Journal of Inequalities in Pure and Applied Mathematics

ON A CONVOLUTION CONJECTURE OF BOUNDED FUNCTIONS

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ISSN (electronic): 1443-5756
229-04

volume 6, issue 2, article 34, 2005.

Received 30 November, 2004; accepted 03 March, 2005.

Communicated by: K. Nikodem

Abstract

We consider the convolution $P(A,B) \star P(C,D)$ of the classes of analytic functions subordinated to the homographies $\frac{1+Az}{1-Bz}$ and $\frac{1+Cz}{1-Dz}$ respectively, where A,B,C,D are some complex numbers. In 1988 J. Stankiewicz and Z. Stankiewicz [11] showed that for certain A,B,C,D there exist X,Y such that $P(A,B) \star P(C,D) \subset P(X,Y)$. In this paper we verify the conjecture that $P(X,Y) \subset (A,B) \star P(C,D)$ for some A,B,C,D,X,Y.

2000 Mathematics Subject Classification: Primary 30C45; Secondary 30C55.
Key words: Hadamard product, Convolution, Subordination, Bounded functions.

Contents

1. Introduction

Let $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$ denote the open unit disc and let $\mathcal{H}$ be the class of functions regular in Δ. We will denote by $\mathcal{N}$ the class of functions $f \in \mathcal{H}$ normalized by $f(0) = 1$. The class of Schwarz functions Ω is the class of functions $\omega \in \mathcal{H}$, such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \Delta$. We say that a function f is subordinate to a function g in Δ (and write $f \prec g$ or $f(z) \prec g(z)$) if there exists a function $\omega \in \Omega$ such that $f(z) = g(\omega(z))$; $z \in \Delta$. If the function g is univalent in Δ, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. In this case we have $\omega(z) = g^{-1}(f(z))$.

Let the functions f and g be of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \Delta.$$

We say that the Hadamard product of f and g is the function $f \star g$ if

$$(f \star g)(z) = f(z) \star g(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

J. Hadamard [2] proved, that the radius of convergence of $f \star g$ is the product of the radii of convergence of the corresponding series f and g. The function $f \star g$ is also called the *convolution* of the functions f and g.

For the classes $Q_1 \subset \mathcal{H}$ and $Q_2 \subset \mathcal{H}$ the convolution $Q_1 \star Q_2$ is defined as

$$Q_1 \star Q_2 = \{h \in \mathcal{H};\ h = f \star g,\ f \in Q_1,\ g \in Q_2\}.$$

The problem of connections between functions f, g and their convolution $f \star g$ or between Q_1, Q_2 and $Q_1 \star Q_2$ has often been investigated. Many conjectures have been given, however, many of them have still not been verified.

In 1958, G. Pólya and I.J. Schoenberg [7] conjectured that the Hadamard product of two convex mappings is a convex mapping. In 1961 H.S. Wilf [12] gave the more general supposition that if F and G are convex mappings in Δ and f is subordinate to F, then the convolution $f \star G$ is subordinate to $F \star G$.

In 1973, S. Rusheweyh and T. Sheil-Small [9] proved both conjectures and more results of this type. Their very important results we may write as:

Theorem A. *If $f \in S^c$ and $g \in S^c$, then $f \star g \in S^c$, where S^c is the class of univalent and convex functions. Moreover $S^c \star S^c = S^c$.*

Theorem B. *If $F \in S^c$, $G \in S^c$ and $f \prec F$, then $f \star G \prec F \star G$.*

In 1985, S. Ruscheweyh and J. Stankiewicz [10] proved some generalizations of Theorems A and B:

Theorem C. *If the functions F and G are univalent and convex in Δ, then for all functions f and g, if $f \prec F$ and $g \prec G$ then $f \star g \prec F \star G$.*

For the given complex numbers A, B such that $A + B \neq 0$ and $|B| \leq 1$ let us denote

$$P(A, B) = \left\{ f \in \mathcal{N} : \; f(z) \prec \frac{1 + Az}{1 - Bz} \right\}.$$

W. Janowski introduced the class $P(A, B)$ in [3] and considered it for some real A and B. If $A = B = 1$ then the class $P(1, 1)$ is the class of functions with positive real part (Carathéodory functions). Note, that for $|B| < 1$ the class $P(A,B)$ is the class of bounded functions. In 1988 J. Stankiewicz and Z. Stankiewicz [11] investigated the convolution properties of the class $P(A,B)$ and proved the following theorem:

