Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 1, Article 5, 2005

TWO ITERATIVE SCHEMES FOR AN H-SYSTEM

PABLO AMSTER AND MARÍA CRISTINA MARIANI

FCEYN - DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDAD DE BUENOS AIRES
CIUDAD UNIVERSITARIA, PABELLÓN I
(1428) BUENOS AIRES, ARGENTINA
CONICET
pamster@dm.uba.ar

DEPARTMENT OF MATHEMATICAL SCIENCES
NEW MEXICO STATE UNIVERSITY LAS CRUCES
NM 88003-0001, USA
mmariani@nmsu.edu

Received 1 November, 2004; accepted 15 December, 2004
Communicated by L. Debnath

ABSTRACT. Two iterative schemes for the solution of an H-system with Dirichlet boundary data for a revolution surface are studied: a Newton imbedding type procedure, which yields the local quadratic convergence of the iteration and a more simple scheme based on the method of upper and lower solutions.

Key words and phrases: H-systems, Newton Imbedding, Upper and Lower solutions, Iterative methods.

2000 *Mathematics Subject Classification.* 34B15, 35J25.

1. INTRODUCTION

The prescribed mean curvature equation with Dirichlet condition for a vector function $X : \overline{\Omega} \longrightarrow \mathbb{R}^3$ is given by the following nonlinear system of partial differential equations:

$$\begin{cases} \triangle X = 2H(X)X_u \wedge X_v & \text{in} \quad \Omega \\ X = X_0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, $\wedge$ denotes the exterior product in $\mathbb{R}^3$, $H : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a given continuous function and X_0 is the boundary data.

The parametric Plateau and Dirichlet problems have been studied by different authors (see [3, 4], [7] – [9]). Nonparametric and more general quasilinear equations are considered in [1, 2, 6].

ISSN (electronic): 1443-5756

243-04

We shall consider the particular case of a revolution surface

$$X(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

with $f, g \in C^2(\overline{I})$ such that $f > 0$ and $g' > 0$ over the interval $I \subset \mathbb{R}$. Without loss of generality we may assume that $I = (0, L)$, and problem (1.1) becomes

$$\begin{cases} f'' - f = -2H(f,g)fg' & \text{in } I \\ g'' = 2H(f,g)ff' & \text{in } I \\ f(0) = \alpha_0 & f(L) = \alpha_L \\ g(0) = \beta_0 & g(L) = \beta_L \end{cases} \tag{1.2}$$

where $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a given continuous function, and $\alpha_0, \alpha_L > 0$, $\beta_0 < \beta_L$ are fixed real numbers.

It is easy to see that any solution of (1.2) verifies the equality

$$(f')^2 + (g')^2 = f^2 + c.$$

Hence, the *isothermal* condition

$$|X_u| - |X_v| = X_u X_v = 0$$

holds if and only if $c = 0$. In this case, H is the mean curvature of the surface parameterized by X (see [8]).

We shall study problem (1.2) for a surface with connected boundary, namely

$$\begin{cases} f'' - f = -2H(f,g)fg' & \text{in } I \\ g'' = 2H(f,g)ff' & \text{in } I \\ f(L) = \alpha_L, & g(L) = \beta_L \\ f(0) = g'(0) = 0. \end{cases} \tag{1.3}$$

In particular, if H depends only on the radius f, from the equality

$$g'' = 2H(f)ff', \qquad g'(0) = 0, \quad g(L) = \beta_L$$

problem (1.3) easily reduces to a single equation; indeed, if $\widetilde{H}(t) = \int_0^t sH(s)ds$, the following integral formula holds for g:

$$g(t) = \beta_L - 2\int_t^L \widetilde{H}(f(s))ds.$$

Thus, problem (1.3) is equivalent to the equation

$$\begin{cases} f'' - f = -4H(f)f\widetilde{H}(f) = -2\left(\widetilde{H}^2\right)'(f) & \text{in } I \\ f = \alpha & \text{on } \partial I \end{cases} \tag{1.4}$$

with $\alpha(t) := \frac{\alpha_L}{L}t$. We remark that if $\alpha_L = 0$ then $g'(L) = 2\widetilde{H}(0) = 0$, which corresponds to a surface without boundary.

