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STRONGLY NONLINEAR ELLIPTIC UNILATERAL PROBLEMS IN ORLICZ SPACE AND L^1 DATA

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ABSTRACT. In this paper, we shall be concerned with the existence result of Unilateral problem associated to the equations of the form,

$$Au + g(x, u, \nabla u) = f,$$

where A is a Leray-Lions operator from its domain $D(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1}E_{\overline{M}}(\Omega)$. On the nonlinear lower order term $g(x, u, \nabla u)$, we assume that it is a Carathéodory function having natural growth with respect to $|\nabla u|$, and satisfies the sign condition. The right hand side f belongs to $L^1(\Omega)$.

Key words and phrases: Orlicz Sobolev spaces, Boundary value problems, Truncations, Unilateral problems.

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1. INTRODUCTION

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with segment property. Let us consider the following nonlinear Dirichlet problem

$$-\operatorname{div}(a(x, u, \nabla u)) + g(x, u, \nabla u) = f, \tag{1.1}$$

where $f \in L^1(\Omega)$, $Au = -\operatorname{div} a(x, u, \nabla u)$ is a Leray-Lions operator defined on its domain $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$, with M an N-function and where g is a nonlinearity with the "natural" growth condition:

$$|g(x, s, \xi)| \leq b(|s|)(h(x) + M(|\xi|))$$

and which satisfies the classical sign condition

$$g(x, s, \xi) \cdot s \geq 0.$$

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In the case where $f \in W^{-1}E_{\overline{M}}(\Omega)$, an existence theorem has been proved in [14] with the nonlinearities g depends only on x and u, and in [4] where g depends also the ∇u.

For the case where $f \in L^1(\Omega)$, the authors in [5] studied the problem (1.1), with the added assumption of exact natural growth

$$|g(x,s,\xi)| \geq \beta M(|\xi|) \text{ for } |s| \geq \mu$$

and in [6] no coercivity condition is assumed on g but the result is restricted to the N-function, M satisfying a Δ_2-condition, while in [11] the authors were concerned about the above problem without assuming a Δ_2-condition on M.

The purpose of this paper is to prove an existence result for unilateral problems associated to (1.1) without assuming the Δ_2-condition in the setting of the Orlicz-Sobolev space.

Further work for the equation (1.1) in the L^p case where there is no restriction can be found in [17], and in [12, 9, 8] in the case of obstacle problems, see also [18].

2. Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$.

Equivalently, M admits the representation: $M(t) = \int_0^t a(s)ds$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0,\ a(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M} = \int_0^t \bar{a}(s)ds$, where $\bar{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{s : a(s) \leq t\}$.

The N-function M is said to satisfy the Δ_2-condition if, for some k

$$M(2t) \leq kM(t), \qquad \forall t \geq 0. \tag{2.1}$$

When (2.1) holds only for $t \geq$ some $t_0 > 0$ then M is said to satisfy the Δ_2-condition near infinity.

We will extend these N-functions to even functions on all $\mathbb{R}$, i.e. $M(t) = M(|t|)$ if $t \leq 0$. Moreover, we have the following Young's inequality

$$\forall s,t \geq 0, \quad st \leq M(t) + \overline{M}(s).$$

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, i.e., for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$.

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$ is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))dx < +\infty \quad \left(\text{resp. } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0\right).$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0, \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \leq 1\right\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives of order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm (for more details see [1]).

We recall some lemmas introduced in [4] which will be used later.

Lemma 2.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial}{\partial x_i}u & \textit{a.e. in} \quad \{x \in \Omega : u(x) \notin D\}, \\ 0 & \textit{a.e. in} \quad \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We assume that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F : W^1L_M(\Omega) \to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.*

We give now the following lemma which concerns operators of Nemytskii type in Orlicz spaces (see [4]).

Lemma 2.3. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P and Q be N-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:*

$$|f(x,s)| \leq c(x) + k_1 P^{-1}M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from

$$\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right) = \left\{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\right\}$$

into $E_Q(\Omega)$.

We define $\mathcal{T}_0^{1,M}(\Omega)$ to be the set of measurable functions $u : \Omega \to \mathbb{R}$ such that $T_k(u) \in W_0^1L_M(\Omega)$, where $T_k(s) = \max(-k, \min(k, s))$ for $s \in \mathbb{R}$ and $k \geq 0$. We give the following lemma which is a generalization of Lemma 2.1 of [2] in Orlicz spaces.

Lemma 2.4. *For every $u \in \mathcal{T}_0^{1,M}(\Omega)$, there exists a unique measurable function $v : \Omega \longrightarrow \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \quad \textit{almost everywhere in } \ \Omega \ \textit{ for every } \ k > 0.$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$.

Lemma 2.5. *Let $\lambda \in \mathbb{R}$ and let u and v be two measurable functions defined on Ω which are finite almost everywhere, and which are such that $T_k(u)$, $T_k(v)$ and $T_k(u+\lambda v)$ belong to $W_0^1 L_M(\Omega)$ for every $k > 0$ then*

$$\nabla(u+\lambda v) = \nabla(u) + \lambda \nabla(v) \quad \textit{a.e. in } \Omega$$

where $\nabla(u)$, $\nabla(v)$ and $\nabla(u+\lambda v)$ are the gradients of u, v and $u+\lambda v$ introduced in Lemma 2.4.

The proof of this lemma is similar to the proof of Lemma 2.12 in [10] for the L^p case.

Below, we will use the following technical lemma.

Lemma 2.6. *Let $(f_n), f \in L^1(\Omega)$ such that*

(i) $f_n \geq 0$ *a.e. in* Ω
(ii) $f_n \to f$ *a.e. in* Ω
(iii) $\int_\Omega f_n(x)dx \to \int_\Omega f(x)dx$
then $f_n \to f$ *strongly in* $L^1(\Omega)$.

3. MAIN RESULTS

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the segment property.

