde{Z}\right)^{\mathbb{C}} \otimes \bigwedge\left(\mathbb{C}^r\right)$ such that $F_1|_L = F_L$ for all $L \subset \Gamma \backslash G' \widetilde{Z}$ compact with $L^\circ \underset{\text{dense}}{\subset} L$.

Step III. **Show that** F_1 **is differentiable along the flow and that for all** $g \in G' \widetilde{Z}$

$$\partial_\tau F_1\left(g a_\tau\right)|_{\tau=0} = \widetilde{f}(g).$$

Let $g \in G'\widetilde{Z}$. It suffices to show that for all $T \in \mathbb{R}$

$$\int_0^T \widetilde{f}\left(ga_\tau\right) d\tau = F_1\left(ga_T\right) - F_1(g).$$

If $g = g_0 a_t$ for some $t \in \mathbb{R}$ then it is clear by construction. For general $g \in G'\widetilde{Z}$, since $\overline{\Gamma g_0 A} = G'\widetilde{Z}$ there exists $(\gamma_n, t_n)_{n\in\mathbb{N}} \in (\Gamma \times \mathbb{R})^{\mathbb{N}}$ such that

$$\lim_{n\to\infty} \gamma_n g_0 a_{t_n} = g,$$

and so

$$\lim_{n\to\infty} \gamma_n g_0 a_{\tau+t_n} = ga_\tau$$

compact in $\tau \in \mathbb{R}$. Finally $\widetilde{f}$ is uniformly LIPSCHITZ continuous. Therefore we can interchange integration and taking the limit $n \rightsquigarrow \infty$:

$$\begin{aligned}\int_0^T \widetilde{f}\left(ga_\tau\right) d\tau &= \lim_{n\to\infty} \int_0^T \widetilde{f}\left(\gamma_n g_0 a_{\tau+t_n}\right) d\tau \\ &= \lim_{n\to\infty} \left(F_1\left(\gamma_n g_0 a_{T+t_n}\right) - F_1\left(\gamma_n g_0 a_{t_n}\right)\right) \\ &= F_1\left(ga_T\right) - F_1(g).\end{aligned}$$

Step IV. **Conclusion.**

Define $F \in \mathcal{C}(G)^{\mathbb{C}} \otimes \bigwedge\left(\mathbb{C}^r\right)$ as

$$F\left(gw\right) := \int_{\widetilde{Z}} F_1\left(gu^{-1}, E_{uw}\eta\right) j(uw)^{k+\rho} du$$

for all $g \in G'\widetilde{Z}$ and $w \in Z_G\left(G'\right)$, where we normalize the HAAR measure on the compact LIE group $\widetilde{Z}$ such that $\operatorname{vol}\widetilde{Z} = 1$. Then we see that F is well defined and fulfills all the desired properties. □

Lemma 7.4.

(i) *For all $L \subset G$ compact there exists $\varepsilon_4 > 0$ such that for all $g, h \in L$ if g and h belong to the same T^--leaf and $d^-(g,h) \leq \varepsilon_4$ then*

$$\lim_{t\to\infty} \left(F\left(ga_t\right) - F\left(ha_t\right)\right) = 0,$$

and if g and h belong to the same T^+-leaf and $d^+(g,h) \leq \varepsilon_4$ then

$$\lim_{t\to-\infty} \left(F\left(ga_t\right) - F\left(ha_t\right)\right) = 0.$$

(ii) *F is continuously differentiable along T^-- and T^+-leafs, more precisely if $\rho : I \to G$ is a continuously differentiable curve in a T^--leaf, then*

$$\partial_t \left(F \circ \rho\right)(t) = -\int_0^\infty \partial_t \widetilde{f}\left(\rho(t)a_\tau\right) d\tau,$$

and if $\rho : I \to G$ is a continuously differentiable curve in a T^+-leaf then

$$\partial_t \left(F \circ \rho\right)(t) = \int_{-\infty}^0 \partial_t \widetilde{f}\left(\rho(t)a_\tau\right) d\tau.$$

Proof. (i) Let $L \subset G$ be compact, and let $L' \subset G$ be a compact neighbourhood of L. Let $T_0 > 0$ be given by Lemma 5.2 and $\varepsilon_2 > 0$ by Theorem 5.3 (ii) both with respect to L'. Define

$$\varepsilon_4 := \frac{1}{3} \min \left(\varepsilon_1, \varepsilon_2, \frac{T_0}{2C_1} \right) > 0,$$

where $\varepsilon_1 > 0$ and $C_1 \geq 1$ are given by Theorem 5.3 (i) with $T_1 := T_0$. Let $\delta_0 > 0$ such that $\overline{U_{\delta_0}(L)} \subset L'$ and let

$$\delta \in]0, \min(\delta_0, \varepsilon_4)[.$$

Let $g, h \in L$ be in the same T^--leaf such that $\varepsilon := d^-(g,h) \leq \varepsilon_4$. Since the splitting of TG is left invariant and $T_1^-(G) \sqsubset \mathfrak{g}'$ we see that there exist $g', h' \in G'$ and $u \in Z_G(G')$ such that $g = g'u$ and $h = h'u$. Fix some $T' > 0$. Again by assumption there exists $g_0 \in G'$ such that $\overline{\Gamma g_0 A} = G'\widetilde{Z}$, and so $g, h \in \overline{\Gamma g_0 u A}$. So there exist $\gamma_g, \gamma_h \in \Gamma$ and $t_g, t_h \in \mathbb{R}$ such that

$$d\left(g a_t, \gamma_g g_0 u a_{t_g+t}\right), d\left(h a_t, \gamma_h g_0 u a_{t_h+t}\right) \leq \delta$$

for all $t \in [0, T']$, and so in particular $\gamma_g g_0 u a_{t_g}, \gamma_h g_0 u a_{t_h} \in L'$. We will show that for all $t \in [0, T']$

$$\left| F\left(\gamma_g g_0 u a_{t_g+t}\right) - F\left(\gamma_h g_0 u a_{t_h+t}\right) \right| \leq C_3' \left(\varepsilon e^{-t} + 2\delta\right)$$

with the same constant $C_3' \geq 0$ as in Step I of the proof of Lemma 7.3 with respect to L'.