The paper is organized as follows. In Section 2, a Newton imbedding type procedure for problem (1.4) is considered, which yields the local quadratic convergence of the iteration. In Section 3 we construct a more simple convergent scheme based on the existence of an ordered pair of a lower and an upper solution.

2. A NEWTON IMBEDDING TYPE PROCEDURE

Throughout this section we shall assume the following condition:

$$\left(\widetilde{H}^2\right)''(x) \le \frac{1+\frac{\pi}{L}\left(\frac{\pi}{L}-\delta_0\right)}{2} \qquad \text{for } -k \le x \le k+\alpha_L, \tag{2.1}$$

where $\delta_0 < \frac{\pi}{L}$ and k satisfies:

$$k\delta_0 > L^{1/2}\left\|2\left(\widetilde{H}^2\right)'(\alpha)-\alpha\right\|_{L^2}.$$

Remark 2.1. A straightforward application of Leray-Schauder degree theory proves that if condition (2.1) holds then there exists at least one solution of (1.4), which is unique in the set

$$\mathcal{K} = \{u \in H^2(I) : -k \le u \le k+\alpha_L\}.$$

Remark 2.2. As $\left(\widetilde{H}^2\right)''(0)=0$ and $\left\|2\left(\widetilde{H}^2\right)'(\alpha)-\alpha\right\|_{L^2} \to 0$ for $\alpha_L \to 0$, we deduce that for any H there exists a positive number α^* such that (2.1) holds when $\alpha_L < \alpha^*$.

In order to solve equation (1.4) in an iterative manner, we shall embed it in a family of problems

$$\begin{cases} f'' + \lambda\left[2\left(\widetilde{H}^2\right)'(f)-f\right] = 0 \\ f(0)=0, \quad f(L)=\alpha_L. \end{cases} \tag{1.4_λ}$$

A simple computation shows that if S_λ is the semilinear operator given by

$$S_\lambda(f) = f'' + \lambda\left[2\left(\widetilde{H}^2\right)'(f)-f\right],$$

then the following estimate holds for any $f, g \in \mathcal{K}$ such that $f = g$ on ∂I:

$$\|f'-g'\|_{L^2} \le \frac{1}{\delta_0}\|S_\lambda(f)-S_\lambda(g)\|_{L^2}. \tag{2.2}$$

Hence, if f^λ is the (unique) solution of (1.4_λ) in $\mathcal{K}$, we have that

$$\|(f^\lambda-\alpha)'\|_{L^2} \le \frac{1}{\delta_0}\|S_\lambda(\alpha)\|_{L^2} \le \frac{1}{\delta_0}\left\|2\left(\widetilde{H}^2\right)'(\alpha)-\alpha\right\|_{L^2}.$$

Thus, setting $k_0 = \frac{L^{1/2}}{\delta_0}\|2\left(\widetilde{H}^2\right)'(\alpha)-\alpha\|_{L^2}$ we obtain:

$$-k_0 \le f^\lambda \le k_0 + \alpha_L.$$

We first present a sketch of the method: given $\lambda < 1$, and assuming that $f^\lambda \in \mathcal{K}$ is known, we shall prove the existence of a positive ε and a recursive sequence $\{f_n\}$ which converges quadratically to the unique solution of ($1.4_{\lambda+\varepsilon}$) in $\mathcal{K}$. As ε can be chosen independently of f^λ and λ, starting at $f^0 = \alpha$, we deduce the existence of a sequence

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_N = 1,$$

where $f^{\lambda_k} \in \mathcal{K}$ is obtained iteratively from $f^{\lambda_{k-1}}$.

