SOME INEQUALITIES FOR THE q -GAMMA FUNCTION

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Abstract: Recently, Shabani [4, Theorem 2.4] established some inequalities involving the gamma function. In this paper we present the q-analogues of these inequalities involving the q-gamma function.

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1. Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} e^{x-1} dt.$$

The psi or digamma function, the logarithmic derivative of the gamma function is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

Alsina and Tomás [1] proved that

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1,$$

for all $x \in [0,1]$ and nonnegative integers n. This inequality can be generalized to

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \leq 1,$$

for all $a \geq 1$ and $x \in [0,1]$, see [3]. Recently, Shabani [4] using the series representation of the function $\psi(x)$ and the ideas used in [3] established some double inequalities involving the gamma function. In particular, Shabani [4, Theorem 2.4] proved

$$\frac{\Gamma(a)^c}{\Gamma(b)^d} \leq \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d} \leq \frac{\Gamma(a+b)^c}{\Gamma(a+b)^d}, \tag{1.1}$$

for all $x \in [0,1]$, $a \geq b > 0$, c, d are positive real numbers such that $bc \geq ad > 0$, and $\psi(b+ax) > 0$.

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In this paper we give the q-inequalities of the above results by using similar techniques to those in [4]. The main ideas of Shabani's paper, as well as of the present one, are contained in paper [3] by Sándor. More precisely, we define the q-psi function as $(0 < q < 1)$

$$\psi_q(x) = \frac{d}{dx} \log \Gamma_q(x),$$

where the q-gamma function $\Gamma_q(x)$ is defined by $(0 < q < 1)$

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{i=1}^{\infty} \frac{1-q^i}{1-q^{x+i}}.$$

Many properties of the q-gamma function were derived by Askey [2]. The explicit form of q-psi function $\psi_q(x)$ is

$$\psi_q(x) = -\log(1-q) + \log q \sum_{i=0}^{\infty} \frac{q^{x+i}}{1-q^{x+i}}. \tag{1.2}$$

In this paper we extend (1.1) to the case of $\Gamma_q(x)$. In particular, by using the facts that $\lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x)$ and $\lim_{q \to 1^-} \psi_q(x) = \psi(x)$ we obtain all the results of Shabani [4].

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2. Main Results

In order to establish the proof of the theorems, we need the following lemmas.

Lemma 2.1. *Let $x \in [0,1]$, $q \in (0,1)$, and a, b be any two positive real numbers such that $a \geq b$. Then*

$$\psi_q(a+bx) \geq \psi_q(b+ax).$$

Proof. Clearly, $a+bx, b+ax > 0$. The series presentation of $\psi_q(x)$, see (1.2), gives

$$\begin{aligned}\psi_q(a+bx) - \psi_q(b+ax) &= \log q \sum_{i=0}^{\infty} \left(\frac{q^{a+bx+i}}{1-q^{a+bx+i}} - \frac{q^{b+ax+i}}{1-q^{b+ax+i}} \right) \\ &= \log q \sum_{i=0}^{\infty} \frac{q^i(q^{a+bx} - q^{b+ax})}{(1-q^{a+bx+i})(1-q^{b+ax+i})} \\ &= \log q \sum_{i=0}^{\infty} \frac{q^{b+bx+i}(q^{a-b} - q^{(a-b)x})}{(1-q^{a+bx+i})(1-q^{b+ax+i})}.\end{aligned}$$

Since $0 < q < 1$ we have that $\log q < 0$. In addition, since $a \geq b$ we get that $q^{a-b} \leq q^{(a-b)x}$. Hence,

$$\psi_q(a+bx) - \psi_q(b+ax) \geq 0,$$

which completes the proof. □

Lemma 2.2. *Let $x \in [0,1]$, $q \in (0,1)$, a, b be any two positive real numbers such that $a \geq b$ and $\psi_q(b+ax) > 0$. Let c, d be any two positive real numbers such that $bc \geq ad > 0$. Then*

$$bc\psi_q(a+bx) - ad\psi_q(b+ax) \geq 0.$$

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Proof. Lemma 2.1 together with $\psi_q(b+ax) > 0$ give that $\psi_q(a+bx) > 0$. Thus Lemma 2.1 obtains

$$bc\psi_q(a+bx) \geq ad\psi_q(a+bx) \geq ad\psi_q(b+ax),$$

as required. □

Now we present the q-inequality of (1.1) .