Theorem D. *If A, B, C, D are some complex numbers such that $A+B \neq 0$, $C+D \neq 0$, $|B| \leq 1$, $|D| \leq 1$, then*

$$P(A,B) \star P(C,D) \subset P(AC+AD+BC, BD),$$

moreover, if $|B| = 1$ or $|D| = 1$, then

$$P(A,B) \star P(C,D) = P(AC+AD+BC, BD).$$

The equality between the class $P(A,B) \star P(C,D)$ and the class $P(AC+AD+BC, BD)$ for $|B| < 1$ and $|D| < 1$ was an open problem. In [5] K. Piejko, J. Sokół and J. Stankiewicz proved that the above mentioned classes are different. In this paper we give an extension of this result. First we need two theorems.

Theorem E (G. Eneström [1], S. Kakeya [4]). *If $a_0 > a_1 > \cdots > a_n > 0$, where $n \in \mathbb{N}$, then the polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ has no roots in $\overline{\Delta} = \{z \in \mathbb{C} : \ |z| \leq 1\}$.*

Theorem F (W. Rogosinski [8]). *If the function $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ is subordinated to the function $F(z) = \sum_{n=0}^{\infty} \beta_n z^n$ in Δ, then $\sum_{n=0}^{\infty} |\alpha_n|^2 \leq \sum_{n=0}^{\infty} |\beta_n|^2$.*

2. Main Result

We prove the following theorem.

Theorem 2.1. *Let* A, B, C, D *be some complex numbers such that* $B + A \neq 0$, $C + D \neq 0$, $|B| < 1$, $|D| < 1$, *then there are not complex numbers* X, Y, $X + Y \neq 0$, $|Y| \leq 1$ *such that* $P(X, Y) \subset P(A, B) \star P(C, D)$.

Proof. As in [5], the proof will be divided into three steps. First we give a family of bounded functions ω^ν; $\nu = 1, 2, 3 \dots$ having special properties of coefficients. Afterwards we construct, using ω^ν, a function belonging to the class $P(X, Y)$ and finally we will show that such a function is not in the class $P(A, B) \star P(C, D)$.

Now we use a method of E. Landau [6] and find some functions ω^ν, $\nu = 1, 2, 3, \dots$, which are basic in this proof. We observe that

$$\frac{1}{1-z} = \left(\frac{1}{\sqrt{1-z}}\right)^2 = \left(\sum_{k=0}^{\infty} p_k z^k\right)^2 = 1 + z + z^2 + z^3 + \cdots,$$

where

$$p_0 = 1 \text{ and } p_k = \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot \dots \cdot 2k}; \quad k = 1, 2, 3, \dots \tag{2.1}$$

For some $\nu \in \mathbb{N}$ we set

$$K_\nu(z) = \sum_{k=0}^{\nu} p_k z^k = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots + p_\nu z^\nu$$

and note that

$$K_\nu^2(z) = 1 + z + z^2 + \cdots + z^\nu + b_{\nu+1}z^{\nu+1} + \cdots + b_{2\nu}z^{2\nu},$$

where $b_{\nu+1}, \ldots, b_{2\nu} \in \mathbb{C}$. Let ω^ν be given by

$$\omega^\nu(z) = \frac{z^{\nu+1}K_\nu\left(\frac{1}{z}\right)}{K_\nu(z)} = z\frac{z^\nu + p_1z^{\nu-1} + p_2z^{\nu-2} + \cdots + p_\nu}{1 + p_1z + p_2z^2 + \cdots + p_\nu z^\nu}. \tag{2.2}$$

Since

$$p_k > p_k\frac{2k+1}{2k+2} = p_{k+1} \quad \text{where} \quad k = 0, 1, 2, 3, \ldots,$$

then for $\nu \in \mathbb{N}$ we have $1 > p_1 > p_2 > p_3 > \cdots > p_\nu > 0$.