Without loss of generality we may assume $T := t_h - t_g \geq 0$. Define $\gamma := \gamma_g \gamma_h^{-1} \in \Gamma$. Then for all $t \in [0, T']$

$$d\left(\gamma \gamma_h g_0 u a_{t_g+t}, \gamma_h g_0 u a_{t_g+t+T}\right) \leq \varepsilon e^{-t} + 2\delta.$$

First assume $T \geq T_0$ and fix $t \in [0, T']$. Then by Theorem 5.3 (i), since $\varepsilon e^{-t} + 2\delta \leq \varepsilon + 2\delta \leq \min\left(\varepsilon_1, \frac{T_0}{2C_1}\right)$, there exist $z \in G$, $t_0 \in \mathbb{R}$ and $w \in M$ such that $\gamma z = z a_{t_0} w$,

$$d\left((t_0, w), (T, 1)\right) \leq C_1 \left(2\delta + \varepsilon e^{-t}\right),$$

and for all $\tau \in [0, T]$

$$d\left(\gamma_g g_0 u a_{t_g+t+\tau}, z a_\tau\right) \leq C_1 \left(\varepsilon e^{-t} + 2\delta\right) \left(e^{-\tau} + e^{-(T-\tau)}\right).$$

And so by the same calculations as in the proof of Lemma 7.3 we obtain the estimate

$$\left| F\left(\gamma_g g_0 u a_{t_g+t}\right) - F\left(\gamma_h g_0 u a_{t_h+t}\right) \right| \leq C_3' \left(\varepsilon e^{-t} + 2\delta\right).$$

Now assume $T \leq T_0$. Then by Theorem 5.3 (ii) since $\gamma_g g_0 m a_{t_g} \in L'$ and $\varepsilon + 2\delta \leq \varepsilon_2$ we obtain $\gamma = 1$ and so by the left invariance of the metric on G

$$d(1, a_T) \leq \varepsilon e^{-T'} + 2\delta,$$

therefore $T \leq \varepsilon e^{-T'} + 2\delta$. So as in the proof of Lemma 7.3,

$$\begin{aligned} \left| F\left(\gamma_g g_0 u a_{t_g+t}\right) - F\left(\gamma_h g_0 u a_{t_h+t}\right) \right| &\leq \left|\left| \widetilde{f} \right|\right|_\infty \left(\varepsilon e^{-T'} + 2\delta\right) \\ &\leq C_3' \left(\varepsilon e^{-t} + 2\delta\right). \end{aligned}$$

Now let us take the limit $\delta \rightsquigarrow 0$. Then $\gamma_g g_0 u a_{t_g} \rightsquigarrow g$ and $\gamma_h g_0 u a_{t_h} \rightsquigarrow h$, so since F is continuous

$$\left| F(g a_t) - F(h a_t) \right| \leq C_3' \varepsilon e^{-t}$$

for all $t \in [0, T']$, and since $T' > 0$ has been arbitrary, we obtain this estimate for all $t \geq 0$ and so $\lim_{t \to \infty} F(g a_t) - F(h a_t) = 0$. By similar calculations we can prove that $\lim_{t \to -\infty} F(g a_t) - F(h a_t) = 0$ if g and h belong to the same T^+-leaf and $d^+(g,h) \leq \varepsilon_4$.

(ii) Let $\rho : I \to G$ be a continuously differentiable curve in a T^--leaf, and let $t_0, t_1 \in I$, $t_1 > t_0$. It suffices to show that

$$F(\rho(t_1)) - F(\rho(t_0)) = -\int_{t_0}^{t_1}\int_0^\infty \partial_t \widetilde{f}(\rho(t)a_\tau)\, d\tau dt.$$

Let $C' \geq 0$ such that $||\partial_t \rho(t)|| \leq C'$ for all $t \in [t_0, t_1]$. Then since ρ lies in a T^--leaf we have $||\partial_t(\rho(t)a_\tau)|| \leq C'e^{-\tau}$ and so

$$\left|\partial_t \widetilde{f}(\rho(t)a_\tau)\right| \leq cC'e^{-\tau}$$

for all $\tau \geq 0$ and $t \in [t_0, t_1]$ where $c \geq 0$ is the LIPSCHITZ constant of $\widetilde{f}$. So the double integral on the right side is absolutely convergent and so we can interchange the order of integration:

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^\infty \partial_t \widetilde{f}(\rho(t)a_\tau)\, d\tau dt &= \int_0^\infty \int_{t_0}^{t_1} \partial_t \widetilde{f}(\rho(t)a_\tau)\, dt d\tau \\ &= \int_0^\infty \left(\widetilde{f}(\rho(t_1)a_\tau) - \widetilde{f}(\rho(t_0)a_\tau)\right) d\tau \\ &= \lim_{T\to\infty} (F(\rho(t_1)a_T) - F(\rho(t_0)a_T)) \\ &\quad - F(\rho(t_1)) + F(\rho(t_0)).\end{aligned}$$