Let $\lambda < 1$, and $f^\lambda \in \mathcal{K}$ be a solution of (1.4_λ). Define the constants:

$$R = \frac{k-k_0}{L^{1/2}},$$

$$k_1 = \left\|f^\lambda - 2\left(\widetilde{H}^2\right)'(f^\lambda)\right\|_2,$$

$$k_2 = \sup_{-k\le\xi\le k+\alpha_L} \left| \left(\widetilde{H}^2\right)'''(\xi) \right|$$

and define a sequence $\{f_n\}$ in the following way:

- $f_0 := f^\lambda$.
- f_{n+1} the unique element of $H^2(I)$ that solves the linear problem

$$\begin{cases} f''_{n+1} = (\lambda+\varepsilon)\left[\left(1-2\left(\widetilde{H}^2\right)''(f_n)\right)(f_{n+1}-f_n) + f_n - 2\left(\widetilde{H}^2\right)'(f_n)\right] \\ f_{n+1}(0)=0, \quad f_{n+1}(L)=\alpha_L. \end{cases}$$

We shall prove that if ε is small enough, then $\{f_n\}$ is well defined. More precisely:

Theorem 2.3. *Assume that* (2.1) *holds, and that H is twice continuously differentiable on $[-k, k+\alpha_L]$. Then $\{f_n\}$ is well defined and converges quadratically for the H^1-norm to the unique solution of ($1.4_{\lambda+\varepsilon}$) in $\mathcal{K}$ for any $\varepsilon \le 1-\lambda$ satisfying:*

$$\varepsilon\left(1+\frac{RL^{3/2}k_2}{\pi\delta_0}\right) < \frac{\delta_0 R}{k_1}.$$

Proof. As $f_0\in\mathcal{K}$, f_1 is well defined, and by an estimate analogous to (2.2) we obtain:

$$\begin{aligned}\|f'_1-f'_0\|_{L^2} &\le \frac{1}{\delta_0}\left\|(f_1-f_0)''-(\lambda+\varepsilon)\left(1-2\left(\widetilde{H}^2\right)''(f_0)\right)(f_1-f_0)\right\|_{L^2} \\ &= \frac{\varepsilon}{\delta_0}\left\|f_0-2\left(\widetilde{H}^2\right)'(f_0)\right\|_{L^2} \\ &= \frac{\varepsilon}{\delta_0}k_1 \le R.\end{aligned}$$

We shall assume as inductive hypothesis that f_k is well defined for $k=1,\dots,n$, and that $\|f'_k-f'_0\|_{L^2}\le R$. Thus, $f_n\in\mathcal{K}$ and f_{n+1} is well defined. Moreover, for $k=1,\dots,n$ we have that

$$\begin{aligned}(f_{k+1}-f_k)''-(\lambda+\varepsilon)\left(1-2\left(\widetilde{H}^2\right)''(f_k)\right)(f_{k+1}-f_k) \\ = -(\lambda+\varepsilon)\left(\widetilde{H}^2\right)'''(\xi_k)(f_k-f_{k-1})^2\end{aligned}$$

for some mean value $\xi_k(x)$, and hence

$$\|(f_{k+1}-f_k)'\|_{L^2} \le \frac{k_2}{\delta_0}\left\|(f_k-f_{k-1})^2\right\|_{L^2} \le \frac{k_2L^{3/2}}{\delta_0\pi}\|(f_k-f_{k-1})'\|^2_{L^2}.$$

By induction,

$$\|(f_{k+1}-f_k)'\|_{L^2} \le \left(\frac{k_2L^{3/2}}{\delta_0\pi}\|(f_1-f_0)'\|_{L^2}\right)^{2^k-1}\|(f_1-f_0)'\|_{L^2} \le A^{2^k-1}\|(f_1-f_0)'\|_{L^2},$$

where

$$A = \frac{\varepsilon k_1k_2L^{3/2}}{\delta_0^2\pi} < 1.$$

Hence,

$$\|(f_{n+1}-f_0)'\|_{L^2} \le \sum_{k=0}^{n}\|(f_{k+1}-f_k)'\|_{L^2} \le \frac{1}{1-A}\|(f_1-f_0)'\|_{L^2} \le \frac{\varepsilon k_1}{\delta_0(1-A)}.$$