Applying Theorem E to the polynomial K_ν we obtain $K_\nu(z) \neq 0$ for $|z| \leq 1$, hence the function ω^ν is regular in Δ. Moreover $\omega^\nu(0) = 0$ and on the circle $|z| = 1$ we have

$$\left|\omega^\nu\left(e^{it}\right)\right| = \left|\frac{e^{(\nu+1)it}K_\nu\left(e^{-it}\right)}{K_\nu\left(e^{it}\right)}\right| = \frac{\left|\overline{K_\nu\left(e^{it}\right)}\right|}{\left|K_\nu\left(e^{it}\right)\right|} = 1; \quad t \in \mathbb{R}.$$

In this way we conclude that for $\nu \in \{1, 2, 3, \ldots\}$

$$\omega^\nu \in \Omega. \tag{2.3}$$

Let, for a certain $\nu \in \mathbb{N}$, the function ω^ν be represented by following power series expansions: $\omega^\nu(z) = \gamma_1^\nu z + \gamma_2^\nu z^2 + \gamma_3^\nu z^3 + \cdots$ and let $s_n^\nu(z)$ denote the partial sum

$$s_n^\nu(z) = \sum_{k=1}^{n} \gamma_k^\nu z^k = \gamma_1^\nu z + \gamma_2^\nu z^2 + \gamma_3^\nu z^3 + \cdots + \gamma_n^\nu z^n$$

for all $n \in \{1, 2, 3, \dots, \nu+1\}$.

Now we will estimate $s_n^\nu(1) = \gamma_1^\nu + \gamma_2^\nu + \gamma_3^\nu + \cdots + \gamma_n^\nu$. If we integrate on a circle $C : |z| = r$ in a counterclockwise direction with $0 < r < 1$, then we obtain

$$\int_C az^m dz = 0 \quad \text{and} \quad \int_C \frac{a}{z} dz = 2\pi a i, \tag{2.4}$$

for all integers $m \neq -1$ and $a \in \mathbb{C}$. Hence

$$\begin{aligned} s_n^\nu(1) &= \gamma_1^\nu + \gamma_2^\nu + \gamma_3^\nu + \cdots + \gamma_n^\nu \\ &= \frac{1}{2\pi i} \int_C \omega^\nu(z) \left(\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots + \frac{1}{z^{n+1}} \right) dz \\ &= \frac{1}{2\pi i} \int_C \frac{\omega^\nu(z)}{z^{n+1}} \left(1 + z + z^2 + z^3 + \cdots + z^{n-1}\right) dz \\ &= \frac{1}{2\pi i} \int_C \frac{\omega^\nu(z)}{z^{n+1}} Q(z) dz, \end{aligned}$$

where

$$Q(z) = 1 + z + z^2 + \cdots + z^{n-1} + d_n z_n + d_{n+1} z^{n+1} + \cdots + d_s z^s$$

is any polynomial whose first n terms are equal to $1 + z + z^2 + \cdots + z^{n-1}$.

If for $n \leq \nu + 1$ we set

$$Q(z) = K_\nu^2(z) = 1+z+z^2+\cdots+z^{n-1}+z^n+\cdots+z^\nu+b_{\nu+1}z^{\nu+1}+\cdots+b_{2\nu}z^{2\nu},$$

then we obtain

$$s_n^\nu(1) = \frac{1}{2\pi i}\int_C \frac{\omega^\nu(z)}{z^{n+1}}K_\nu^2(z)dz.$$

From (2.2) it follows that

$$\begin{aligned} s_n^\nu(1) &= \frac{1}{2\pi i}\int_C \frac{\frac{z^{\nu+1}K_\nu\left(\frac{1}{z}\right)}{K_\nu(z)}}{z^{n+1}}K_\nu^2(z)dz \\ &= \frac{1}{2\pi i}\int_C z^{\nu-n}K_\nu\left(\frac{1}{z}\right)K_\nu(z)dz \\ &= \frac{1}{2\pi i}\int_C z^{\nu-n}\left(1+p_1\frac{1}{z}+p_2\frac{1}{z^2}+\cdots+p_\nu\frac{1}{z^\nu}\right) \\ &\qquad \times\left(1+p_1z+p_2z^2+\cdots+p_\nu z^\nu\right)dz. \end{aligned}$$

Using (2.4) we get

$$s_n^\nu(1) = p_{\nu-n+1} + p_{\nu-n+2}p_1 + p_{\nu-n+3}p_2 + \cdots + p_\nu p_{n-1},$$

where (p_k) are given by (2.1), and so

$$s_n^\nu(1) = \sum_{k=1}^n \gamma_k^\nu = \sum_{k=1}^n p_{\nu-n+k}p_{k-1}. \tag{2.5}$$