Now let $L \subset G$ be compact such that $\rho([t_1, t_2]) \subset L$ and let $\varepsilon_4 > 0$ as in (i). Without loss of generality we may assume that $d^-(\rho(t_0), \rho(t_1)) \leq \varepsilon_4$. Then

$$\lim_{T\to\infty} (F(\rho(t_1)a_T) - F(\rho(t_0)a_T)) = 0$$

by (i). By similar calculations one can also prove

$$\partial_t (F \circ \rho)(t) = \int_{-\infty}^0 \partial_t \widetilde{f}(\rho(t)a_\tau)\, d\tau$$

in the case when $\rho : I \to G$ is a continuously differentiable curve in a T^+-leaf. □

Lemma 7.5.
(i) $F \in L^2(\Gamma\backslash G) \otimes \bigwedge(\mathbb{C}^r)$,
(ii) $\xi F \in L^2(\Gamma\backslash G) \otimes \bigwedge(\mathbb{C}^r)$ *for all* $\xi \in \mathbb{R}\mathcal{D} \oplus \mathfrak{g} \cap (T^+ \oplus T^-)$.

Proof. (i) If $\Gamma\backslash G$ is compact then the assertion is trivial. So assume that $\Gamma\backslash G$ is not compact, then we use the unbounded realization $\mathcal{H}$ of $\mathcal{B}$ introduced in Section 6. Since $\mathrm{vol}(\Gamma\backslash G) < \infty$, it suffices to prove that F is bounded, and by Corollary 6.3 it is even enough to show that $F(g\diamondsuit)$ is bounded on $NA_{>t_0}K$ for all $g \in \Xi$, where $t_0 \in \mathbb{R}$ and $\Xi \subset G'$ are given by Theorem 6.2. So let $g \in \Xi$.

Step I. **Show that** $F(g\diamondsuit)$ **is bounded on** $Na_{t_0}K$**.**

Let $\eta \subset N$ also be given by Theorem 6.2. Then $F(g\diamondsuit)$ is clearly bounded on the compact set $\overline{\eta}a_{t_0}K$. On the other hand $F(g\diamondsuit)$ is left-$g^{-1}\Gamma g$ -invariant, so it is also bounded on

$$Na_{t_0}K = \left(g\Gamma g^{-1} \cap NZ_G(G')\right)\eta a_{t_0}K$$

by Theorem 6.2 (i).

Step II. **Show that there exists** $C' \geq 0$ **such that for all** $g' \in NA_{>t_0}K$

$$\left|\widetilde{f}(gg')\right| \leq \frac{C'}{\Delta'(Rg'\mathbf{0}, Rg'\mathbf{0})}.$$

As in Section 6, let $q_I \in \mathcal{O}(H)$ such that $f|_g|_{R^{-1}} = \sum_{I\in\wp(r)} q_I \vartheta^I$. Then since $\widetilde{f}(g\diamondsuit) \in L^2(\eta A_{>t_0}K)\otimes\bigwedge(\mathbb{C}^r)$, by Theorem 6.4 we have FOURIER expansions

$$q_I(\mathbf{w}) = \sum_{m\in\frac{1}{\lambda_0}(\mathbb{Z}-\mathrm{tr}_I D-(k+|I|)\chi)\cap\mathbb{R}_{<0}} c_{I,m}(\mathbf{w}_2)\, e^{2\pi m w_1} \tag{7.1}$$

for all $I \in \wp(I)$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ \mathbf{w}_2 \end{pmatrix} \begin{matrix} \leftarrow 1 \\ \}n-1 \end{matrix} \in H$, where $c_{I,m} \in \mathcal{O}(\mathbb{C}^{n-1})$, $I \in \wp(r)$, $m \in \frac{1}{\lambda_0}(\mathbf{z} - \mathrm{tr}_I D - (k+|I|)\chi)\cap\mathbb{R}_{<0}$. Define

$$M_0 := \max \bigcup_{I\in\wp(r)} \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi)\cap\mathbb{R}_{<0} < 0.$$

$R\overline{\eta}a_{t_0}\mathbf{0} \subset H$ is compact, and so since the convergence of the FOURIER series (7.1) is absolute and compact we can define

$$C'' := e^{-2\pi M_0 e^{2t_0}} \times \max_{I\in\wp(r)} \sum_{m\in\frac{1}{\lambda_0}(\mathbb{Z}-\mathrm{tr}_I D-(k+|I|)\chi)\cap\mathbb{R}_{<0}} \left|\left|c_{I,m}(\mathbf{w}_2)\, e^{2\pi m w_1}\right|\right|_{\infty, R\overline{\eta}a_{t_0}\mathbf{0}} < \infty.$$

Then we have

$$|q_I(\mathbf{w})| \le C'' e^{\pi M_0 \Delta'(\mathbf{w},\mathbf{w})}$$

for all $I \in \wp(r)$ and $\mathbf{w} \in R\eta A_{>t_0}\mathbf{0}$. Now let

$$g' = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E' \end{array}\right) \in \eta A_{>0}K,$$