By hypothesis, we conclude that $\|(f_{n+1}-f_0)'\|_{L^2} \le R$. Thus, f_n is well defined for every n, and the inequality

$$\|(f_{n+1}-f_n)'\|_{L^2} \le A^{2^n-1}\|(f_1-f_0)'\|_{L^2}$$

holds, proving that $\{f_n\}$ is a Cauchy sequence in $H^1(I)$. Furthermore, if $f = \lim_{n\to\infty} f_n$, it is immediate that $f_n \to f$ in $H^2(I)$, and $f \in \mathcal{K}$ solves $(1.4_{\lambda+\varepsilon})$. $\square$

Remark 2.4. A uniform choice of ε can be obtained if we set

$$k_1 = L^{1/2} \sup_{-k_0 \le x \le k_0+\alpha_L} \left| x - 2\left(\widetilde{H}^2\right)'(x) \right|.$$

3. UPPER AND LOWER SOLUTIONS FOR PROBLEM (1.4)

In this section we define a convergent sequence based on the existence of an upper solution of the problem: namely, a nonnegative function β such that

$$\beta'' - \beta \le -2\left(\widetilde{H}^2\right)'(\beta), \qquad \beta(L) \ge \alpha_L. \tag{3.1}$$

We remark that it suffices to consider this assumption, since 0 is a lower solution of (1.4).

Theorem 3.1. *Assume that $\beta \ge 0$ satisfies* (3.1) *and that H is continuously differentiable for $0 \le x \le \|\beta\|_\infty$. Set*

$$C = 1 - 2 \min_{0 \le x \le \|\beta\|_\infty} \left(\widetilde{H}^2\right)''(x)$$

and define the sequences $\{f_n^\pm\}$ given by:

- $f_0^- \equiv 0 \qquad f_0^+ = \beta$
- $\{f_{n+1}^\pm\}$ *the unique solution of the linear problem*

$$\begin{cases} (f_{n+1}^\pm)'' - Cf_{n+1}^\pm = (1-C)f_n^\pm - 2\left(\widetilde{H}^2\right)'(f_n^\pm) \\ f_{n+1}^\pm = \alpha \quad \textit{on} \quad \partial I. \end{cases}$$

Then $\{f_n^-\}$ (respectively $\{f_n^+\}$) is nondecreasing (nonincreasing), and converges pointwise to a solution of (1.2). Moreover, the respective limits $f^\pm$ satisfy: $0 \le f^- \le f^+ \le \beta$.

Proof. Let us first note that $C \ge 0$ (in fact, $C \ge 1$), which implies that both sequences are well defined. Furthermore, from the choice of C, it is immediate that the function $\psi(x) = (1-C)x - 2\left(\widetilde{H}^2\right)'(x)$ is nonincreasing for $0 \le x \le \|\beta\|_\infty$. By definition,

$$(f_1^+)'' - Cf_1^+ = (1-C)\beta - 2\left(\widetilde{H}^2\right)'(\beta) \ge \beta'' - C\beta$$

and using the maximum principle it follows that $f_1^+ \le \beta$. On the other hand,

$$(f_1^+)'' - Cf_1^+ = \psi(\beta) \le \psi(0) = 0$$

and as $f_1^+ \ge 0$ on ∂I we deduce that $f_1^+ \ge 0$ over I. Assume as inductive hypothesis that

$$0 \le f_n^+ \le f_{n-1}^+.$$

Then

$$(f_{n+1}^+)'' - Cf_{n+1}^+ = \psi(f_n^+) \ge \psi(f_{n-1}^+) = (f_n^+)'' - Cf_n^+$$

and

$$(f_{n+1}^+)'' - Cf_{n+1}^+ = \psi(f_n^+) \le \psi(0) = 0$$

which implies that $0 \le f_{n+1}^+ \le f_n^+$. Thus, $\{f_n^+\}$ is nonincreasing and converges pointwise to a function $f^+ \ge 0$. By standard apriori estimates, we have that

$$\|f_{n+1}^+ - \alpha\|_{H^2} \le c\|(f_{n+1}^+ - \alpha)'' - C(f_{n+1}^+ - \alpha)\|_{L^2} = c\|\psi(f_n^+) - \alpha\|_{L^2} \le M$$

for some constant M. By compactness, $\{f_n^+\}$ admits a convergent subsequence in $C^1(\overline{I})$, proving that $f^+ \in C^1(\overline{I})$. Furthermore,

$$(f_{n+1}^+)'(x) - (f_{n+1}^+)'(0) = \int_0^x Cf_{n+1}^+ + \psi(f_n^+),$$

and by dominated convergence we conclude that

$$(f^+)'(x) - (f^+)'(0) = \int_0^x Cf^+ + \psi(f^+) = \int_0^x f^+ - 2\left(\widetilde{H}^2\right)'(f^+).$$

