journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 10 (2009), Issue 2, Article 35, 11 pp.

FOURIER RESTRICTION ESTIMATES TO MIXED HOMOGENEOUS SURFACES

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Received 17 September, 2008; accepted 13 February, 2009
Communicated by L. Pick

ABSTRACT. Let a, b be real numbers such that $2 \leq a < b$, and let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ a mixed homogeneous function. We consider polynomial functions φ and also functions of the type $\varphi(x_1, x_2) = A|x_1|^a + B|x_2|^b$. Let $\Sigma = \{(x, \varphi(x)) : x \in B\}$ with the Lebesgue induced measure. For $f \in S(\mathbb{R}^3)$ and $x \in B$, let $(\mathcal{R}f)(x, \varphi(x)) = \widehat{f}(x, \varphi(x))$, where $\widehat{f}$ denotes the usual Fourier transform.

For a large class of functions φ and for $1 \leq p < \frac{4}{3}$ we characterize, up to endpoints, the pairs (p, q) such that $\mathcal{R}$ is a bounded operator from $L^p(\mathbb{R}^3)$ on $L^q(\Sigma)$. We also give some sharp $L^p \to L^2$ estimates.

Key words and phrases: Restriction theorems, Fourier transform.

2000 *Mathematics Subject Classification.* Primary 42B10, 26D10.

1. INTRODUCTION

Let a, b be real numbers such that $2 \leq a < b$, let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a mixed homogeneous function of degree one with respect to the non isotropic dilations $r \cdot (x_1, x_2) = \left(r^{\frac{1}{a}} x_1, r^{\frac{1}{b}} x_2\right)$, i.e.

$$\varphi\left(r^{\frac{1}{a}} x_1, r^{\frac{1}{b}} x_2\right) = r\varphi(x_1, x_2), \quad r > 0. \tag{1.1}$$

We also suppose φ to be smooth enough. We denote by B the closed unit ball of $\mathbb{R}^2$, by

$$\Sigma = \{(x, \varphi(x)) : x \in B\}$$

and by σ the induced Lebesgue measure. For $f \in S(\mathbb{R}^3)$, let $\mathcal{R}f : \Sigma \to \mathbb{C}$ be defined by

$$(\mathcal{R}f)(x, \varphi(x)) = \widehat{f}(x, \varphi(x)), \quad x \in B, \tag{1.2}$$

Research partially supported by Secyt-UNC, Agencia Nacional de Promoción Científica y Tecnológica.
The authors wish to thank Professor Fulvio Ricci for fruitful conversations about this subject.
254-08

where $\widehat{f}$ denotes the usual Fourier transform of f. We denote by E the type set associated to $\mathcal{R}$, given by

$$E = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in [0,1] \times [0,1] : \|\mathcal{R}\|_{L^p(\mathbb{R}^3), L^q(\Sigma)} < \infty \right\}.$$

Our aim in this paper is to obtain as much information as possible about the set E, for certain surfaces Σ of the type above described.

In the general n-dimensional case, the $L^p\left(\mathbb{R}^{n+1}\right) - L^q\left(\Sigma\right)$ boundedness properties of the restriction operator $\mathcal{R}$ have been studied by different authors. A very interesting survey about recent progress in this research area can be found in [11]. The $L^p\left(\mathbb{R}^{n+1}\right) - L^2\left(\Sigma\right)$ restriction theorems for the sphere were proved by E. Stein in 1967, for $\frac{3n+4}{4n+4} < \frac{1}{p} \leq 1$; for $\frac{n+4}{2n+4} < \frac{1}{p} \leq 1$ by P. Tomas in [12] and then in the same year by Stein for $\frac{n+4}{2n+4} \leq \frac{1}{p} \leq 1$. The last argument has been used in several related contexts by R. Strichartz in [9] and by A. Greenleaf in [6]. This method provides a general tool to obtain, from suitable estimates for $\widehat{\sigma}$, $L^p\left(\mathbb{R}^{n+1}\right) - L^2\left(\Sigma\right)$ estimates for $\mathcal{R}$. Moreover, a general theorem, due to Stein, holds for smooth enough hypersurfaces with never vanishing Gaussian curvature ([8], pp.386). There it is shown that in this case, $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$ if $\frac{n+4}{2n+4} \leq \frac{1}{p} \leq 1$ and $-\frac{n+2}{n}\frac{1}{p} + \frac{n+2}{n} \leq \frac{1}{q} \leq 1$, also that this last relation is the best possible and that no restriction theorem of any kind can hold for $f \in L^p\left(\mathbb{R}^{n+1}\right)$ when $\frac{1}{p} \leq \frac{n+2}{2n+2}$ ([8, pp.388]). The cases $\frac{n+2}{2n+2} < \frac{1}{p} < \frac{n+4}{2n+4}$ are not completely solved. The best results for surfaces with non vanishing curvature like the paraboloid and the sphere are due to T. Tao [10]. Restriction theorems for the Fourier transform to homogeneous polynomial surfaces in $\mathbb{R}^3$ are obtained in [4]. Also, in [1] the authors obtain sharp $L^p\left(\mathbb{R}^{n+l}\right) - L^2\left(\Sigma\right)$ estimates for certain homogeneous surfaces Σ of codimension l in $\mathbb{R}^{n+l}$.

