

ON THE ERDÖS-DEBRUNNER INEQUALITY

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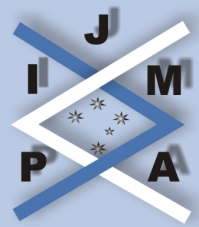
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Abstract: We confirm two recent conjectures of W. Janous and thereby state the best possible form of the Erdős-Debrunner inequality for triangles.

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Erdős-Debrunner Inequality

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Fix a triangle ABC and, on each of the sides BC , CA , AB fix arbitrary interior points D , E , F . Label the areas of the resulting triangles DEF , AEF , BDF , CED as F_0 , F_1 , F_2 , F_3 . F_0 is thus the area of the central triangle, while the other three are the areas of the “corner” triangles. The Erdős-Debrunner inequality states that at least one of the corner triangles has no greater area than the central triangle:

$$(1) \quad \min\{F_1, F_2, F_3\} \leq F_0.$$

Walther Janous [1] generalized (1), proving that

$$(2) \quad M_{-1}(F_1, F_2, F_3) \leq F_0,$$

where $M_{-1}(F_1, F_2, F_3)$ denotes the harmonic mean of the areas F_1, F_2, F_3 (for notation and properties of general power means, see the standard reference [2]). Moreover, Janous [1] also proves that if an inequality of the form

$$(3) \quad M_p(F_1, F_2, F_3) \leq F_0$$

should generally hold (with $p \geq -1$) then we must necessarily have

$$-1 \leq p \leq -\frac{\ln(3/2)}{\ln(2)}.$$

Prompted by these results, Janous formulates the following conjecture

Conjecture 1 (Janous [1]). *The best possible value of p for which (3) generally holds is $p = -\frac{\ln(3/2)}{\ln(2)}$.*

In this note we will confirm this conjecture, and thereby state the best possible form of the Erdős-Debrunner inequality as a theorem:



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Theorem 2. *It is always true that*

$$M_p(F_1, F_2, F_3) \leq F_0$$

with $p = -\frac{\ln(3/2)}{\ln(2)}$, and this value of p is best possible, in the sense that with any greater p there are examples that contradict the inequality.

In [1] Janous develops a useful notation to simplify the Erdős-Debrunner problem, and we will adopt it as our starting point. First, he selects $t, u, v > 0$ so that the sides BC, CA, AB are divided by the points D, E, F in the ratios $t : 1 - t, u : 1 - u, v : 1 - v$. Then, defining

$$x = \frac{t}{1 - u}, \quad y = \frac{u}{1 - v}, \quad z = \frac{v}{1 - t},$$

and setting $q := -p$, Janous shows that the inequality (3) for $p < 0$ is equivalent to

$$(4) \quad f(x, y, z) \geq 3,$$

where f is defined by

$$(5) \quad f(x, y, z) := \left(\frac{1}{z} + x - 1\right)^q + \left(\frac{1}{x} + y - 1\right)^q + \left(\frac{1}{y} + z - 1\right)^q.$$

Here we require that

$$(6) \quad x, y, z > 0, \quad \frac{1}{z} + x - 1 \geq 0 \quad \frac{1}{x} + y - 1 \geq 0 \quad \frac{1}{y} + z - 1 \geq 0.$$

This new x, y, z notation and the related conditions, and the fact that we are only interested in exponents q with $\ln(3/2)/\ln(2) \leq q < 1$, is all we need to know. In reference to the function f , Janous formulates a second “minor” conjecture:



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Conjecture 3 (Janous [1]). Under conditions (6) and for any $q > 0$, the minimum of $f(x, y, z)$ is attained at points satisfying $xyz = 1$.

To prove Theorem 2 we would only need to consider the smallest possible q . However, we will start with a proof of this conjecture for the relevant interval of exponents $\ln(3/2)/\ln(2) \leq q < 1$.

Lemma 4. Under the conditions (6) and if $\ln(3/2)/\ln(2) \leq q < 1$, the function $f(x, y, z)$ can only attain a minimum at (x, y, z) when $xyz = 1$.

Proof. The inequalities in (6) define a region in \mathbb{R}^3 , and we first want to consider points on its boundary. That is, we first assume that one of the last three inequalities is actually an identity; without loss of generality, we assume that

$$\frac{1}{y} + z - 1 = 0.$$

Thus, since $z = (y - 1)/y$ and since $z > 0$, we conclude that $y > 1$. The function f defined in (5) simplifies to

$$g(x, y) := \left(\frac{1}{y-1} + x \right)^q + \left(\frac{1}{x} + y - 1 \right)^q.$$

After the change of variables $s = x^q$, $t = \frac{1}{(y-1)^q}$, $p = 1/q$ this takes the more symmetric form

$$(7) \quad g(s, t) := (s^p + t^p)^{1/p} + \left(\frac{1}{s^p} + \frac{1}{t^p} \right)^{1/p}.$$

Using the definition of general power means, we can rewrite g as

$$g(s, t) = 2^{1/p} \left(M_p(s, t) + \frac{1}{M_{-p}(s, t)} \right).$$



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Thus, estimating both summands within parentheses via the geometric mean $M_0(s, t)$, we get

$$g(s, t) \geq 2^{1/p} \left(M_0(s, t) + \frac{1}{M_0(s, t)} \right) \geq 2^{1+1/p} = 2^{1+q},$$

because of the well-known inequality $a + 1/a \geq 2$. We can now see, working backwards through the previous steps, that the minimum 2^{1+q} can only be attained if $s = t$, which in turn means that $x = 1/(y - 1)$. Therefore,

$$xyz = \frac{1}{y-1} y \frac{y-1}{y} = 1$$

as claimed. Further, we notice that 2^{1+q} is greater than or equal to 3, where equality holds when $q = \ln(3/2)/\ln(2)$.