Given an obstacle function $\psi : \Omega \to \mathbb{R}$, we consider

$$K_\psi = \{u \in W_0^1 L_M(\Omega);\ u \geq \psi \text{ a.e. in } \Omega\}, \tag{3.1}$$

this convex set is sequentially $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closed in $W_0^1 L_M(\Omega)$ (see [15]). We now state conditions on the differential operator

$$Au = -\operatorname{div}(a(x,u,\nabla u)) \tag{3.2}$$

(A_1) $a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function.

(A_2) There exist two N-functions M and P with $P \ll M$, function $c(x)$ in $E_{\overline{M}}(\Omega)$, constants k_1, k_2, k_3, k_4 such that, for a.e. x in Ω and for all $s \in \mathbb{R}, \zeta \in \mathbb{R}^N$

$$|a(x,s,\zeta)| \leq c(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\zeta|).$$

(A_3) $[a(x,s,\zeta) - a(x,s,\zeta')](\zeta - \zeta') > 0$ for a.e. x in Ω, s in $\mathbb{R}$ and ζ, ζ' in $\mathbb{R}^N$, with $\zeta \neq \zeta'$.

(A_4) There exist $\delta(x)$ in $L^1(\Omega)$, strictly positive constant α such that, for some fixed element v_0 in $K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$,

$$a(x,s,\zeta)(\zeta - Dv_0) \geq \alpha M(|\zeta|) - \delta(x)$$

for a.e. x in Ω, and all $s \in \mathbb{R}$, $\zeta \in \mathbb{R}^N$.

(A_5) For each $v \in K_\psi \cap L^\infty(\Omega)$ there exists a sequence $v_n \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that $v_n \to u$ for the modular convergence.

Furthermore let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}, \zeta \in \mathbb{R}^N$

(G_1) $g(x,s,\zeta)s \geq 0$

(G_2) $|g(x,s,\zeta)| \leq b(|s|)\,(h(x) + M(|\zeta|))$,
where $b : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non decreasing function, and h is a given nonegative function in $L^1(\Omega)$.

Consider the following Dirichlet problem:

$$A(u) + g(x,u,\nabla u) = f \text{ in } \Omega. \tag{3.3}$$

Remark 3.1. The condition (A_5) holds if one of the following conditions is verified.

(1) There exist $\overline{\psi} \in K_\psi$ such that $\psi - \overline{\psi}$ is continuous in Ω, (see [15, Proposition 9]).

(2) $\psi \in W_0^1 E_M(\Omega)$, (see [15, Proposition 10]).

We shall prove the following existence theorem.

Theorem 3.2. *Assume that (A_1) – (A_5), (G_1) and (G_2) hold and $f \in L^1(\Omega)$. Then there exists at least one solution of the following unilateral problem,*

$$
(P) \qquad \begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega), u \geq \psi \ a.e. \ \text{in} \ \Omega, \\ g(x,u,\nabla u) \in L^1(\Omega) \\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\ dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\ dx \\ \qquad\qquad\qquad \leq \int_\Omega f T_k(u-v)\ dx, \\ \forall\ v \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0. \end{cases}
$$

4. PROOF OF THEOREM 3.2

To prove the existence theorem, we proceed by steps.

STEP 1: *Approximate unilateral problems.*

Let us define

$$
g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}|g(x,s,\xi)|}
$$

and let us consider the approximate unilateral problems:

$$
(P_n) \qquad \begin{cases} u_n \in K_\psi \cap D(A), \\ \langle Au_n, u_n - v\rangle + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)dx \leq \int_\Omega f_n(u_n - v)dx, \\ \forall v \in K_\psi. \end{cases}
$$

where f_n is a regular function such that f_n strongly converges to f in $L^1(\Omega)$.

From Gossez and Mustonen ([15, Proposition 5]), the problem (P_n) has at least one solution.

STEP 2: *A priori estimates.*

Let $k \geq \|v_0\|_\infty$ and let $\varphi_k(s) = se^{\gamma s^2}$, where $\gamma = \left(\frac{b(k)}{\alpha}\right)^2$.

It is well known that

$$
\varphi_k'(s) - \frac{b(k)}{\alpha}|\varphi_k(s)| \geq \frac{1}{2}, \ \ \forall s \in \mathbb{R} \ \text{(see [9])}. \tag{4.1}
$$

Taking $u_n - \eta\varphi_k(T_l(u_n - v_0))$ as test function in (P_n), where $l = k + \|v_0\|_\infty$, we obtain,

$$
\int_\Omega a(x,u_n,\nabla u_n)\nabla T_l(u_n - v_0)\varphi_k'(T_l(u_n - v_0))dx \\ + \int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(T_l(u_n - v_0))dx \\ \leq \int_\Omega f_n \varphi_k(T_l(u_n - v_0))dx.
$$

Since

$$
g_n(x,u_n,\nabla u_n)\varphi_k(T_l(u_n - v_0)) \geq 0
$$

on the subset $\{x \in \Omega : |u_n(x)| > k\}$, then

$$\int_{\{|u_n - v_0| \le l\}} a(x, u_n, \nabla u_n)\nabla(u_n - v_0)\varphi'_k(T_l(u_n - v_0))dx$$
$$\le \int_{\{|u_n| \le k\}} |g_n(x, u_n, \nabla u_n)||\varphi_k(T_l(u_n - v_0))|dx + \int_\Omega f_n\varphi_k(T_l(u_n - v_0))dx.$$

By using (A_4) and (G_1), we have

$$\alpha \int_{\{|u_n - v_0| \le l\}} M(|\nabla u_n|)\varphi'_k(T_l(u_n - v_0))\, dx$$
$$\le b(|k|)\int_\Omega (h(x) + M(\nabla T_k(u_n)))\, |\varphi_k(T_l(u_n - v_0))|\, dx$$
$$+ \int_\Omega \delta(x)\varphi'_k(T_l(u_n - v_0))\, dx + \int_\Omega f_n\varphi_k(T_l(u_n - v_0))\, dx.$$