Theorem 2.3. *Let $x \in [0,1]$, $q \in (0,1)$, $a \geq b > 0, c, d$ positive real numbers with $bc \geq ad > 0$ and $\psi_q(b+ax) > 0$. Then*

$$\frac{\Gamma_q(a)^c}{\Gamma_q(b)^d} \leq \frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d} \leq \frac{\Gamma_q(a+b)^c}{\Gamma_q(a+b)^d}.$$

Proof. Let $f(x) = \frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d}$ and $g(x) = \log f(x)$. Then

$$g(x) = c\log\Gamma_q(a+bx) - d\log\Gamma_q(b+ax),$$

which implies that

$$\begin{aligned} g'(x) &= \frac{d}{dx}g(x) \\ &= bc\frac{\Gamma_q'(a+bx)}{\Gamma_q(a+bx)} - ad\frac{\Gamma_q'(b+ax)}{\Gamma(b+ax)} \\ &= bc\psi_q(a+bx) - ad\psi_q(b+ax). \end{aligned}$$

Thus, Lemma 2.2 gives $g'(x) \geq 0$, that is, $g(x)$ is an increasing function on $[0,1]$. Therefore, $f(x)$ is an increasing function on $[0,1]$. Hence, for all $x \in [0,1]$ we have that $f(0) \leq f(x) \leq f(1)$, which is equivalent to

$$\frac{\Gamma_q(a)^c}{\Gamma_q(b)^d} \leq \frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d} \leq \frac{\Gamma_q(a+b)^c}{\Gamma_q(a+b)^d},$$

as requested. □

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Similarly as in the argument proofs of Lemmas 2.1 – 2.2 and Theorem 2.3 we obtain the following results.

Lemma 2.4. *Let $x \geq 1$, $q \in (0,1)$, and a, b be any two positive real numbers with $b \geq a$. Then*

$$\psi_q(a+bx) \geq \psi_q(b+ax).$$

Lemma 2.5. *Let $x \geq 1$, $q \in (0,1)$, a, b be any two positive real numbers with $b \geq a$ and $\psi_q(b+ax) > 0$, and c, d be any two real numbers such that $bc \geq ad > 0$. Then*

$$bc\psi_q(a+bx) - ad\psi_q(b+ax) \geq 0.$$

Using similar techniques to the ones in the proof of Theorem 2.3 with Lemmas 2.4 and 2.5, instead of Lemmas 2.1 and 2.2, we can prove the following result.

Theorem 2.6. *Let $x \geq 1$, $q \in (0,1)$, a, b be any two positive real numbers with $b \geq a > 0$ and $\psi_q(b+ax) > 0$, and c, d be any two real numbers such that $bc \geq ad > 0$. Then $\frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d}$ is an increasing function on $[1, +\infty)$.*

In addition, similar arguments as in the proof of Lemma 2.2 will obtain the following lemmas.

Lemma 2.7. *Let $x \in [0,1]$, $q \in (0,1)$, a, b be any two positive real numbers with $a \geq b > 0$ and $\psi_q(a+bx) < 0$, and c, d be any two real numbers such that $ad \geq bc > 0$. Then*

$$bc\psi_q(a+bx) - ad\psi_q(b+ax) \geq 0.$$

Lemma 2.8. *Let $x \geq 1$, $q \in (0,1)$, a, b be any two positive real numbers with $b \geq a$ and $\psi_q(a+bx) < 0$, and c, d be any two real numbers such that $ad \geq bc > 0$. Then*

$$bc\psi_q(a+bx) - ad\psi_q(b+ax) \geq 0.$$

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Using similar techniques to the ones in the proof of Theorem 2.3, with Lemmas 2.2 and 2.7, we obtain the following.

Theorem 2.9. *Let* $x \in [0,1]$, $q \in (0,1)$, a,b *be any two positive real numbers with* $a \geq b > 0$ *and* $\psi_q(a+bx) < 0$, *and* c,d *be any two real numbers such that* $ad \geq bc > 0$. *Then* $\frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d}$ *is an increasing function on* $[0,1]$.

Using similar techniques to the ones in the proof of Theorem 2.3, with Lemmas 2.4 and 2.8, we obtain the following.

Theorem 2.10. *Let* $x \geq 1$, $q \in (0,1)$, a,b *be any two positive real numbers with* $b \geq a > 0$ *and* $\psi_q(a+bx) < 0$, *and* c,d *be any two real numbers such that* $ad \geq bc > 0$. *Then* $\frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d}$ *is an increasing function on* $[1,+\infty)$.

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