By the above we have $\gamma_1^\nu = s_1^\nu(1) = p_\nu$ and for all $\nu \in \{1, 2, 3, \ldots\}$ and $n \in \{2, 3, 4, \ldots, \nu + 1\}$

$$\gamma_n^\nu = s_n^\nu(1) - s_{n-1}^\nu(1) = \sum_{k=1}^{n} p_{\nu-n+k} p_{k-1} - \sum_{k=1}^{n-1} p_{\nu-n+1+k} p_{k-1}.$$

Therefore

(2.6) $$\gamma_n^\nu = \sum_{k=1}^{n-1} (p_{\nu-n+k} - p_{\nu-n+1+k})\, p_{k-1} + p_\nu p_{n-1}.$$

The sequence (p_n) is positive and decreasing, then from (2.6) it follows that

(2.7) $$\gamma_n^\nu > 0 \quad \text{for all} \quad \nu \in \mathbb{N} \quad \text{and} \quad n \in \{1, 2, 3, \ldots, \nu + 1\}.$$

We conclude from (2.5) that for $n = \nu + 1$

$$s_{\nu+1}^\nu(1) = \gamma_1^\nu + \gamma_2^\nu + \cdots + \gamma_\nu^\nu + \gamma_{\nu+1}^\nu = \sum_{k=1}^{\nu+1} p_{k-1} p_{k-1} = 1 + \sum_{k=1}^{\nu} p_k^2.$$

Since $\sum_{k=1}^{\infty} p_k^2 = \infty$ then

(2.8) $$\lim_{\nu\to\infty} s_{\nu+1}^\nu(1) = \lim_{\nu\to\infty} \left(1 + \sum_{k=1}^{\nu} p_k^2\right) = +\infty.$$

Now using the properties of ω^ν we construct the function belonging to the class $P(X, Y)$ which is not in the class $P(A, B) \star P(C, D)$, where $|B| < 1$ and $|D| < 1$.

Since for $|x| = 1$ we have $P(A, B) = P(Ax, Bx)$, we can assume without loss of generality that $B \in [0; 1)$, $D \in [0; 1)$ and $Y \in [0; 1]$.

For fixed $\nu \in \mathbb{N}$ let h^ν be given by

$$h^\nu(z) = \frac{1 + X\omega^\nu(z)}{1 - Y\omega^\nu(z)}, \tag{2.9}$$

where $w^\nu \in \Omega$ is given by (2.2). It is clearly that $h^\nu \in P(X, Y)$. Suppose that there exist the functions $f \in P(A, B)$, $g \in P(C, D)$ such that

$$f(z) \star g(z) = h^\nu(z). \tag{2.10}$$

Let the functions f and g have the following form:

$$f(z) = \frac{1 + A\omega_1(z)}{1 - B\omega_1(z)} = 1 + (A + B)\frac{\omega_1(z)}{1 - B\omega_1(z)}$$

and

$$g(z) = \frac{1 + C\omega_2(z)}{1 - D\omega_2(z)} = 1 + (C + D)\frac{\omega_2(z)}{1 - D\omega_2(z)},$$

where $\omega_1, \omega_2 \in \Omega$. For simplicity of notation we write

$$f(z) = 1 + (A + B)\tilde{f}(z), \qquad g(z) = 1 + (C + D)\tilde{g}(z),$$

where

$$\tilde{f}(z) = \frac{\omega_1(z)}{1 - B\omega_1(z)} \quad \text{and} \quad \tilde{g}(z) = \frac{\omega_2(z)}{1 - D\omega_2(z)}. \tag{2.11}$$

Using these notations we can rewrite (2.10) as

$$\left[1+(A+B)\tilde{f}(z)\right] \star [1+(C+D)\tilde{g}(z)] = \frac{1+X\omega^{\nu}(z)}{1-Y\omega^{\nu}(z)}$$

and so

$$\tilde{f}(z) \star \tilde{g}(z) = \frac{X+Y}{(A+B)(C+D)}\tilde{h}^{\nu}(z), \tag{2.12}$$

where $\tilde{h}^{\nu}(z) = \frac{\omega^{\nu}(z)}{1-BD\omega^{\nu}(z)}$.