Since

$$\{x \in \Omega, |u_n(x)| \le k\} \subseteq \{x \in \Omega : |u_n - v_0| \le l\}$$

and the fact that $h, \delta \in L^1(\Omega)$, further f_n is bounded in $L^1(\Omega)$, then

$$\int_\Omega M(|\nabla T_k(u_n)|)\varphi'_k(T_l(u_n - v_0))\, dx \le \frac{b(k)}{\alpha}\int_\Omega M(|\nabla T_k(u_n)|)|\varphi_k(T_l(u_n - v_0))|\, dx + c_k,$$

where c_k is a positive constant depending on k, which implies that

$$\int_\Omega M(|\nabla T_k(u_n)|)\left[\varphi'_k(T_l(u_n - v_0)) - \frac{b(k)}{\alpha}|\varphi_k(T_l(u_n - v_0))|\right] dx \le c_k.$$

By using (4.1), we deduce,

$$\int_\Omega M(|\nabla T_k(u_n)|)\, dx \le 2c_k. \tag{4.2}$$

Since $T_k(u_n)$ is bounded in $W_0^1 L_M(\Omega)$, there exists some $v_k \in W_0^1 L_M(\Omega)$ such that

$$\begin{array}{ll} T_k(u_n) \rightharpoonup v_k \ \text{ weakly in } \ W_0^1 L_M(\Omega) \ \text{ for } \ \sigma(\prod L_M, \prod E_{\overline{M}}), \\ T_k(u_n) \to v_k \ \text{ strongly in } \ E_M(\Omega) \ \text{ and a.e. in } \ \Omega. \end{array} \tag{4.3}$$

STEP 3: *Convergence in measure of* u_n

Let $k_0 \ge \|v_0\|_\infty$ and $k > k_0$, taking $v = u_n - T_k(u_n - v_0)$ as test function in (P_n) gives,

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_0)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)T_k(u_n - v_0)\, dx \le \int_\Omega f_n T_k(u_n - v_0)\, dx, \tag{4.4}$$

since $g_n(x, u_n, \nabla u_n)T_k(u_n - v_0) \ge 0$ on the subset $\{x \in \Omega, |u_n(x)| > k_0\}$, hence (4.4) implies that,

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_0)\, dx \le k\int_{\{|u_n| \le k_0\}} |g_n(x, u_n, \nabla u_n)|\, dx + k\|f\|_{L^1(\Omega)}$$

which gives, by using (G_1),

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx \leq kb(k_0)\left[\int_\Omega |h(x)|\,dx+\int_\Omega M(|\nabla T_{k_0}(u_n)|)\,dx\right]+kc. \tag{4.5}$$

Combining (4.2) and (4.5), we have,

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx \leq k[c_{k_0}+c].$$

By (A_4), we obtain,

$$\int_{\{|u_n-v_0|\leq k\}} M(|\nabla u_n|)\,dx \leq kc_1,$$

where c_1 is independent of k, since k is arbitrary, we have

$$\int_{\{|u_n|\leq k\}} M(|\nabla u_n|)\,dx \leq \int_{\{|u_n-v_0|\leq k+\|v_0\|_\infty\}} M(|\nabla u_n|)dx \leq kc_2,$$

i.e.,

$$\int_\Omega M(|\nabla T_k(u_n)|)\,dx \leq kc_2. \tag{4.6}$$

Now, we prove that u_n converges to some function u in measure (and therefore, we can always assume that the convergence is a.e. after passing to a suitable subsequence). We shall show that u_n is a Cauchy sequence in measure.

Let $k>0$ large enough, by Lemma 5.7 of [13], there exist two positive constants c_3 and c_4 such that

$$\int_\Omega M(c_3T_k(u_n))\,dx \leq c_4\int_\Omega M(|\nabla T_k(u_n)|)\,dx \leq kc_5, \tag{4.7}$$

then, we deduce, by using (4.7) that

$$M(c_3k)\,\mathrm{meas}\{|u_n|>k\} = \int_{\{|u_n|>k\}} M(c_3T_k(u_n))\,dx \leq c_5k,$$

hence

$$\mathrm{meas}(|u_n|>k) \leq \frac{c_5k}{M(c_3k)} \quad \forall n, \forall k. \tag{4.8}$$

Letting k to infinity, we deduce that, $\mathrm{meas}\{|u_n|>k\}$ tends to 0 as k tends to infinity.

For every $\lambda>0$, we have

$$\mathrm{meas}(\{|u_n-u_m|>\lambda\}) \leq \mathrm{meas}(\{|u_n|>k\})+\mathrm{meas}(\{|u_m|>k\}) + \mathrm{meas}(\{|T_k(u_n)-T_k(u_m)|>\lambda\}). \tag{4.9}$$

Consequently, by (4.3) we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.

Let $\epsilon>0$ then, by (4.9) there exists some $k(\epsilon)>0$ such that $\mathrm{meas}(\{|u_n-u_m|>\lambda\})<\epsilon$ for all $n,m\geq h_0(k(\epsilon),\lambda)$. This proves that (u_n) is a Cauchy sequence in measure in Ω , thus converges almost everywhere to some measurable function u. Then

$$\begin{aligned} &T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^1L_M(\Omega) \text{ for } \sigma(\textstyle\prod L_M, \prod E_{\overline{M}}),\\ &T_k(u_n) \to T_k(u) \text{ strongly in } E_M(\Omega) \text{ and a.e. in } \Omega. \end{aligned} \tag{4.10}$$

Step 4: *Boundedness of* $(a(x,T_k(u_n),\nabla T_k(u_n))_n$ *in* $(L_{\overline{M}}(\Omega))^N$.

