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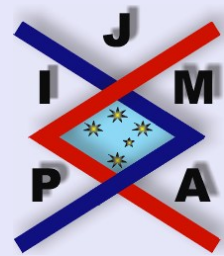
A NOTE ON OSTROWSKI LIKE INEQUALITIES

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Abstract

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Abstract

The aim of this note is to establish new Ostrowski like inequalities by using a fairly elementary analysis.

2000 Mathematics Subject Classification: 26D10, 26D15.

Key words: Ostrowski like inequalities, Estimates, Grüss type inequality, Čebyšev inequality.

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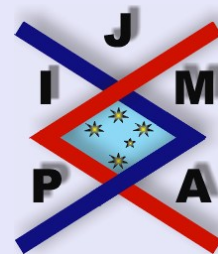
1. Introduction

In an elegant note [5], A.M. Ostrowski proved the following interesting and useful inequality (see also [3, p. 468]):

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty,$$

for all $x \in [a, b]$, where $f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) , whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty = \sup_{x \in (a, b)} |f'(x)| < \infty$.

In the last few years, the study of such inequalities has been the focus of great attention to many researchers and a number of papers have appeared which deal with various generalizations, extensions and variants, see [2, 3, 6] and the references given therein. Inspired and motivated by the recent work going on related to the inequality (1.1), in the present note, we establish new inequalities of the type (1.1) involving two functions and their derivatives. An interesting feature of our results is that they are presented in an elementary way and provide new estimates on these types of inequalities.



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2. Main Results

Our main result is given in the following theorem.

Theorem 2.1. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous functions on $[a, b]$ and differentiable on (a, b) , whose derivatives $f', g' : (a, b) \rightarrow \mathbb{R}$ are bounded on (a, b) , i.e., $\|f'\|_\infty = \sup_{x \in (a,b)} |f'(x)| < \infty$, $\|g'\|_\infty = \sup_{x \in (a,b)} |g'(x)| < \infty$. Then

$$(2.1) \quad \left| f(x)g(x) - \frac{1}{2(b-a)} \left[g(x) \int_a^b f(y) dy + f(x) \int_a^b g(y) dy \right] \right| \\ \leq \frac{1}{2} \{ |g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty \} \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a),$$

for all $x \in [a, b]$.

Proof. For any $x, y \in [a, b]$ we have the following identities:

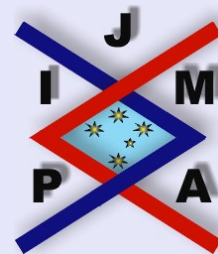
$$(2.2) \quad f(x) - f(y) = \int_y^x f'(t) dt$$

and

$$(2.3) \quad g(x) - g(y) = \int_y^x g'(t) dt.$$

Multiplying both sides of (2.2) and (2.3) by $g(x)$ and $f(x)$ respectively and adding we get

$$(2.4) \quad 2f(x)g(x) - [g(x)f(y) + f(x)g(y)]$$



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$$= g(x) \int_y^x f'(t) dt + f(x) \int_y^x g'(t) dt.$$

Integrating both sides of (2.4) with respect to y over $[a, b]$ and rewriting we have

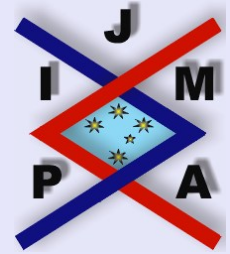
$$(2.5) \quad f(x)g(x) - \frac{1}{2(b-a)} \left[g(x) \int_a^b f(y) dy + f(x) \int_a^b g(y) dy \right] \\ = \frac{1}{2(b-a)} \int_a^b \left\{ g(x) \int_y^x f'(t) dt + f(x) \int_y^x g'(t) dt \right\} dy.$$

From (2.5) and using the properties of modulus we have

$$\left| f(x)g(x) - \frac{1}{2(b-a)} \left[g(x) \int_a^b f(y) dy + f(x) \int_a^b g(y) dy \right] \right| \\ \leq \frac{1}{2(b-a)} \int_a^b \{ |g(x)| \|f'\|_\infty |x-y| + |f(x)| \|g'\|_\infty |x-y| \} dy \\ = \frac{1}{2(b-a)} \{ |g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty \} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right] \\ = \frac{1}{2} \{ |g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty \} \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a).$$

The proof is complete. \square

Remark 1. We note that, by taking $g(x) = 1$ and hence $g'(x) = 0$ in Theorem 2.1, we recapture the well known Ostrowski's inequality in (1.1).



