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AN EXAMPLE OF A STABLE FUNCTIONAL EQUATION WHEN THE HYERS METHOD DOES NOT WORK

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ABSTRACT. We show that the functional equation

$$g\left(\frac{x+y}{2}\right) = \sqrt[4]{g(x)g(y)}$$

is stable in the classical sense on arbitrary $\mathbb{Q}$-algebraically open convex sets, but the Hyers method does not work.

For the convenience of the reader, we have included an extensive list of references where stability theorems for functional equations were obtained using the direct method of Hyers.

Key words and phrases: Cauchy's functional equation, Stability, Hyers iteration.

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1. INTRODUCTION

The basic problem of the stability of functional equations asks whether an 'approximate solution' of the Cauchy functional equation $g(x+y) = g(x) + g(y)$ 'can be approximated' by a solution of this equation. This problem was formulated (and also solved) in Gy. Pólya and G. Szegő's book [60] (Teil I, Aufgabe 99) for functions defined on the set of positive integers, it was reformulated in a more general form by S. Ulam in 1940 (see [86], [87]). In 1941, D. H. Hyers [40] gave the following solution to this problem: *If X and Y are Banach spaces, ε is a nonnegative real number and a function $f : X \to Y$ fulfills the inequality $||f(x+y) - f(x) - f(y)|| \le \varepsilon$ $(x, y \in X)$, then there exists a unique solution $g : X \to Y$ of the Cauchy equation for which $||f(x) - g(x)|| \le \varepsilon$ $(x \in X)$.* Stability problems of this type were investigated by several authors during the last decades, most of them used the idea of Hyers,

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which will be described below. For surveys on these developments see, e.g., the papers by Forti [16], Ger [30], Székelyhidi [85] and the book [43].

Let X and Y be non-empty sets and let $*, \diamond$ be binary operations on X and Y, respectively. The Cauchy equation concerning these general structures is the functional equation

$$g(x * y) = g(x) \diamond g(y) \qquad (x, y \in X), \tag{1.1}$$

where $g : X \to Y$ is considered as an unknown function.

Assuming, in addition, that Y is a metric space with metric d, we can speak about approximate solutions of (1.1): A function $f : X \to Y$ is called an *ε-approximate solution of* (1.1) if it satisfies the following so-called stability inequality

$$d\big(f(x * y), f(x) \diamond f(y)\big) \le \varepsilon \qquad (x, y \in X) \tag{1.2}$$

for some $\varepsilon \ge 0$.

The Cauchy equation (1.1) is said to be *stable in the sense of Hyers and Ulam* if, for all positive δ, there exists $\varepsilon > 0$ such that, for an arbitrary solution f of (1.2), there exists a solution g of (1.1) satisfying $d(f(x), g(x)) \le \delta$ for all $x \in X$.

The most general results concerning this stability problem were obtained in the context of square-symmetric groupoids, i.e., when the operations $*$ and $\diamond$ satisfy the algebraic identities

$$(x * y) * (x * y) = (x * x) * (y * y) \qquad \text{and} \qquad (u \diamond v) \diamond (u \diamond v) = (u \diamond u) \diamond (v \diamond v)$$

for all x, y in X and u, v in Y. (Cf. [78], [15], [59], [58], [2].)

Let us denote $x * x$ by $\sigma_*(x)$ $(x \in X)$, and $u \diamond u$ by $\sigma_\diamond(u)$ $(u \in Y)$ (i.e., σ_* and $\sigma_\diamond$ stand for the squaring in the corresponding structures). The square-symmetry of the operations $*$ and $\diamond$ simply means that σ_* and $\sigma_\diamond$ are endomorphisms. Substituting $x = y$ into (1.1) and (1.2), we get the following single-variable functional equation and functional inequality:

$$g \circ \sigma_*(x) = \sigma_\diamond \circ g(x) \qquad (x \in X), \tag{1.3}$$

and

$$d\big(f \circ \sigma_*(x), \sigma_\diamond \circ f(x)\big) \le \varepsilon \qquad (x \in X). \tag{1.4}$$

Assuming that a function $f : X \to Y$ satisfies (1.2), we see that it also satisfies (1.4). In order to construct the solution g of (1.1) which is close to f, the idea of the Hyers method is to consider one of the following two iterations:

$$g_1 := f, \qquad g_{n+1} = \sigma_\diamond \circ g_n \circ \sigma_*^{-1} \quad (n \in \mathbb{N}), \tag{1.5}$$

$$g_1 := f, \qquad g_{n+1} = \sigma_\diamond^{-1} \circ g_n \circ \sigma_* \quad (n \in \mathbb{N}). \tag{1.6}$$

(assuming that σ_* and $\sigma_\diamond$ is invertible, respectively) and then to show that for all solutions f of (1.4), one of these sequences of functions converges to a limit function g, which is a solution of (1.3) and of (1.1), moreover, $d(f(x), g(x)) \le c\varepsilon$ for some $c \in \mathbb{R}$.