Let $w \in (E_M(\Omega))^N$ be arbitrary, by (A_3) we have

$$(a(x,u_n,\nabla u_n) - a(x,u_n,w))(\nabla u_n - w) \geq 0,$$

which implies that

$$a(x,u_n,\nabla u_n)(w-\nabla v_0) \leq a(x,u_n,\nabla u_n)(\nabla u_n - \nabla v_0) - a(x,u_n,w)(\nabla u_n - w)$$

and integrating on the subset $\{x \in \Omega, |u_n - v_0| \leq k\}$, we obtain,

$$\begin{aligned}(4.11)\quad \int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(w-\nabla v_0)\,dx &\leq \int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(\nabla u_n-\nabla v_0)\,dx \\ &\quad + \int_{\{|u_n-v_0|\leq k\}} a(x,u_n,w)(w-\nabla u_n)\,dx.\end{aligned}$$

We claim that,

$$(4.12)\qquad \int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(\nabla u_n - v_0)\,dx \leq c_{10},$$

where c_{10} is a positive constant depending on k.

Indeed, if we take $v = u_n - T_k(u_n - v_0)$ as test function in (P_n), we get,

$$\int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(\nabla u_n-\nabla v_0)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v_0)\,dx \leq \int_\Omega f_n T_k(u_n-v_0)\,dx.$$

Since $g_n(x,u_n,\nabla u_n)T_k(u_n-v_0) \geq 0$ on the subset $\{x\in\Omega,\ |u_n| \geq \|v_0\|_\infty\}$, which implies

$$\begin{aligned}(4.13)\quad \int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(\nabla u_n-\nabla v_0)\,dx \\ \leq b(\|v_0\|_\infty)\left(\int_\Omega h(x)\,dx + \int_\Omega M(\nabla T_{\|v_0\|_\infty}(u_n)\right)dx + k\|f\|_{L^1(\Omega)}.\end{aligned}$$

Combining (4.2) and (4.13), we deduce (4.12).

On the other hand, for λ large enough, we have by using (A_2)

$$(4.14)\qquad \int_{\{|u_n-v_0|\leq k\}} \overline{M}\left(\frac{a(x,u_n,w)}{\lambda}\right)dx \leq \overline{M}\left(\frac{c(x)}{\lambda}\right) + \frac{k_3}{\lambda}M(k_4|w|) + c \leq c_{11},$$

hence, $|a(x,u_n,w)|$ bounded in $L_{\overline{M}}(\Omega)$, which implies that the second term of the right hand side of (4.11) is bounded

Consequently, we obtain,

$$(4.15)\qquad \int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(w-\nabla v_0)\,dx \leq c_{12},$$

with c_{12} is positive constant depending of k.

Hence, by the Theorem of Banach-Steinhaus, the sequence $(a(x,u_n,\nabla u_n))\chi_{\{|u_n-v_0|\leq k\}})_n$ remains bounded in $(L_{\overline{M}}(\Omega))^N$. Since k is arbitrary, we deduce that $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is also bounded in $(L_{\overline{M}}(\Omega))^N$, which implies that, for all $k>0$ there exists a function $h_k \in (L_{\overline{M}}(\Omega))^N$, such that,

$$(4.16)\qquad a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}(\Omega), \Pi E_M(\Omega)).$$

STEP 5: *Almost everywhere convergence of the gradient.*

We fix $k > \|v_0\|_\infty$. Let $\Omega_r = \{x \in \Omega, |\nabla T_k(u(x))| \leq r\}$ and denote by χ_r the characteristic function of Ω_r. Clearly, $\Omega_r \subset \Omega_{r+1}$ and $\text{meas}(\Omega \backslash \Omega_r) \longrightarrow 0$ as $r \longrightarrow \infty$.

Fix r and let $s \geq r$, we have,

$$
\begin{aligned}
(4.17)\quad 0 &\leq \int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]dx \\
&\leq \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]dx \\
&= \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]dx \\
&\leq \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]dx.
\end{aligned}
$$

By (A_5) there exists a sequence $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular converge in $W_0^1 L_M(\Omega)$.

Here, we define

$$
\begin{aligned}
w_{n,j}^h &= T_{2k}(u_n - v_0 - T_h(u_n - v_0) + T_k(u_n) - T_k(v_j)), \\
w_j^h &= T_{2k}(u - v_0 - T_h(u - v_0) + T_k(u) - T_k(v_j))
\end{aligned}
$$

and

$$
w^h = T_{2k}(u - v_0 - T_h(u - v_0)),
$$

where $h > 2k > 0$.
For $\eta = \exp(-4\gamma k^2)$, we defined the following function as

$$
v_{n,j}^h = u_n - \eta \varphi_k \left(w_{n,j}^h\right). \tag{4.18}
$$

We take $v_{n,j}^h$ as test function in (P_n), we obtain,

$$
\left\langle A(u_n), \eta\varphi_k \left(w_{n,j}^h\right)\right\rangle + \int_\Omega g_n(x, u_n, \nabla u_n)\eta\varphi_k \left(w_{n,j}^h\right) dx \leq \int_\Omega f_n \eta \varphi_k \left(w_{n,j}^h\right) dx.
$$

Which, implies that

$$
\left\langle A(u_n), \varphi_k \left(w_{n,j}^h\right)\right\rangle + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_k \left(w_{n,j}^h\right) dx \leq \int_\Omega f_n \varphi_k \left(w_{n,j}^h\right) dx. \tag{4.19}
$$

It follows that

$$
\begin{aligned}
(4.20)\quad \int_\Omega a(x, u_n, \nabla u_n)\nabla w_{n,j}^h \varphi_k' \left(w_{n,j}^h\right) dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_k \left(w_{n,j}^h\right) dx \\
\leq \int_\Omega f_n \varphi_k \left(w_{n,j}^h\right) dx.
\end{aligned}
$$

Note that, $\nabla w_{n,j}^h = 0$ on the set where $|u_n| > h + 5k$, therefore, setting $m = 5k + h$, and denoting by $\epsilon(n, j, h)$ any quantity such that

$$
\lim_{h \to +\infty} \lim_{j \to +\infty} \lim_{n \to +\infty} \epsilon(n, j, h) = 0.
$$