Thus, the result follows. The proof is analogous for $\{f_n^-\}$. □

As a simple consequence we have:

Corollary 3.2. *Assume there exists a number $k \ge \alpha_L$ such that*

$$\widetilde{H}(k)H(k) \le \frac{1}{4}.$$

Then $\beta \equiv k$ is an upper solution, and the schemes defined in the previous theorem converge.

Example 3.1. For $H(x) = rx^n$, we have that $\widetilde{H}(x) = \frac{r}{n+2}x^{n+2}$, and the conditions of the previous corollary hold for

$$\alpha_L \le k \le \left(\frac{n+2}{4r^2}\right)^{\frac{1}{2n+2}}.$$

However, it is possible to find a sharper bound for α_L, if we consider the parabola

$$\beta(x) = \alpha_L\left[1 - \left(\frac{x-L}{L}\right)^2\right].$$

Indeed, in this case we have that β is an upper solution if and only if

$$-\frac{2\alpha_L}{L^2} - \beta \le -\frac{4r^2}{n+2}\beta^{2n+3}$$

or equivalently

$$\phi(\beta) \le \frac{2\alpha_L}{L^2}$$

for $\phi(x) = x\left(\frac{4r^2}{n+2}x^{2n+2} - 1\right)$. Note that for $0 \le x \le \alpha_L$ we have:

$$\phi(x) \le \max\{0, \phi(\alpha_L)\}.$$

Thus, it suffices to assume that

$$0 < \alpha_L^{2n+2} \le \frac{n+2}{4r^2}\left(1 + \frac{2}{L^2}\right).$$

Remark 3.3. In the previous example, equation (1.4) is *superlinear*, namely:

$$f'' + \frac{4r^2}{n+2}f^{2n+3} - f = 0, \qquad f(0) = 0, \quad f(L) = \alpha_L.$$

It can be proved (see e.g. [5]) that this problem admits infinitely many solutions. More precisely, there exists $k_0 \in \mathbb{N}$ such that for any $j > k_0$ the problem has at least two solutions crossing the line $\alpha(t) = \frac{\alpha_L}{L}t$ exactly j times in $(0, L)$.

REFERENCES

[1] P. AMSTER, M.M. CASSINELLI AND M.C. MARIANI, Solutions to general quasilinear elliptic second order problems, *Nonlinear Studies*, **7**(2) (2000), 283–289.

[2] P. AMSTER, M.M. CASSINELLI AND M.C. MARIANI, Solutions to quasilinear equations by an iterative method, *Bulletin of the Belgian Math. Society*, **7** (2000), 435–441.

[3] P. AMSTER, M.M. CASSINELLI AND D.F RIAL, Existence and uniqueness of H-System's solutions with Dirichlet conditions, *Nonlinear Analysis, Theory, Methods, and Applications*, **42**(4) (2000), 673–677.

[4] H. BREZIS AND J.M. CORON, Multiple solutions of H systems and Rellich's conjecture, *Comm. Pure Appl. Math.*, **37** (1984), 149–187.

[5] A. CAPPIETO, J. MAWHIN AND F. ZANOLIN, Boundary value problems for forced superlinear second order ordinary differential equations, Séminaire du Collége de France.

[6] D. GILBARG AND N.S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag (1983).

[7] S. HILDEBRANDT, On the Plateau problem for surfaces of constant mean curvature, *Comm. Pure Appl. Math.*, **23** (1970), 97–114.

[8] M. STRUWE, *Plateau's Problem and the Calculus of Variations*, Lecture Notes, Princeton Univ. Press (1988).

[9] GUOFANG WANG, The Dirichlet problem for the equation of prescribed mean curvature, *Analyse Nonlinéaire*, **9** (1992), 643–655.