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Integrating both sides of (2.5) with respect to x over $[a, b]$, rewriting the resulting identity and using the properties of modulus, we obtain the following Grüss type inequality:

$$(2.6) \quad \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \leq \frac{1}{2(b-a)^2} \int_a^b \left[\int_a^b \{ |g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty \} |x-y| dy \right] dx.$$

For other inequalities of the type (2.6), see the book [3], where many other references are given.

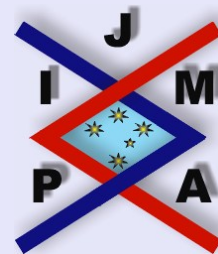
A slight variant of Theorem 2.1 is embodied in the following theorem.

Theorem 2.2. *Let f, g, f', g' be as in Theorem 2.1. Then*

$$(2.7) \quad \left| f(x) g(x) - \frac{1}{b-a} \left[g(x) \int_a^b f(y) dy + f(x) \int_a^b g(y) dy \right] + \frac{1}{b-a} \int_a^b f(y) g(y) dy \right| \leq \frac{1}{b-a} \|f'\|_\infty \|g'\|_\infty \left[\frac{(x-a)^3 + (b-x)^3}{3} \right],$$

for all $x \in [a, b]$.

Proof. From the hypotheses, the identities (2.2) and (2.3) hold. Multiplying the



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left and right sides of (2.2) and (2.3) we get

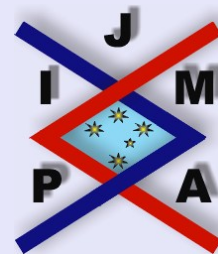
$$(2.8) \quad f(x)g(x) - [g(x)f(y) + f(x)g(y)] + f(y)g(y) \\ = \left\{ \int_y^x f'(t) dt \right\} \left\{ \int_y^x g'(t) dt \right\}.$$

Integrating both sides of (2.8) with respect to y over $[a, b]$ and rewriting we have

$$(2.9) \quad f(x)g(x) - \frac{1}{b-a} \left[g(x) \int_a^b f(y) dy \right. \\ \left. + f(x) \int_a^b g(y) dy \right] + \frac{1}{b-a} \int_a^b f(y)g(y) dy \\ = \frac{1}{b-a} \int_a^b \left\{ \int_y^x f'(t) dt \right\} \left\{ \int_y^x g'(t) dt \right\} dy.$$

From (2.9) and using the properties of modulus we obtain

$$\left| f(x)g(x) - \frac{1}{b-a} \left[g(x) \int_a^b f(y) dy \right. \right. \\ \left. \left. + f(x) \int_a^b g(y) dy \right] + \frac{1}{b-a} \int_a^b f(y)g(y) dy \right| \\ \leq \frac{1}{b-a} \|f'\|_\infty \|g'\|_\infty \int_a^b |x-y|^2 dy \\ = \frac{1}{b-a} \|f'\|_\infty \|g'\|_\infty \left[\frac{(x-a)^3 + (b-x)^3}{3} \right].$$



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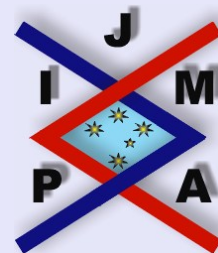
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The proof is complete. \square

Remark 2. Integrating both sides of (2.9) with respect to x over $[a, b]$, rewriting the resulting identity, using the properties of modulus and by elementary calculations we get

$$(2.10) \quad \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \leq \frac{1}{12} (b-a)^2 \|f'\|_{\infty} \|g'\|_{\infty}.$$

Here, it is to be noted that the inequality (2.10) is the well known Čebyšev inequality (see [4, p. 297]).



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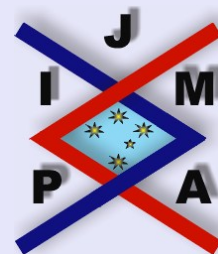
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