If the quantity we consider does not depend on one parameter among n, j and h, we will omit the dependence on the corresponding parameter: as an example, $\epsilon(n, h)$ is any quantity such that

$$
\lim_{h \to +\infty} \lim_{n \to +\infty} \epsilon(n, h) = 0.
$$

Finally, we will denote (for example) by $\epsilon_h(n,j)$ a quantity that depends on n, j, h and is such that

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon_h(n,j)=0$$

for any fixed value of $h.$

We get, by (4.20),

$$\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w^h_{n,j}\varphi'_k\left(w^h_{n,j}\right)dx+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k\left(w^h_{n,j}\right)dx \leq \int_\Omega f_n\varphi_k\left(w^h_{n,j}\right)dx,$$

In view of (4.10), we have $\varphi_k\left(w^h_{n,j}\right)\to\varphi_k(w^h_j)$ weakly* as $n\to+\infty$ in $L^\infty(\Omega)$, and then

$$\int_\Omega f_n\varphi_k\left(w^h_{n,j}\right)dx\to\int_\Omega f\varphi_k(w^h_j)dx \quad \text{as } n\to+\infty.$$

Again tends j to infinity, we get

$$\int_\Omega f\varphi_k(w^h_j)\,dx\to\int_\Omega f\varphi_k(w^h)dx \quad \text{as } j\to+\infty,$$

finally letting h the infinity, we deduce by using the Lebesgue Theorem $\int_\Omega f\varphi_k(w^h)dx\to 0.$

So that

$$\int_\Omega f_n\varphi_k\left(w^h_{n,j}\right)dx=\epsilon(n,j,h).$$

Since in the set $\{x\in\Omega,|u_n(x)|>k\}$, we have $g(x,u_n,\nabla u_n)\varphi_k\left(w^h_{n,j}\right)\geq 0$, we deduce from (4.20) that

$$\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w^h_{n,j}\varphi'_k\left(w^h_{n,j}\right)dx + \int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k\left(w^h_{n,j}\right)dx\leq\epsilon(n,j,h). \tag{4.21}$$

Splitting the first integral on the left hand side of (4.21) where $|u_n|\leq k$ and $|u_n|>k$, we can write,

$$\begin{aligned}\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w^h_{n,j}\varphi'_k\left(w^h_{n,j}\right)dx &= \int_{\{|u_n|\leq k\}} a(x,T_m(u_n),\nabla T_m(u_n))[\nabla T_k(u_n)-\nabla T_k(v_j)]\varphi'_k\left(w^h_{n,j}\right)dx\\ &\quad+\int_{\{|u_n|>k} a(x,T_m(u_n),\nabla T_m(u_n))\nabla w^h_{n,j}\varphi'_k\left(w^h_{n,j}\right)dx.\end{aligned} \tag{4.22}$$

The first term of the right hand side of the last inequality can write as

$$\begin{aligned}\int_{\{|u_n|\leq k\}} a(x,T_m(u_n),\nabla T_m(u_n))[\nabla T_k(u_n)-\nabla T_k(v_j)]\varphi'_k\left(w^h_{n,j}\right)dx &\geq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(v_j)]\varphi'_k\left(w^h_{n,j}\right)dx\\ &\quad-\varphi'_k(2k)\int_{\{|u_n|>k\}}|a(x,T_k(u_n),0)||\nabla T_k(v_j)|dx.\end{aligned} \tag{4.23}$$

Recalling that, $|a(x, T_k(u_n), 0)|\chi_{\{|u_n|>k\}}$ converges to $|a(x, T_k(u), 0)|\chi_{\{|u|>k\}}$ strongly in $L_{\overline{M}}(\Omega)$, moreover, since $|\nabla T_k(v_j)|$ modular converges to $|\nabla T_k(u)|$, then

$$-\varphi'_k(2k)\int_{\{|u_n|>k\}} |a(x, T_k(u_n), 0)||\nabla T_k(v_j)|dx = \epsilon(n, j).$$

For the second term of the right hand side of (4.14) we can write, using (A_3)

$$\begin{aligned}(4.24)\quad \int_{\{|u_n|>k\}} a(x, T_m(u_n), \nabla T_m(u_n))\nabla w^h_{n,j}\varphi'_k\left(w^h_{n,j}\right) dx \\ \geq -\varphi'_k(2k)\int_{\{|u_n|>k\}} |a(x, T_m(u_n), \nabla T_m(u_n))|\nabla T_k(v_j)|dx \\ - \varphi'(2k)\int_{\{|u_n-v_0|>h} \delta(x)dx.\end{aligned}$$

Since $|a(x, T_m(u_n), \nabla T_m(u_n))|$ is bounded in $L_{\overline{M}}(\Omega)$, we have, for a subsequence

$$|a(x, T_m(u_n), \nabla T_m(u_n))| \rightharpoonup l_m$$

weakly in $L_{\overline{M}}(\Omega)$ in $\sigma(L_{\overline{M}}, E_M)$ as n tends to infinity, and since

$$|\nabla T_k(v_j)|\chi_{\{|u_n|>k\}} \to |\nabla T_k(v_j)|\chi_{\{|u|>k\}}$$

strongly in $E_M(\Omega)$ as n tends to infinity, we have

$$-\varphi'_k(2k)\int_{\{|u_n|>k\}} |a(x, T_m(u_n), \nabla T_m(u_n))|\nabla T_k(v_j)|dx \to -\varphi'_k(2k)\int_{\{|u|>k\}} l_m|\nabla T_k(v_j)|dx$$

as n tends to infinity.

Using now, the modular convergence of (v_j), we get

$$-\varphi'_k(2k)\int_{\{|u|>k\}} l_m|\nabla T_k(v_j)|dx \to -\varphi'_k(2k)\int_{\{|u|>k\}} l_m|\nabla T_k(u)|dx = 0$$

as j tends to infinity.