Let the functions $\tilde{h}^{\nu}$, $\tilde{f}$ and $\tilde{g}$ have the following expansions in Δ:

$$\tilde{h}^{\nu}(z) = \sum_{n=1}^{\infty} c_n^{\nu} z^n, \quad \tilde{f}(z) = \sum_{n=1}^{\infty} a_n z^n, \quad \tilde{g}(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{2.13}$$

From (2.11) it follows, that $\tilde{f}(z) \prec \frac{z}{1-Bz}$ and $\tilde{g}(z) \prec \frac{z}{1-Dz}$, and hence by Theorem F (and since $0 \le B < 1$ and $0 \le D < 1$) we obtain

$$\sum_{n=1}^{\infty} |a_n|^2 \le \frac{1}{1-B^2} \quad \text{and} \quad \sum_{n=1}^{\infty} |b_n|^2 \le \frac{1}{1-D^2}.$$

Let us note, that

$$\begin{aligned}\sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2\right) &= \sum_{n=1}^{\infty} \left[(|a_n| - |b_n|)^2 + 2|a_n||b_n|\right] \\ &\le \frac{1}{1-|B|^2} + \frac{1}{1-|D|^2}.\end{aligned}$$

From (2.12) and (2.13) we obtain

$$a_n b_n = \frac{X+Y}{(A+B)(C+D)} c_n^\nu, \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

therefore

$$\sum_{n=1}^{\infty} (|a_n| - |b_n|)^2 \leq \frac{1}{1-B^2} + \frac{1}{1-D^2} - \left| \frac{2(X+Y)}{(A+B)(C+D)} \right| \sum_{n=1}^{\infty} |c_n^\nu|. \tag{2.14}$$

Now we observe that

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n^\nu z^n = \tilde{h}^\nu(z) &= \frac{\omega^\nu(z)}{1 - Y\omega^\nu(z)} \\
&= \omega^\nu(z) + Y\left[\omega^\nu(z)\right]^2 + Y^2\left[\omega^\nu(z)\right]^3 + \cdots \\
&= \left(\gamma_1^\nu z + \gamma_2^\nu z^2 + \cdots\right) + Y\left(\gamma_1^\nu z + \gamma_2^\nu z^2 + \cdots\right)^2 \\
&\qquad + Y^2\left(\gamma_1^\nu z + \gamma_2^\nu z^2 + \cdots\right)^3 + \cdots \\
&= \gamma_1^\nu z + \left(\gamma_2^\nu + Y(\gamma_1^\nu)^2\right) z^2 + \left(\gamma_3^\nu + 2Y\gamma_1^\nu\gamma_2^\nu + Y^2(\gamma_1^\nu)^3\right) z^3 + \cdots .
\end{aligned}$$

Since $Y \in [0;1]$ and since (2.7) we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |c_n^\nu| &= |\gamma_1^\nu| + \left|\gamma_2^\nu + Y(\gamma_1^\nu)^2\right| + \left|\gamma_3^\nu + 2Y\gamma_1^\nu\gamma_2^\nu + Y^2(\gamma_1^\nu)^3\right| + \cdots \\
&\geq \sum_{n=1}^{\nu+1} |c_n^\nu| \geq \gamma_1^\nu + \gamma_2^\nu + \gamma_3^\nu + \ldots + \gamma_{\nu+1}^\nu = s_{\nu+1}^\nu(1).
\end{aligned}$$

From the above we have for all $\nu \in \{1, 2, 3, \dots\}$

$$\sum_{n=1}^{\infty} |c_n^\nu| \geq s_{\nu+1}^\nu(1). \tag{2.15}$$

Combining (2.14) and (2.15) we obtain

$$\sum_{n=1}^{\infty} (|a_n| - |b_n|)^2 \leq \frac{1}{1-B^2} + \frac{1}{1-D^2} - \left| \frac{2(X+Y)}{(A+B)(C+D)} \right| s_{\nu+1}^\nu(1). \tag{2.16}$$

From (2.8) it follows that we are able to choose a suitable ν such that the right side of (2.16) is negative. In this way (2.16) follows the contradiction and the proof is complete. □

From Theorem 2.1 and Theorem D we immediately have the following

Corollary 2.2. *The classes* $P(A, B) \star P(C, D)$ *and* $P(AD + AC + BC, BD)$ *are equal if and only if* $|B| = 1$ *or* $|D| = 1$.

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