$E' \in U(r)$. Then

$$\begin{aligned} \widetilde{f}(gg') &= f|_g|_{R^{-1}}\big|_{RgR^{-1}}(\mathbf{e}_1) \\ &= f|_g|_{R^{-1}}\left(Rg'R^{-1}\left(\frac{\mathbf{e}_1}{\eta}\right)\right) j\left(Rg'R^{-1}, \mathbf{e}_1\right)^k \\ &= f|_g|_{R^{-1}}\left(\frac{Rg'\mathbf{0}}{E\eta j(Rg'R^{-1})}\right) j\left(Rg'R^{-1}, \mathbf{e}_1\right)^k \\ &= \sum_{I\in\wp(r)} q_I(Rg'\mathbf{0})(E\eta)^I\, j\left(Rg'R^{-1}, \mathbf{e}_1\right)^{k+|I|}. \end{aligned}$$

Therefore since $|j(Rg'R^{-1}, \mathbf{e}_1)| = \sqrt{\Delta'(Rg'\mathbf{0}, Rg'\mathbf{0})}$ we get

$$\begin{aligned} \left|\widetilde{f}(gg')\right| &\le 2^r C'' e^{\pi M_0 \Delta'(Rg'\mathbf{0}, Rg'\mathbf{0})} \\ &\quad \times \left(\Delta'(Rg'\mathbf{0}, Rg'\mathbf{0})^{\frac{k}{2}} + \Delta'(Rg'\mathbf{0}, Rg'\mathbf{0})^{\frac{k+r}{2}}\right). \end{aligned}$$

So we see that there exists $C' > 0$ such that

$$\left|\widetilde{f}(gg')\right| \le \frac{C'}{\Delta'(Rg'\mathbf{0}, Rg'\mathbf{0})}$$

for all $g' \in \eta A_{>t_0}K$. However, on the one hand $\widetilde{f}(g\diamondsuit)$ is left- $g^{-1}\Gamma g$ -invariant, and on the other hand Δ' is $RNZ_G(G')R^{-1}$ -invariant. Therefore the estimate is correct even for all

$$g' \in NA_{>t_0}K = \left(g\Gamma g^{-1} \cap NZ_G(G')\right)\eta A_{>t_0}K$$

by Theorem 6.2 (i).

Step III. **Conclusion: Prove that**

$$|F(g\diamondsuit)| \leq ||F(g\diamondsuit)||_{\infty,Na_{t_0}K} + 2C'e^{-2t_0}$$

on $NA_{>t_0}K$**.**

Let $g' \in G$ be arbitrary. We will show the estimate on $g'A \cap NA_{>t_0}K$.

$$\mathbb{R} \to H, t \mapsto \mathbf{w}_t := Rg'a_t\mathbf{0}$$

is a geodesic in H, and for all $t \in \mathbb{R}$ we have $g'a_t \in NA_{>t_0}K$ if and only if $\Delta'(\mathbf{w}_t, \mathbf{w}_t) > 2e^{2t_0}$. Now we have to distinguish two cases.

In the first case the geodesic connects ∞ with a point in ∂H. First assume that $\lim_{t\to\infty} \mathbf{w}_t = \infty$ and $\lim_{t\to-\infty} \mathbf{w}_t \in \partial H$. Then $\lim_{t\to\infty} \Delta'(\mathbf{w}_t, \mathbf{w}_t) = \infty$ and $\lim_{t\to-\infty} \Delta'(\mathbf{w}_t, \mathbf{w}_t) = 0$. So we may assume without loss of generality that $\Delta'(\mathbf{w}_0, \mathbf{w}_0) = 2e^{2t_0}$, and therefore $g' = g'a_0 \in Na_{t_0}K$ and $g'a_t \in NA_{>t_0}K$ if and only if $t > 0$. So let $t > 0$. Then

$$F(gg'a_t) = F(gg') + \int_0^t \widetilde{f}(gg'a_\tau)\, d\tau,$$

and so

$$|F(gg'a_t)| \leq ||F(g\diamondsuit)||_{\infty,Na_{t_0}K} + \int_0^t \left|\widetilde{f}(gg'a_\tau)\right| d\tau.$$

By Step II and Lemma 6.1 (i),

$$\begin{aligned}\int_0^t \left|\widetilde{f}(gg'a_\tau)\right| d\tau &\leq C' \int_0^t \frac{d\tau}{\Delta'(\mathbf{w}_\tau, \mathbf{w}_\tau)} \\ &= \frac{C'}{\Delta'(\mathbf{w}_0, \mathbf{w}_0)} \int_0^t e^{-2\tau} d\tau \\ &\leq C'e^{-2t_0}.\end{aligned}$$

The case where $\lim_{t\to-\infty} = \infty$ and $\lim_{t\to\infty} \in \partial H$ is done similarly.