Results concerning stability of various functional equations in several variables using these kinds of iterations can be found in a huge number of recent works (see the extensive list of references at the end of the paper).

In this note we present an example of a stable Cauchy-type functional equation with square-symmetric operations, where for all solutions f of (1.4), the limit function of the corresponding Hyers-sequences either does not exist or is a solution of the single variable functional equation (1.3) but it does not solve (1.1) and it is not close to the original function f.

2. RESULTS

Let X denote a vector space over the field of rational numbers throughout this paper. In what follows, we deal with the stability of the two-variable functional equation

$$g\Big(\frac{x+y}{2}\Big) = \sqrt[4]{g(x)g(y)} \qquad (x, y \in H), \tag{2.1}$$

where H is a midpoint-convex set of X, i.e., $\frac{x+y}{2} \in H$ for all $x, y \in H$. A function $f : H \to [0, \infty[$ is called an ε-approximate solution of (2.1) if it satisfies the functional inequality

$$\Big|f\Big(\frac{x+y}{2}\Big) - \sqrt[4]{f(x)f(y)}\Big| \le \varepsilon \qquad (x, y \in H). \tag{2.2}$$

Observe that, with the notations

$$x * y := \frac{x+y}{2} \qquad \text{and} \qquad x \diamond y := \sqrt[4]{xy},$$

the operations $*$ and $\diamond$ are square-symmetric (over H and $[0, \infty[$), furthermore, (2.1) and (2.2) are particular cases of (1.1) and (1.2), respectively.

With the substitution $y = x$ one obtains the following single variable functional equation and functional inequality from (2.1) and (2.2):

$$g(x) = \sqrt{g(x)} \qquad (x \in H), \tag{2.3}$$

and

$$\big|f(x) - \sqrt{f(x)}\big| \le \varepsilon \qquad (x \in H), \tag{2.4}$$

respectively.

In this setting, for the iteration (1.6), we get

$$g_n(x) = \big(f(x)\big)^{2^{n-1}} \qquad (x \in X, n \in \mathbb{N}),$$

which is not convergent for those elements $x \in X$ where $f(x) > 1$, otherwise

$$g(x) = \lim_{n\to\infty} g_n(x) = \begin{cases} 0 & \text{if } f(x) < 1, \\ 1 & \text{if } f(x) = 1. \end{cases}$$

Clearly, g is a solution of (2.3). Assume that H has at least two elements and $0 < \varepsilon \le 1$. Let $x_0 \in H$ be fixed. Define $f_1 : H \to [0, \infty[$ by

$$f_1(x) = \begin{cases} 1 & \text{if } x = x_0, \\ 1+\varepsilon & \text{if } x \ne x_0, \end{cases}$$

and $f_2 : H \to [0, \infty[$ by

$$f_2(x) = \begin{cases} 1 & \text{if } x = x_0, \\ 1-\varepsilon & \text{if } x \ne x_0. \end{cases} \tag{2.5}$$

It is not so difficult to prove, that f_1 and f_2 satisfy inequality (2.2). It is clear, that the corresponding iteration (1.6) referring to f_1 is not convergent when $x \ne x_0$. The iteration (1.6) referring to f_2 converges to

$$g(x) = \begin{cases} 1 & \text{if } x = x_0, \\ 0 & \text{if } x \ne x_0, \end{cases}$$

which is a solution of (2.3), but as we see later, does not necessarily solve (2.1). It is obvious that there does not exist a $c \in \mathbb{R}$, for which $d(f_2(x), g(x)) \le c\varepsilon$ for an arbitrary ε. Moreover, if $\varepsilon \approx 0$, then $d(f_2(x), g(x)) \approx 1$, when $x \ne x_0$.

Similarly, for the iteration (1.5), we get

$$g_n(x) = (f(x))^{\frac{1}{2^{n-1}}} \qquad (x \in X,\, n \in \mathbb{N}),$$

so we have

$$g(x) := \lim_{n\to\infty} g_n(x) = \begin{cases} 0 & \text{if } f(x) = 0, \\ 1 & \text{if } f(x) \neq 0, \end{cases}$$

which is a solution of (2.3). Assume that H has at least two elements, $0 < \varepsilon \leq 1$ and define $f_3 : H \to [0, \infty[$ by

$$f_3(x) = \begin{cases} 0 & \text{if } x = x_0, \\ \varepsilon^2 & \text{if } x \neq x_0, \end{cases} \tag{2.6}$$

where $x_0 \in H$ is fixed. It is obvious, that (2.2) holds for the function f_3, but the Hyers iteration now converges to

$$g(x) = \begin{cases} 0 & \text{if } x = x_0, \\ 1 & \text{if } x \neq x_0, \end{cases}$$

which solves (2.3) but does not necessarily solve (2.1). Again as before, there does not exist a $c \in \mathbb{R}$, for which $d(f_3(x), g(x)) \leq c\varepsilon$ for an arbitrary ε, because if ε is approximately zero, then $d(f_3(x), g(x))$ is approximately 1 when $x \neq x_0$.