Finally

$$(4.25)\qquad -\varphi'_k(2k)\int_{\{|u_n|>k\}} |a(x, T_m(u_n), \nabla T_m(u_n))|\nabla T_k(v_j)|dx = \epsilon_h(n, j).$$

On the other hand, since $\delta \in L^1(\Omega)$ it is easy to see that

$$(4.26)\qquad -\varphi'_k(2k)\int_{\{|u_n-v_0|>h} \delta(x)dx = \epsilon(n, h).$$

Combining (4.23) – (4.26), we deduce

$$\begin{aligned}(4.27)\quad \int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))\nabla w^h_{n,j}\varphi'_k\left(w^h_{n,j}\right) dx \\ \geq \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(v_j)]\varphi'_k\left(w^h_{n,j}\right) dx \\ + \epsilon(n, h) + \epsilon(n, j) + \epsilon_h(n, j),\end{aligned}$$

which implies that

$$\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w^h_{n,j}\varphi'_k\left(w^h_{n,j}\right)dx \tag{4.28}$$
$$\geq \int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a\left(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s\right)\right]$$
$$\times\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi^j_s\right]\varphi'_k\left(w^h_{n,j}\right)dx$$
$$+\int_\Omega a\left(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s\right)\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi^j_s\right]\varphi'_k\left(w^h_{n,j}\right)dx$$
$$-\int_{\Omega\setminus\Omega^j_s} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\varphi'_k\left(w^h_{n,j}\right)dx$$
$$+\epsilon(n,h)+\epsilon(n,j)+\epsilon_h(n,j),$$

where χ^j_s denotes the characteristic function of the subset $\Omega^j_s = \{x\in\Omega : |\nabla T_k(v_j)|\leq s\}$.

By (4.16) and the fact that $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega^j_s}\varphi'_k\left(w^h_{n,j}\right)$ tends to $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega^j_s}\varphi'_k(w^h_j)$ strongly in $(E_M(\Omega))^N$, the third term on the right hand side of (4.28) tends to the quantity

$$\int_{\Omega\setminus\Omega^j_s} h_k\nabla T_k(v_j)\varphi'_k\left(w^h_j\right)dx$$

as n tends to infinity.

Letting now j tend to infinity, by using the modular convergence of v_j, we have

$$\int_\Omega h_k\nabla T_k(v_j)\chi_{\Omega\setminus\Omega^j_s}\varphi'_k\left(w^h_j\right)dx \to \int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\varphi'_k(w^h)dx$$

as j tends to infinity.

Finally

$$-\int_{\Omega\setminus\Omega^j_s} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\varphi'_k\left(w^h_{n,j}\right)dx \tag{4.29}$$
$$= -\int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\varphi'_k(w^h)dx + \epsilon_h(n,j).$$

Concerning the second term on the right hand side of (4.28) we can write

$$\int_\Omega a\left(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s\right)\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi^j_s\right]\varphi'_k\left(w^h_{n,j}\right)dx \tag{4.30}$$
$$= \int_\Omega a\left(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s\right)\nabla T_k(u_n)\varphi'_k(T_k(u_n)-T_k(v_j))dx$$
$$-\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)\nabla T_k(v_j)\chi^j_s\varphi'_k\left(w^h_{n,j}\right)dx.$$

The first term on the right hand side of (4.30) tends to the quantity

$$\int_\Omega a\left(x,T_k(u),\nabla T_k(v_j)\chi^j_s\right)\nabla T_k(u)\varphi'_k(T_k(u)-T_k(v_j))dx \text{ as } n\to\infty$$

since

$$a\left(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s\right)\varphi'_k(T_k(u_n)-T_k(v_j))$$
$$\to a\left(x,T_k(u),\nabla T_k(v_j)\chi^j_s\right)\varphi'_k(T_k(u)-T_k(v_j))$$

strongly in $(E_{\overline{M}}(\Omega))^N$ by Lemma 2.3 and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(\Omega))^N$ for $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$.

For the second term on the right hand side of (4.30) it is easy to see that

$$\int_\Omega a\left(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j\right) \nabla T_k(v_j)\chi_s^j \varphi_k'\left(w_{n,j}^h\right) dx \longrightarrow \int_\Omega a\left(x, T_k(u), \nabla T_k(v_j)\chi_s^j\right) \nabla T_k(v_j)\chi_s^j \varphi_k'\left(w_j^h\right) dx. \tag{4.31}$$

as $n \to \infty$.

Consequently, we have

$$\begin{aligned}\int_\Omega a\left(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j\right) \left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\right] \varphi_k'\left(w_{n,j}^h\right) dx \\ = \int_\Omega a\left(x, T_k(u), \nabla T_k(v_j)\chi_s^j\right) \left[\nabla T_k(u) - \nabla T_k(v_j)\chi_s^j\right] \varphi_k'\left(w_j^h\right) dx + \epsilon_{j,h}(n)\end{aligned} \tag{4.32}$$

since

$$\nabla T_k(v_j)\chi_s^j \varphi_k'(w_j^h) \to \nabla T_k(u)\chi_s \varphi_k'(w^h)$$

strongly in $(E_M(\Omega))^N$ as $j \to +\infty$, it is easy to see that

$$\int_\Omega a\left(x, T_k(u), \nabla T_k(v_j)\chi_s^j\right) \left[\nabla T_k(u) - \nabla T_k(v_j)\chi_s^j\right] \varphi_k'\left(w_j^h\right) dx \longrightarrow \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0)\nabla T_k(u)\varphi_k'(w^h)dx$$

as $j \to +\infty$, thus

$$\begin{aligned}\int_\Omega a\left(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j\right) \left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\right] \varphi_k'\left(w_{n,j}^h\right) dx \\ = \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0)\nabla T_k(u)\varphi_k'(0)dx + \epsilon(n, j).\end{aligned} \tag{4.33}$$

Combining (4.28), (4.29) and (4.32), we get

$$\begin{aligned}\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))\nabla w_{n,j}^h \varphi_k'\left(w_{n,j}^h\right) dx \\ \geq \int_\Omega \left[a(x, T_k(u_n), \nabla T_k(u_n)) - a\left(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j\right)\right] \\ \times \left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\right] \varphi_k'\left(w_{n,j}^h\right) dx \\ - \int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)\varphi_k'(0)dx + \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0)\nabla T_k(u)\varphi_k'(0)dx + \epsilon(n, j, h).\end{aligned} \tag{4.34}$$