In the second case the geodesic connects two points in ∂H. Then without loss of generality we may assume that $\Delta'(R\mathbf{w}_t, R\mathbf{w}_t)$ is maximal for $t = 0$. So if $\Delta'(\mathbf{w}_0, \mathbf{w}_0) < 2e^{2t_0}$, we have $g'A \cap NA_{>t_0}K = \emptyset$. Otherwise by Lemma 6.1 (ii) there exists $T \geq 0$ such that $\Delta'(\mathbf{w}_T, \mathbf{w}_T) = \Delta'(\mathbf{w}_{-T}, \mathbf{w}_{-T}) = 2e^{2t_0}$, and since $\Delta'(\mathbf{w}_T, \mathbf{w}_T) \leq \frac{4}{e^{2|T|}}\Delta'(\mathbf{w}_0, \mathbf{w}_0)$, we see that

$$T \leq \frac{1}{2}\log(2\Delta'(\mathbf{w}_T, \mathbf{w}_T)) - t_0.$$

So $g'a_T, g'a_{-T} \in Na_{t_0}K$ and $g'a_t \in NA_{>t_0}K$ if and only if $t \in]-T, T[$. Let $t \in]-T, T[$ and assume $t \geq 0$ first. Then

$$F(gg'a_t) = F(gg'a_T) - \int_t^T \widetilde{f}(gg'a_\tau)\, d\tau,$$

and so

$$|F(gg'a_t)| \leq ||F(g\diamondsuit)||_{\infty,Na_{t_0}K} + \int_0^T \left|\widetilde{f}(gg'a_\tau)\right| d\tau.$$

By Step II and Lemma 6.1 (ii), now

$$\begin{aligned}\int_0^T \left|\widetilde{f}\left(gg'a_\tau\right)\right| d\tau &\leq C' \int_0^T \frac{d\tau}{\Delta'\left(\mathbf{w}_\tau, \mathbf{w}_\tau\right)} \\ &\leq \frac{C'}{\Delta'\left(\mathbf{w}_0, \mathbf{w}_0\right)} \int_0^T e^{2\tau} d\tau \\ &\leq \frac{C'}{2\Delta'\left(\mathbf{w}_0, \mathbf{w}_0\right)} e^{2T} \\ &\leq 2C' e^{-2t_0}.\end{aligned}$$

The case $t \leq 0$ is done similarly.

(ii) Since on one hand $\partial_\tau F\left(\diamondsuit a_\tau\right)|_{\tau=0} = \widetilde{f} \in L^2\left(\Gamma\backslash G\right) \otimes \bigwedge\left(\mathbb{C}^r\right)$ and on the other hand $\mathrm{vol}\ \left(\Gamma\backslash G\right) < \infty$, it suffices to show that ξF is bounded for all $\alpha \in \Phi \setminus \{0\}$ and $\xi \in \mathfrak{g}^\alpha$. So let $\alpha \in \Phi \setminus \{0\}$ and $\xi \in \mathfrak{g}^\alpha$. First assume $\alpha > 0$, which clearly implies that $\alpha \geq 1$ and $\xi \in T^-$. So there exists a continuously differentiable curve $\rho : I \to G$ contained in the T^--leaf containing 1 such that $0 \in I$, $\rho(0) = 1$ and $\partial_t \rho(t)|_{t=0} = \xi$. Let $g \in G$. Then by Theorem 7.4 (ii), we have

$$\begin{aligned}\left(\xi F\right)(g) &= \partial_t F\left(g\rho(t)\right)|_{t=0} \\ &= -\int_0^\infty \partial_t \widetilde{f}\left(g\rho(t)a_\tau\right)\Big|_{t=0} d\tau \\ &= -\int_0^\infty \partial_t \widetilde{f}\left(ga_\tau a_{-\tau}\rho(t)a_\tau\right)\Big|_{t=0} d\tau \\ &= -\int_0^\infty \left(\left(\mathrm{Ad}_{a_{-\tau}}(\xi)\right)\widetilde{f}\right)\left(ga_\tau\right) d\tau \\ &= -\int_0^\infty e^{-\alpha\tau}\left(\xi\widetilde{f}\right)\left(ga_\tau\right) d\tau,\end{aligned}$$

so

$$\left|\left(\xi F\right)(g)\right| \leq c\left|\left|\xi\right|\right|_2 < \infty,$$

where c is the LIPSCHITZ constant of $\widetilde{f}$. The case $\alpha < 0$ is done similarly. □

Therefore by the FOURIER decomposition described above we have

$$F = \sum_{I \in \wp(r), |I| = \rho} \sum_{\nu \in \mathbb{Z}} F_{I\nu} \eta^I,$$

where $F_{I\nu} \in H_\nu$ for all $I \in \wp(r)$, $|I| = \rho$, and $\nu \in \mathbb{Z}$. $\mathcal{D} = \mathcal{D}^+ + \mathcal{D}^-$, and a simple calculation shows that $\mathcal{D}^+$ and $\mathcal{D}^- \in \mathbb{R}\mathcal{D} \oplus \mathfrak{g} \cap \left(T^+ \oplus T^-\right)$, and so $\mathcal{D}^+F, \mathcal{D}^-F \in L^2\left(\Gamma\backslash G\right) \otimes \bigwedge\left(\mathbb{C}^r\right)$ by Lemma 7.5 (ii). So we get the FOURIER decomposition of $\widetilde{f}$ as

$$\widetilde{f} = \mathcal{D}F = \sum_{I \in \wp(r), |I| = \rho} \sum_{\nu \in \mathbb{Z}} \left(\mathcal{D}^+ F_{I,\nu-2} + \mathcal{D}^- F_{I,\nu+2}\right) \eta^I$$

with $\mathcal{D}^+ F_{I,\nu-2} + \mathcal{D}^- F_{I,\nu+2} \in H_\nu$ for all $\nu \in \mathbb{Z}$. But since $f \in sS_k^\rho(\Gamma)$ the FOURIER decomposition of $\widetilde{f}$ is exactly

$$\widetilde{f} = \sum_{I \in \wp(r), |I| = \rho} q_I \eta^I$$

with $q_I \in \mathcal{C}^\infty(G)^{\mathbb{C}} \cap H_{k+\rho}$, and so for all $I \in \wp(r)$, $|I| = \rho$, and $\nu \in \mathbb{Z}$

$$\mathcal{D}^+ F_{I,\nu-2} + \mathcal{D}^- F_{I,\nu+2} = \begin{cases} q_I & \text{if } \nu = k+\rho \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 7.6. *$F_{I,\nu}=0$ for $I\in\wp(r)$, $|I|=\rho$, and $\nu\geq k+\rho$.*