Next, we will look for the extrema of f under the set of strict conditions

$$(8) \quad x, y, z > 0, \quad \frac{1}{z} + x - 1 > 0 \quad \frac{1}{x} + y - 1 > 0 \quad \frac{1}{y} + z - 1 > 0,$$

which together define an *open* region in \mathbb{R}^3 . The extrema in this region must occur where the gradient of f vanishes. We compute the partial derivative with respect to x , and obtain

$$\frac{\partial f}{\partial x} = q \left(\frac{1}{z} + x - 1 \right)^{q-1} - q \left(\frac{1}{x} + y - 1 \right)^{q-1} \frac{1}{x^2}.$$

The condition $\frac{\partial f}{\partial x} = 0$ can be rewritten as (remembering that $\ln(3/2)/\ln(2) \leq q < 1$)

$$(9) \quad \left(\frac{1}{x} + y - 1 \right)^q = \left(\frac{1}{z} + x - 1 \right)^q \frac{1}{x^{2q/(1-q)}}.$$



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By permuting the variables x, y, z cyclically, we obtain from (9) the corresponding equations equivalent to $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 0$, that is,

$$(10) \quad \left(\frac{1}{y} + z - 1\right)^q = \left(\frac{1}{x} + y - 1\right)^q \frac{1}{y^{2q/(1-q)}}$$

and

$$(11) \quad \left(\frac{1}{z} + x - 1\right)^q = \left(\frac{1}{y} + z - 1\right)^q \frac{1}{z^{2q/(1-q)}}.$$

It should be now clear that the product of the three equations (9), (10), (11) implies $xyz = 1$ in this case, too. The lemma is thus proved. \square

Proof of Theorem 2. Using Lemma 4, finding the minimum of f becomes a two-variable problem after setting $z = 1/xy$. Accordingly, we consider a new function

$$h(x, y) := (xy + x - 1)^q + \left(\frac{1}{x} + y - 1\right)^q + \left(\frac{1}{y} + \frac{1}{xy} - 1\right)^q,$$

and henceforth we will also fix q to be $\ln(3/2)/\ln(2)$, recalling Janous' proof that the inequality is invalid for $q < \ln(3/2)/\ln(2)$. Our ultimate target is to show that with $q = \ln(3/2)/\ln(2)$ and under conditions (6) the minimum of h is 3 (see (4) and replace z with $1/xy$ in (6)).

Now, if any of the last three inequalities in (6) is an identity, the proof of Lemma 4 already shows that the minimum of h is 2^{q+1} , and this number is identical to 3 given the choice $q = \ln(3/2)/\ln(2)$. We thus want to examine possible extrema of h under the more restrictive conditions

$$(12) \quad x, y > 0, \quad xy + x - 1 > 0 \quad \frac{1}{x} + y - 1 > 0 \quad \frac{1}{y} + \frac{1}{xy} - 1 > 0$$



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which result from (8) after replacing z with $1/xy$. Rewriting (12) as

$$(13) \quad x, y > 0, \quad y + 1 > \frac{1}{x} \quad \frac{1}{x} + y > 1 \quad 1 + \frac{1}{x} > y$$

it follows that $1/x$, y and 1 must be the lengths of the three sides of a triangle. After the change of variables $s = 1/x$, $t = y$, h can be written as

$$(14) \quad h(s, t) = \left(\frac{1+t-s}{s} \right)^q + (s+t-1)^q + \left(\frac{1+s-t}{t} \right)^q,$$

where the quantities $s, t, 1$ are the sides of a (non-degenerate) triangle.

If we now look at

$$(15) \quad H(a, b, c) := \left(\frac{b+c-a}{a} \right)^q + \left(\frac{c+a-b}{b} \right)^q + \left(\frac{a+b-c}{c} \right)^q,$$

where a, b, c are the sides of a triangle, and realize that the function H is invariant under a common scaling of a, b, c , we see that the problem of minimizing $h(s, t)$ in (14) is equivalent to minimizing $H(a, b, c)$ in (15). Let us now use elementary trigonometric relations to rewrite H as a function of the angles α, β, γ (defined as the angles opposite the sides of length a, b, c). The result is

$$H(\alpha, \beta, \gamma) = 2^q \left[\left(\frac{\sin(\beta/2) \sin(\gamma/2)}{\sin(\alpha/2)} \right)^q + \left(\frac{\sin(\gamma/2) \sin(\alpha/2)}{\sin(\beta/2)} \right)^q + \left(\frac{\sin(\alpha/2) \sin(\beta/2)}{\sin(\gamma/2)} \right)^q \right].$$