We now, turn to the second term on the left hand side of (4.21), we have

$$
\begin{aligned}
(4.35)\qquad & \int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k\left(w_{n,j}^h\right)dx \\
&\le b(k)\int_\Omega (h(x)+M(|\nabla T_k(u_n)|))\left|\varphi_k\left(w_{n,j}^h\right)\right|dx \\
&\le b(k)\int_\Omega h(x)|\varphi_k\left(w_{n,j}^h\right)|dx+\frac{b(k)}{\alpha}\int_\Omega \delta(x)\left|\varphi_k\left(w_{n,j}^h\right)\right|dx \\
&\qquad +\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\left|\varphi_k\left(w_{n,j}^h\right)\right|dx \\
&\qquad -\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla v_0\left|\varphi_k\left(w_{n,j}^h\right)\right|dx \\
&\le \epsilon(n,j,h)+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\left|\varphi_k\left(w_{n,j}^h\right)\right|dx.
\end{aligned}
$$

The last term on the last side of this inequality reads as

$$
\begin{aligned}
(4.36)\qquad & \frac{b(k)}{\alpha}\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n))-a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\right] \\
&\qquad \times\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]\left|\varphi_k\left(w_{n,j}^h\right)\right|dx \\
&+\frac{b(k)}{\alpha}\int_\Omega a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]\left|\varphi_k\left(w_{n,j}^h\right)\right|dx \\
&\qquad +\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_s^j\left|\varphi_k\left(w_{n,j}^h\right)\right|dx
\end{aligned}
$$

and reasoning as above, it is easy to see that

$$
\frac{b(k)}{\alpha}\int_\Omega a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]\left|\varphi_k\left(w_{n,j}^h\right)\right|dx=\epsilon(n,j)
$$

and

$$
-\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_s^j\left|\varphi_k\left(w_{n,j}^h\right)\right|dx=\epsilon(n,j,h).
$$

So that

$$
\begin{aligned}
(4.37)\qquad & \left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k\left(w_{n,j}^h\right)dx\right| \\
&\le \frac{b(k)}{\alpha}\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n))-a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\right] \\
&\qquad \times\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]\left|\varphi_k\left(w_{n,j}^h\right)\right|dx+\epsilon(n,j,h).
\end{aligned}
$$

Combining (4.21), (4.34) and (4.37), we obtain

$$
\begin{aligned}
(4.38)\qquad & \int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n))-a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\right] \\
&\qquad \times\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]\left(\varphi_k'\left(w_{n,j}^h\right)-\frac{b(k)}{\alpha}\left|\varphi_k\left(w_{n,j}^h\right)\right|\right)dx \\
&\le \int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\varphi_k'(0)dx+\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx+\epsilon(n,j,h),
\end{aligned}
$$

which implies that, by using (4.1)

$$\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\right]\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]dx \leq 2\int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)\varphi_k'(0)dx + 2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx + \epsilon(n,j,h). \tag{4.39}$$

Now, remark that

$$\begin{aligned}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]dx \\ = \int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\right]\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]dx \\ + \int_\Omega a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]dx \\ - \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]dx \\ + \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\left[\nabla T_k(v_j)\chi_s^j - \nabla T_k(u)\chi_s\right]dx.\end{aligned} \tag{4.40}$$

We shall pass to the limit in n and j in the last three terms of the right hand side of the last inequality, we get

$$\int_\Omega a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]dx = \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)dx + \epsilon(n,j),$$

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]dx = \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)dx + \epsilon(n),$$

and

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\left[\nabla T_k(v_j)\chi_s^j - \nabla T_k(u)\chi_s\right]dx = \epsilon(n,j),$$

which implies that

$$\begin{aligned}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]dx \\ = \int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a\left(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j\right)\right] \\ \times \left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]dx + \epsilon(n,j).\end{aligned} \tag{4.41}$$

Combining (4.17), (4.39) and (4.41), we have

$$\begin{aligned}(4.42)\quad &\int_{\Omega_r}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dx\\ &\le \int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]dx\\ &\le 2\int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\varphi_k'(0)dx+2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx+\epsilon(n,j,h).\end{aligned}$$

By passing to the $\limsup$ over n, and letting j,h,s tend to infinity, we obtain

$$\lim_{n\to+\infty}\int_{\Omega_r}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dx=0,$$

thus implies by the same method used in [5] that

$$(4.43)\qquad \nabla u\to\nabla u_n \text{ a.e. in } \Omega.$$

Step 6: *Modular convergence of the truncation*:
By (4.16) and (4.43), we have $h_k=a(x,T_k(u),\nabla T_k(u))$, which implies by using (4.42)

$$\begin{aligned}(4.44)\quad &\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\ &\le \int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u)\chi_s-\nabla v_0)+\delta(x)]dx\\ &\quad+\int_{\Omega} a(x,T_k(u_n),\nabla T_k(u)\chi_s)(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)dx\\ &\quad+2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\varphi_k'(0)dx\\ &\quad+2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx+\epsilon(n,j,h),\end{aligned}$$

which implies, by using Fatou's Lemma,

$$\begin{aligned}(4.45)\quad &\int_{\Omega}[a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+\delta(x)]dx\\ &\le \liminf_{n\to+\infty}\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\ &\le \limsup_{n\to+\infty}\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\ &\le \limsup_{n\to+\infty}\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u)\chi_s-\nabla v_0)+\delta(x)]dx\\ &\quad+\limsup_{n\to+\infty}\int_{\Omega} a(x,T_k(u_n),\nabla T_k(u)\chi_s)(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)dx\\ &\quad+2\int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\varphi_k'(0)dx\\ &\quad+2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx+\epsilon(n,j,h).\end{aligned}$$