In what follows, we prove the stability of the functional equations (2.3) and (2.1). It can be immediately seen that the solutions of (2.3) are functions with values 0 and 1, that is, characteristic functions of a certain subset of H. The next result shows that if f is a solution of (2.4) then it is close to a certain characteristic function as well. Thus, the functional equation (2.3) is stable in the Hyers-Ulam sense.

Theorem 2.1. *Let H be a nonempty set and let $f : H \to [0, \infty[$ be a solution of the functional inequality* (2.4) *with $0 \leq \varepsilon \leq \varepsilon_0 := 4/25$. Then, for all $x \in H$,*

$$\textit{either} \qquad f(x) \leq \frac{25\varepsilon^2}{16} \qquad \textit{or} \qquad -\frac{9\varepsilon}{4} \leq f(x) - 1 \leq 2\varepsilon.$$

Proof. Define the subsets A and B of H by

$$A := \left\{ x \in H : -\frac{9\varepsilon}{4} \leq f(x) - 1 \leq 2\varepsilon \right\}, \qquad B := \left\{ x \in H : f(x) \leq \frac{25\varepsilon^2}{16} \right\}.$$

The proof of the theorem is equivalent to showing that A and B form a partition of H.

Let $x \in H$ be arbitrary. Inequality (2.4) is equivalent to the quadratic inequalities

$$-\varepsilon \leq \left(\sqrt{f(x)}\right)^2 - \sqrt{f(x)} \leq \varepsilon. \tag{2.7}$$

Since $\varepsilon \leq \varepsilon_0 = 4/25$, we have the estimate

$$\frac{1 - \sqrt{1 - 4\varepsilon}}{2} = \frac{4\varepsilon}{2\left(1 + \sqrt{1 - 4\varepsilon}\right)} \leq \frac{2\varepsilon}{1 + \sqrt{1 - 4\varepsilon_0}} = \frac{2\varepsilon}{1 + \sqrt{9/25}} \leq \frac{5}{4}\varepsilon. \tag{2.8}$$

From (2.7), using (2.8), we obtain that either

$$0 \leq f(x) = \left(\sqrt{f(x)}\right)^2 \leq \left(\frac{1 - \sqrt{1 - 4\varepsilon}}{2}\right)^2 \leq \frac{25\varepsilon^2}{16},$$

i.e., $x \in B$, or

$$\frac{1 + \sqrt{1 - 4\varepsilon}}{2} \leq \sqrt{f(x)} \leq \frac{1 + \sqrt{1 + 4\varepsilon}}{2},$$

consequently, in view of the estimate (2.8) and

$$\frac{\sqrt{1+4\varepsilon}-1}{2} = \frac{4\varepsilon}{2\left(\sqrt{1+4\varepsilon}+1\right)} \leq \varepsilon, \tag{2.9}$$

we have

$$\begin{aligned} 1-\frac{9\varepsilon}{4} &\leq 1-\frac{1-\sqrt{1-4\varepsilon}}{2}-\varepsilon = \left(\frac{1+\sqrt{1-4\varepsilon}}{2}\right)^2 \\ &\leq f(x) \\ &\leq \left(\frac{1+\sqrt{1+4\varepsilon}}{2}\right)^2 = 1+\frac{\sqrt{1+4\varepsilon}-1}{2}+\varepsilon \leq 1+2\varepsilon, \end{aligned}$$

which means that $x \in A$.

Thus we have showed that $A \cup B = H$. On the other hand, since $\varepsilon \leq \varepsilon_0 = 4/25$, it easily follows that $A \cap B = \emptyset$. □

In order to investigate the stability of the two-variable functional equation (2.1), we need the notion of an ideal of midpoint-convex sets. We say that a set $I \subset H$ is an *ideal in the midpoint-convex set H with respect to the midpoint operation* if

$$x \in H \text{ and } y \in I \quad \Longrightarrow \quad \frac{x+y}{2} \in I. \tag{2.10}$$

Trivially, $\emptyset$ and H are always ideals in H. However, in general, there could exist further ideals in H. For instance, if H is the closed unit interval $H = [0,1] \subset \mathbb{R}$ then the sets $]0,1[$, $[0,1[$, and $]0,1]$ are also ideals for H. As we shall see below, if H enjoys a certain openness property then it can have only trivial ideals.