Since we are dealing with (positive) angles satisfying $\alpha + \beta + \gamma = \pi$, we have $\sin(\gamma/2) = \cos((\alpha + \beta)/2)$, and so a further dose of trigonometry transforms H



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into a function of the two variables α, β which we nevertheless call $H(\alpha, \beta)$, since the value is the same:

$$H(\alpha, \beta) = 2^q (\sin(\alpha/2)^{2q} + \sin(\beta/2)^{2q}) (\cot(\alpha/2) \cot(\beta/2) - 1)^q + \frac{1}{(\cot(\alpha/2) \cot(\beta/2) - 1)^q}.$$

Next, using the identity $\sin^2(\xi) = 1/(1 + \cot^2(\xi))$ we can express H as a function of $\cot(\alpha/2)$ and $\cot(\beta/2)$. After one more change of variables, namely $u = \cot(\alpha/2)$ and $v = \cot(\beta/2)$, we obtain our final expression for H :

$$(16) \quad H(u, v) = 2^q \left[\left(\frac{1}{(1+u^2)^q} + \frac{1}{(1+v^2)^q} \right) (uv-1)^q + \frac{1}{(uv-1)^q} \right]$$

where u and v are only required to be positive and such that $uv > 1$. We are now able to minimize (16) with traditional methods. Any critical point in the open domain specified must satisfy the conditions $\frac{\partial H}{\partial u} = \frac{\partial H}{\partial v} = 0$. To spare the reader the rather unpleasant complete calculation of these partial derivatives, let us just state that, for some function $M(u, v)$ (whose details are not needed here), we have

$$\frac{1}{q2^q(uv-1)^q} \frac{\partial H}{\partial u} = -2u \frac{1}{(1+u^2)^{\ln(3)/\ln(2)}} + vM(u, v)$$

and

$$\frac{1}{q2^q(uv-1)^q} \frac{\partial H}{\partial v} = -2v \frac{1}{(1+v^2)^{\ln(3)/\ln(2)}} + uM(u, v).$$

If both partial derivatives are zero, we can solve the resulting equations for $M(u, v)$, eliminate $M(u, v)$, and obtain

$$(17) \quad \frac{u}{v} \frac{1}{(1+u^2)^{\ln(3)/\ln(2)}} = \frac{v}{u} \frac{1}{(1+v^2)^{\ln(3)/\ln(2)}}.$$

Introducing the function

$$\phi(z) := \frac{z}{(1+z)^{\ln(3)/\ln(2)}},$$

condition (17) simplifies to

$$\phi(u^2) = \phi(v^2).$$

We first consider the case where $u \neq v$. The function ϕ is easily seen to be strictly increasing for $z \in [0, \ln(2)/\ln(3/2)]$ and strictly decreasing for $z > \ln(2)/\ln(3/2)$. $u \neq v$ implies that $u^2 < 1/q < v^2$. Since we assume that $uv > 1$, we also have $1/v^2 < u^2$ (and thus $\phi(1/v^2) < \phi(u^2)$). Now, elementary algebra shows that

$$\phi(1/v^2) = \phi(v^2)v^{2(\ln(3)/\ln(2)-1)}.$$

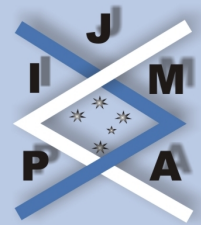
Since $v^2 > 1/q > 1$, this implies that

$$\phi(1/v^2) > \phi(v^2) = \phi(u^2),$$

which is a contradiction. Therefore, the case $u \neq v$ is impossible, and we are left with the analysis of the “isosceles” case $u = v$. Indeed, backtracking through our last change of variables, $u = v$ means that $\alpha = \beta$, and thus $a = b$ in the original expression (15) for $H(a, b, c)$. Thus, we should consider the function $h(s, t)$ from (14), for the case when $s = t$ (and $2s > 1$, to preserve the triangle condition). Our last task is thus to minimize

$$(18) \quad h(s, s) = 2\frac{1}{s^q} + (2s - 1)^q$$

for $s \in (1/2, \infty)$. An analysis of the derivative of $h(s, s)$ shows that it has exactly two zeros for $s > 1/2$, and since the function initially increases (with infinite derivative at $s = 1/2$), the second critical point, at $s = 1$, must be a minimum, which corresponds to the equilateral case. When $s = 1$, $h(1, 1) = 3$. This and $h(1/2, 1/2) = 3$ complete the proof. \square



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Remark 1. Based on our proof, the following corollary can be stated, which is a consequence of $H(a, b, c) \geq 3$ and the general power means inequality:

Corollary 5. *Let $p \geq \ln(3/2)/\ln(2)$ be an arbitrary real number. Then for all triangles with sides a, b and c and semi-perimeter s the inequality*

$$\left(\frac{s-a}{a}\right)^p + \left(\frac{s-b}{b}\right)^p + \left(\frac{s-c}{c}\right)^p \geq \frac{3}{2^p}$$

is valid.

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