Reasoning as above, we have

$$\text{(4.46)}\quad \limsup_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u)\chi_s-\nabla v_0)+\delta(x)]dx = \int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)\chi_s-\nabla v_0)+\delta(x)]dx,$$

and

$$\limsup_{n\to+\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)dx = \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)dx.$$

Which implies that

$$\begin{aligned}\text{(4.47)}\quad &\int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+\delta(x)]dx\\ &\le \liminf_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\ &\le \limsup_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\ &\le \int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)\chi_s-\nabla v_0)+\delta(x)]dx\\ &\quad + 2\int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\varphi_k'(0)dx + \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)dx\\ &\quad + 2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)dx.\end{aligned}$$

Using the fact that

$[a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)\chi_s-\nabla v_0)+\delta(x)], h_k\nabla T_k(u)\varphi_k'(0)$ and
$a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)$ in $L^1(\Omega)$

and letting $s\to+\infty$, we get, since $\operatorname{meas}(\Omega\setminus\Omega_s)\to 0$,

$$\begin{aligned}\text{(4.48)}\quad &\int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+\delta(x)]dx\\ &\le \liminf_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\ &\le \limsup_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx\\ &\le \int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+\delta(x)]dx.\end{aligned}$$

Finally, we have

$$\text{(4.49)}\quad \lim_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+\delta(x)]dx = \int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+\delta(x)]dx$$

and by using (A_3), one obtains, by Lemma 2.6

$$M(\nabla T_k(u_n)) \to M(\nabla T_k(u)) \ \text{ in } \ L^1(\Omega). \tag{4.50}$$

Step 7: *Equi-integrability of the nonlinearities.*

We need to prove that

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \ \text{ strongly in } \ L^1(\Omega), \tag{4.51}$$

in particular it is enough to prove the equi-integrability of $g_n(x, u_n, \nabla u_n)$. To this purpose. We take $u_n - T_1(u_n - v_0 - T_h(u_n - v_0))$ as test function in (P_n), we obtain

$$\int_{\{|u_n - v_0| > h+1\}} |g_n(x, u_n, \nabla u_n)| dx \leq \int_{\{|u_n - v_0| > h\}} (|f_n| + \delta(x)) dx.$$

Let $\varepsilon > 0$, then there exists $h(\varepsilon) \geq 1$ such that

$$\int_{\{|u_n - v_0| > h(\varepsilon)\}} |g(x, u_n, \nabla u_n)| dx < \frac{\varepsilon}{2}. \tag{4.52}$$

For any measurable subset $E \subset \Omega$, we have

$$\int_E |g_n(x, u_n, \nabla u_n)| dx \leq \int_E b(h(\varepsilon) + \|v_0\|_\infty)(h(x) + M(\nabla T_{h(\varepsilon)+\|v_0\|_\infty}(u_n))) dx + \int_{\{|u_n - v_0| > h(\varepsilon)\}} |g(x, u_n, \nabla u_n)| dx.$$

In view of (4.50) there exists $\eta(\varepsilon) > 0$ such that

$$\int_E b(h(\varepsilon) + \|v_0\|_\infty)(h(x) + M(\nabla T_{h(\varepsilon)+\|v_0\|_\infty}(u_n))) dx < \frac{\varepsilon}{2} \quad \text{for all } \ E \ \text{ such that } \ \operatorname{meas}(E) < \eta(\varepsilon). \tag{4.53}$$

Finally, combining (4.52) and (4.53), one easily has

$$\int_E |g_n(x, u_n, \nabla u_n)| \, dx < \varepsilon \ \text{ for all } \ E \ \text{ such that } \ \operatorname{meas}(E) < \eta(\varepsilon),$$

which implies (4.51).

Step 8: *Passing to the limit.*

Let $v \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$, we take $u_n - T_k(u_n - v)$ as test function in (P_n), we can write

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - v) dx + \int_\Omega g(x, u_n, \nabla u_n) T_k(u_n - v) dx \leq \int_\Omega f_n T_k(u_n - v) dx, \tag{4.54}$$

which implies that

$$\int_{\{|u_n - v| \leq k\}} a(x, u_n, \nabla u_n) \nabla(u_n - v_0) dx + \int_{\{|u_n - v| \leq k\}} a(x, T_{k+\|v\|_\infty} u_n, \nabla T_{k+\|v\|_\infty}(u_n)) \nabla(v_0 - v) dx + \int_\Omega g(x, u_n, \nabla u_n) T_k(u_n - v) dx \leq \int_\Omega f_n T_k(u_n - v) dx. \tag{4.55}$$

By Fatou's lemma and the fact that

$$a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u))$$

weakly in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ one can easily see that

$$\int_{\{|u-v|\le k\}} a(x,u,\nabla u)\nabla(u-v_0)dx + \int_{\{|u-v|\le k\}} a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u))\nabla(v_0 - v)dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)dx \le \int_\Omega fT_k(u-v)dx. \tag{4.56}$$

Hence

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)dx \le \int_\Omega fT_k(u-v)dx. \tag{4.57}$$

Now, let $v \in K_\psi \cap L^\infty(\Omega)$, by the condition (A_5) there exists $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that v_j converges to v modular, let $h > \|v_0\|_\infty$, taking $v = T_h(v_j)$ in (4.57), we have

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u - T_h(v_j))dx + \int_\Omega g(x,u,\nabla u)T_k(u - T_h(v_j))dx \le \int_\Omega fT_k(u - T_h(v_j))dx. \tag{4.58}$$

We can easily pass to the limit as $j \to +\infty$ to get

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u - T_h(v))dx + \int_\Omega g(x,u,\nabla u)T_k(u - T_h(v))dx \le \int_\Omega fT_k(u - T_h(v))dx \quad \forall\, v \in K_\psi \cap L^\infty(\Omega), \tag{4.59}$$

the same, we pass to the limit as $h \to +\infty$, we deduce

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)dx \le \int_\Omega fT_k(u-v)dx \quad \forall v \in K_\psi \cap L^\infty(\Omega), \forall k > 0. \tag{4.60}$$

Thus, the proof of the theorem is now complete. □

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