We say that a set H is $\mathbb{Q}$-*algebraically open* if, for each point $p \in H$ and vector $v \in X$, there exists a positive number τ such that $p + tv \in H$ for all $t \in [0,\tau] \cap \mathbb{Q}$. It is obvious, that every open set (of a topological linear space) is $\mathbb{Q}$-algebraically open, but the reversed statement is not true in general.

Lemma 2.2. *Let $H \subset X$ be a $\mathbb{Q}$-algebraically open midpoint-convex set. Then H has only trivial ideals, i.e., the only ideals in H are the sets $\emptyset$ and H.*

Proof. Assume that $I \subset H$ is a nonempty ideal with respect to the midpoint operation, and let $y \in I$ be fixed. It easily follows by induction that $\frac{2^n-1}{2^n}x + \frac{1}{2^n}y \in I$ for all $x \in H$.

Now let $x \in H$ be arbitrary. Since H is $\mathbb{Q}$-algebraically open, for large $n \in \mathbb{N}$, we have that

$$x_n = x + \frac{1}{2^n-1}(x-y) \in H.$$

Then

$$\frac{2^n-1}{2^n}x_n + \frac{1}{2^n}y = x,$$

which, in view of the ideal property of I, yields that $x \in I$. Therefore, $H \subset I$ follows. □

Our next result concerns the stability of the functional equation (2.1).

Theorem 2.3. *Let $H \subset X$ be a $\mathbb{Q}$-algebraically open midpoint-convex set and let $f : H \to [0,\infty[$ be a solution of the functional inequality* (2.2) *with $0 \leq \varepsilon \leq \varepsilon_0 := 4/25$. Then, either*

$$f(x) \leq \frac{25\varepsilon^2}{16} \qquad (x \in H)$$

or

$$-\frac{9\varepsilon}{4} \leq f(x) - 1 \leq 2\varepsilon \qquad (x \in H).$$

Proof. Define the sets A and B as in the proof of Theorem 2.1. Then, by Theorem 2.1, these sets form a partition of H. In order to complete the proof, we have to show that one of the sets A or B is empty. To do that, we prove that B is an ideal in H with respect to the midpoint operation.

Let $x \in H$ and $y \in B$. It suffices to prove that $\frac{x+y}{2} \notin A$, because then we have $\frac{x+y}{2} \in H \setminus A = B$. From inequality (2.2) it follows that

$$\begin{aligned}
f\Big(\frac{x+y}{2}\Big) &\le \varepsilon + \sqrt{\sqrt{f(x)}\sqrt{|f(y)|}} \\
&\le \varepsilon + \sqrt{\sqrt{1+2\varepsilon}\sqrt{\frac{25\varepsilon^2}{16}}} \\
&\le \varepsilon_0 + \sqrt{\sqrt{1+2\varepsilon_0}\frac{5\varepsilon_0}{4}} \\
&= \frac{4}{25} + \frac{\sqrt[4]{33}}{5} < \frac{16}{25} = 1 - \frac{9\varepsilon_0}{4} \le 1 - \frac{9\varepsilon}{4}.
\end{aligned}$$

In view of Lemma 2.2, we have that B is a trivial ideal, i.e., either $B = H$ or $B = \emptyset$ which means that $A = H$, and the statement of the theorem follows from this. $\square$

The functions $g \equiv 0$ and $g \equiv 1$ are trivially the solutions of the functional equation (2.1). Choosing $\varepsilon = 0$ in Theorem 2.3, we immediately get that the reversed statement is also true, i.e., we have the following result:

Corollary 2.4. *Let X be a real linear space, $H \subset X$ be a $\mathbb{Q}$-algebraically open midpoint-convex set. Then a function $g : H \to [0,\infty[$ is a solution of the functional equation* (2.1) *if and only if either $g \equiv 0$ or $g \equiv 1$.*

Now Theorem 2.3 can be interpreted as the stability theorem of (2.1) since it states that if f solves the stability inequality (2.2), then it is close to one of the solutions of the functional equation (2.1). Thus, (2.1) is stable in the Hyers-Ulam sense.

On the other hand, Corollary 2.4 shows that if we consider equation (2.1) over a $\mathbb{Q}$-algebraically open midpoint-convex set, then the limits of the corresponding Hyers-sequences referring to the functions f_2 and f_3 defined in (2.5) and (2.6) are not solutions of (2.1), so the stability of this functional equation cannot be proved via the Hyers-method.

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