Proof. Similar to the argument of GUILLEMIN and KAZHDAN in [6]. Let $I\in\wp(r)$ such that $|I|=\rho$. Then by the commutation relations of D^+ and D^- we get for all $n\in\mathbb{Z}$

$$\left|\left|\mathcal{D}^+F_{I,n}\right|\right|_2^2=\left|\left|\mathcal{D}^-F_{I,n}\right|\right|_2^2+\nu\left|\left|F_{I,n}\right|\right|_2^2, \tag{7.2}$$

and for all $n\geq k+\rho+1$ we have $\mathcal{D}^+F_{I,n-2}+\mathcal{D}^-F_{I,n+2}=0$ and so

$$\left|\left|\mathcal{D}^-F_{I,n+2}\right|\right|_2=\left|\left|\mathcal{D}^+F_{I,n-2}\right|\right|_2.$$

Now let $\nu\geq k+\rho$. We will prove that

$$\left|\left|\mathcal{D}^+F_{I,\nu+4l}\right|\right|_2\geq\left|\left|F_{I,\nu}\right|\right|_2$$

for all $l\in\mathbb{N}$ by induction on l:

> If $l=0$ then the inequality is clear by (7.2). So let us assume that the inequality is true for some $l\in\mathbb{N}$. Then again by (7.2) we have
>
> $$\left|\left|\mathcal{D}^+F_{I,\nu+4l+4}\right|\right|_2^2\geq\left|\left|\mathcal{D}^-F_{I,\nu+4l+4}\right|\right|_2^2=\left|\left|\mathcal{D}^+F_{I,\nu+4l}\right|\right|_2^2\geq\left|\left|F_{I,\nu}\right|\right|_2^2.$$

On the other hand, $\mathcal{D}^+F_I\in L^2(\Gamma\backslash G)$ by Lemma 7.5 and so $\left|\left|\mathcal{D}^+F_{I,n}\right|\right|_2\rightsquigarrow 0$ for $n\rightsquigarrow\infty$. This implies $F_\nu=0$. □

So for all $I\in\wp(r)$, $|I|=\rho$, we obtain $\mathcal{D}^+F_{I,k+\rho-2}=q_I$ and finally $\mathcal{D}^-q_I=0$ by Lemma 7.1, since $f\in\mathcal{O}(\mathcal{B})$, so

$$\left|\left|q_I\right|\right|_2^2=\left(q_I,\mathcal{D}^+F_{I,k+\rho-2}\right)=-\left(\mathcal{D}^-q_I,F_{I,k+\rho-2}\right)=0,$$

and so $\widetilde{f}=0$, which completes the proof of our main theorem. □

8. COMPUTATION OF THE $\varphi_{\gamma_0,I,m}$

Fix a regular loxodromic $\gamma_0\in\Gamma$, $g\in G$, $t_0>0$ and $w_0\in M$ such that $E_0:=E_{w_0}$ is diagonal and $\gamma_0=ga_{t_0}w_0g^{-1}\in gAMg^{-1}$. Let $D\in\mathbb{R}^{r\times r}$ be diagonal such that $\exp(2\pi iD)=E_0$ and $\chi\in\mathbb{R}$ such that $j(w_0)=e^{2\pi i\chi}$. Now we will compute $\varphi_{\gamma_0,I,m}\in sS_k(\Gamma)$, $I\in\wp(r)$, $m\in\frac{1}{t_0}\left(\mathbb{Z}-(k+|I|)\chi-\mathrm{tr}_ID\right)$, as a relative POINCARÉ series with respect to $\Gamma_0:=\langle\gamma_0\rangle\sqsubset\Gamma$. Hereby again '$\equiv$' means equality up to a constant $\neq 0$ not necessarily independent of γ_0, I and m.

Theorem 8.1. *Let $I\in\wp(r)$ and $k\geq 2n+1-|I|$. Then for all $m\in\frac{1}{t_0}\left(\mathbb{Z}-(k+|I|)\chi-\mathrm{tr}_ID\right)$*

(i)

$$\varphi_{\gamma_0,I,m}\equiv\sum_{\gamma\in\Gamma_0\backslash\Gamma}q|_\gamma\in sS_k^{(|I|)}(\Gamma),$$

where

$$\begin{aligned}q:=&\int_{-\infty}^{\infty}e^{2\pi imt}\Delta\left(\diamondsuit,ga_t\mathbf{0}\right)^{-k-|I|}\overline{j\left(ga_t,\mathbf{0}\right)}^{k+|I|}dt\left(E_g^{-1}\zeta\right)^I\\&\in sM_k^{(|I|)}\left(\Gamma_0\right)\cap L_k^1\left(\Gamma_0\backslash\mathcal{B}\right).\end{aligned}$$

(ii) *For all $\mathbf{z}\in B$ we have*

$$q\left(\mathbf{z}\right)\equiv\left(\Delta\left(\mathbf{z},\mathbf{X}^+\right)\Delta\left(\mathbf{z},\mathbf{X}^-\right)\right)^{-\frac{k+|I|}{2}}\left(\frac{1+v_1}{1-v_1}\right)^{\pi im}\left(E_g^{-1}\zeta\right)^I,$$

where

$$\mathbf{X}^+ := g\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \quad \textit{and} \quad \mathbf{X}^- := g\begin{pmatrix}-1\\0\\\vdots\\0\end{pmatrix}$$

are the two fixpoints of γ_0 *in* ∂B*, and*

$$\mathbf{v} := g^{-1}\mathbf{z} \in B \subset \mathbb{C}^p.$$

Proof. Let $\rho := |I|$.

(i) Let $f \in sS_k^{(\rho)}(\Gamma)$, and define $h = \sum_{J\in\wp(r),|J|=\rho} h_J\eta^J \in \mathcal{C}^\infty\left(\mathbb{R}\times M\right)^{\mathbb{C}} \otimes \bigwedge(\mathbb{C}^r)$, all $h_J \in \mathcal{C}^\infty\left(\mathbb{R}\times M\right)^{\mathbb{C}}$, and $b_{I,m} \in \mathbb{C}$, $m \in \frac{1}{t_0}\left(\mathbb{Z} - (k+|I|)\,\chi - \mathrm{tr}_I D\right)$, as in Theorem 4.1. Then by standard FOURIER theory and Lemma 2.3 we have

$$\begin{aligned}
b_{I,m} &\equiv \int_0^{t_0} e^{-2\pi imt} h_I(t,1)dt\\
&\equiv \int_0^{t_0} e^{-2\pi imt}\left(\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-\rho}\left(E_g^{-1}\zeta\right)^I, f\right) j\left(ga_t,\mathbf{0}\right)^{k+\rho} dt\\
&= \int_0^{t_0} e^{-2\pi imt}\int_G \left\langle \widetilde{f}, \left(\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-\rho}\left(E_g^{-1}\zeta\right)^I\right)^{\sim}\right\rangle j\left(ga_t,\mathbf{0}\right)^{k+\rho} dt.
\end{aligned}$$

Since by SATAKE's theorem, Theorem 2.1, $\widetilde{f} \in L^\infty(G)\otimes\bigwedge(\mathbb{C}^r)$, and

$$\begin{aligned}
&\int_0^{t_0}\int_G \left|\left(\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-\rho}\left(E_g^{-1}\zeta\right)^I\right)^{\sim} j\left(ga_t,\mathbf{0}\right)^{k+\rho}\right| dt\\
&\quad= \int_0^{t_0}\int_G \left|\left(\Delta\left(\diamondsuit,\mathbf{0}\right)^{-k-\rho}\zeta^I\right)^{\sim}\left(\left(ga_t\right)^{-1}\diamondsuit\right)\right| dt\\
&\quad\equiv \int_G \left|\widetilde{\zeta^I}\right|\\
&\quad= \int_G \left|j\left(\diamondsuit,\mathbf{0}\right)^{k+\rho}\right|\\
&\quad\equiv \int_B \Delta\left(\mathbf{Z},\mathbf{Z}\right)^{\frac{k+\rho}{2}-(p+1)} dV_{\mathrm{Leb}} < \infty,
\end{aligned}$$

by TONELLI's and FUBINI's theorem we can interchange the order of integration:

$$\begin{aligned}
b_{I,m} &\equiv \int_G \left\langle \widetilde{f}, \int_0^{t_0} e^{2\pi imt}\left(\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-\rho}\left(E_g^{-1}\zeta\right)^I\right)^{\sim}\overline{j\left(ga_t,\mathbf{0}\right)}^{k+\rho} dt\right\rangle\\
&= \left(\int_0^{t_0} e^{2\pi imt}\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-\rho}\overline{j\left(ga_t,\mathbf{0}\right)}^{k+\rho} dt\left(E_g^{-1}\zeta\right)^I, f\right)\\
&= (q,f)_{\Gamma_0},
\end{aligned}$$

where

$$\left(\int_0^{t_0} e^{2\pi imt}\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-\rho}\overline{j\left(ga_t,\mathbf{0}\right)}^{k+\rho} dt\left(E_g^{-1}\zeta\right)^I\right)^{\sim} \in L^1(G)\otimes\bigwedge\left(\mathbb{C}^r\right),$$

$$\int_0^{t_0} e^{2\pi imt}\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-\rho}\overline{j\left(ga_t,\mathbf{0}\right)}^{k+\rho} dt\left(E_g^{-1}\zeta\right)^I \in \mathcal{O}(\mathcal{B})$$

since $\Delta(\diamondsuit, \mathbf{w}) \in \mathcal{O}(B)$ for all $\mathbf{w} \in B$ and the convergence of the integral is compact, and so by Lemma 2.2,

$$\begin{aligned} q &:= \sum_{\gamma' \in \Gamma_0} \int_0^{t_0} e^{2\pi i m t} \Delta(\diamondsuit, g a_t \mathbf{0})^{-k-\rho} \overline{j(g a_t, \mathbf{0})}^{k+\rho} dt \left(E_g^{-1}\zeta\right)^I \Big|_{\gamma'} \\ &\in sM_k(\Gamma_0) \cap L_k^1(\Gamma_0 \backslash \mathcal{B}). \end{aligned}$$

Clearly

$$\begin{aligned} &\Delta(\diamondsuit, g a_t \mathbf{0})^{-k-\rho} \left(E_g^{-1}\zeta\right)^I \Big|_{\gamma_0} \\ &\quad = \Delta(\gamma_0 \diamondsuit, g a_t \mathbf{0})^{-k-\rho} \left(E_0 E_g^{-1}\zeta\right)^I j(\gamma_0, \diamondsuit)^{k+\rho} \\ &\quad = \Delta\left(\diamondsuit, \gamma_0^{-1} g a_t \mathbf{0}\right)^{-k-\rho} \left(E_0 E_g^{-1}\zeta\right)^I \overline{j\left(\gamma_0^{-1}, g a_t \mathbf{0}\right)}^{k+\rho}, \end{aligned}$$

so for all $\mathbf{z} \in B$ we can compute $q(\mathbf{z})$ as

$$\begin{aligned} q(\mathbf{z}) &= \sum_{\nu \in \mathbb{Z}} \int_0^{t_0} e^{2\pi i m t} \Delta(\diamondsuit, g a_t \mathbf{0})^{-k-\rho} \left(E_g^{-1}\zeta\right)^I \overline{j(g a_t, \mathbf{0})}^{k+\rho} dt \Big|_{\gamma_0^\nu} (\mathbf{z}) \\ &= \sum_{\nu \in \mathbb{Z}} \int_0^{t_0} e^{2\pi i m t} \Delta\left(\mathbf{z}, \gamma_0^{-\nu} g a_t \mathbf{0}\right)^{-k-\rho} \left(E_0^\nu E_g^{-1}\zeta\right)^I \times \overline{j\left(\gamma_0^{-\nu} g a_t, \mathbf{0}\right)}^{k+\rho} dt \\ &= \sum_{\nu \in \mathbb{Z}} \int_0^{t_0} e^{2\pi i m t} \Delta(\mathbf{z}, g a_{t-\nu t_0} \mathbf{0})^{-k-\rho} \left(E_g^{-1}\zeta\right)^I e^{2\pi i \nu \operatorname{tr}_I D} \overline{j(g a_{t-\nu t_0}, \mathbf{0})}^{k+\rho} e^{2\pi i \nu (k+\rho)\chi} dt \\ &= \sum_{\nu \in \mathbb{Z}} \int_0^{t_0} e^{2\pi i m (t-\nu t_0)} \Delta(\mathbf{z}, g a_{t-\nu t_0} \mathbf{0})^{-k-\rho} \overline{j(g a_{t-\nu t_0}, \mathbf{0})}^{k+\rho} dt \left(E_g^{-1}\zeta\right)^I \\ &= \int_{-\infty}^{\infty} e^{2\pi i m t} \Delta(\mathbf{z}, g a_t \mathbf{0})^{-k-\rho} \overline{j(g a_t, \mathbf{0})}^{k+\rho} dt \left(E_g^{-1}\zeta\right)^I. \end{aligned}$$

Again by Lemma 2.2 we see that $\sum_{\gamma \in \Gamma_0 \backslash \Gamma} q|_\gamma \in sM_k^{(\rho)}(\Gamma) \cap L_k^1(\Gamma \backslash \mathcal{B})$, and so by SATAKE's theorem, Theorem 2.1, it is even an element of $sS_k^{(\rho)}(\Gamma)$, such that

$$b_{I,m} \equiv \left(\sum_{\gamma \in \Gamma_0 \backslash \Gamma} q|_\gamma, f \right)_\Gamma,$$

and so we conclude that $\varphi_{\gamma_0, I, m} \equiv \sum_{\gamma \in \Gamma_0 \backslash \Gamma} q|_\gamma$.

(ii)

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{2\pi i m t} \Delta(\mathbf{z}, g a_t \mathbf{0})^{-k-\rho} \overline{j(g a_t, \mathbf{0})}^{k+\rho} dt \\ &\qquad = j\left(g^{-1}, \mathbf{z}\right)^{k+\rho} \int_{-\infty}^{\infty} e^{2\pi i m t} \Delta\left(g^{-1}\mathbf{z}, a_t \mathbf{0}\right)^{-k-\rho} \overline{j(a_t, \mathbf{0})}^{k+\rho} dt \\ &\qquad = j\left(g^{-1}, \mathbf{z}\right)^{k+\rho} \int_{-\infty}^{\infty} e^{2\pi i m t} (1 - v_1 \tanh t)^{-k-\rho} \frac{1}{(\cosh t)^{k+\rho}} dt \end{aligned}$$

$$
\begin{aligned}
&= j\left(g^{-1},\mathbf{z}\right)^{k+\rho}\int_{-\infty}^{\infty}\frac{e^{2\pi imt}}{\left(\cosh t - v_1 \sinh t\right)^{k+\rho}}dt\\
&\equiv j\left(g^{-1},\mathbf{z}\right)^{k+\rho}\frac{1}{\left(1-v_1^2\right)^{\frac{k+\rho}{2}}}\left(\frac{1+v_1}{1-v_1}\right)^{\pi im}\\
&= j\left(g^{-1},\mathbf{z}\right)^{k+\rho}\left(\left(1-v_1\right)\left(1+v_1\right)\right)^{-\frac{k+\rho}{2}}\left(\frac{1+v_1}{1-v_1}\right)^{\pi im}\\
&\equiv \left(\Delta\left(\mathbf{z},\mathbf{X}^+\right)\Delta\left(\mathbf{z},\mathbf{X}^-\right)\right)^{-\frac{k+\rho}{2}}\left(\frac{1+v_1}{1-v_1}\right)^{\pi im}.
\end{aligned}
$$

□

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