In Section 2 we give some preliminary results.

In Section 3 we consider $\varphi\left(x_1, x_2\right) = A\left|x_1\right|^a + B\left|x_2\right|^b$, $A \neq 0, B \neq 0$. We describe completely, up to endpoints, the pairs $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$ with $\frac{1}{p} > \frac{3}{4}$. A fundamental tool we use is Theorem 2.1 of [2].

In Section 4 we deal with polynomial functions φ. Under certain hypothesis about φ we can prove that if $\frac{3}{4} < \frac{1}{p} \leq 1$ and the pair $\left(\frac{1}{p}, \frac{1}{q}\right)$ satisfies some sharp conditions, then $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$. Finally we obtain some $L^{\frac{4}{3}} - L^q$ estimates and also some sharp $L^p - L^2$ estimates.

2. PRELIMINARIES

We take φ to be a mixed homogeneous and smooth enough function that satisfies (1.1). If V is a measurable set in $\mathbb{R}^2$, we denote $\Sigma^V = \{(x, \varphi(x)) : x \in V\}$ and σ^V as the associated surface measure. Also, for $f \in S\left(\mathbb{R}^3\right)$, we define $\mathcal{R}^V f : \Sigma^V \to \mathbb{C}$ by

$$\left(\mathcal{R}^V f\right)(x, \varphi(x)) = \widehat{f}(x, \varphi(x)) \qquad x \in V;$$

we note that $\mathcal{R}^B = \mathcal{R}$, $\sigma^B = \sigma$ and $\Sigma^B = \Sigma$.

For $x = (x_1, x_2)$ letting $\|x\| = |x_1|^a + |x_2|^b$, we define

$$A_0 = \left\{ x \in \mathbb{R}^2 : \frac{1}{2} \leq \|x\| \leq 1 \right\}$$

and for $j \in \mathbb{N}$,

$$A_j = 2^{-j} \cdot A_0.$$

Thus $B \subseteq \overline{\bigcup_{j\in\mathbb{N}\cup\{0\}} A_j}$. A standard homogeneity argument (see, e.g. [5]) gives, for $1 \le p, q \le \infty$,

$$\left\| \mathcal{R}^{A_j} \right\|_{L^p(\mathbb{R}^3), L^q\left(\Sigma^{A_j}\right)} = 2^{-j\frac{a+b}{ab}\left(\frac{1}{q} - \frac{a+b+ab}{a+b} + \frac{1}{p}\frac{a+b+ab}{a+b}\right)} \left\| \mathcal{R}^{A_0} \right\|_{L^p(\mathbb{R}^3), L^q\left(\Sigma^{A_0}\right)}. \tag{2.1}$$

From this we obtain the following remarks.

Remark 1. If $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$ then $\frac{1}{q} \ge -\frac{a+b+ab}{a+b}\frac{1}{p} + \frac{a+b+ab}{a+b}$.

Remark 2. If $-\frac{a+b+ab}{a+b}\frac{1}{p} + \frac{a+b+ab}{a+b} < \frac{1}{q} \le 1$ and

$$\left\| \mathcal{R}^{A_0} \right\|_{L^p(\mathbb{R}^3), L^q\left(\Sigma^{A_0}\right)} < \infty, \tag{2.2}$$

then $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$.

We will use a theorem due to Strichartz (see [9]), whose proof relies on the Stein complex interpolation theorem, which gives $L^p\left(\mathbb{R}^3\right) - L^2\left(\Sigma^V\right)$ estimates for the operator $\mathcal{R}^V$ depending on the behavior at infinity of $\widehat{\sigma^V}$. In [4] we obtained information about the size of the constants. There we found the following:

Remark 3. If V is a measurable set in $\mathbb{R}^2$ of positive measure and if

$$\left| \widehat{\sigma^V}(\xi) \right| \le A\left(1 + |\xi_3|\right)^{-\tau}$$

for some $\tau > 0$ and for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, then there exists a positive constant c_τ such that

$$\left\| \mathcal{R}^V \right\|_{L^p(\mathbb{R}^3), L^2(\Sigma^V)} \le c_\tau A^{\frac{1}{2(1+\tau)}}$$

for $p = \frac{2+2\tau}{2+\tau}$.

In [2] the authors obtain a result (Theorem 2.1, p.155) from which they also obtain the following consequence

Remark 4 ([2, Corollary 2.2])**.** Let I, J be two real intervals, and let

$$M = \left\{ (x_1, x_2, \psi(x_1, x_2)) : (x_1, x_2) \in I \times J \right\},$$

where $\psi : I \times J \to \mathbb{R}$ is a smooth function such that either $\left| \frac{\partial^2\psi}{\partial x_1^2}(x_1, x_2) \right| \ge c > 0$ or $\left| \frac{\partial^2\psi}{\partial x_2^2}(x_1, x_2) \right| \ge c > 0$, uniformly on $I \times J$. If M has the Lebesgue surface measure, $\frac{1}{q} = 3\left(1 - \frac{1}{p}\right)$ and $\frac{3}{4} < \frac{1}{p} \le 1$ then there exists a positive constant c such that

$$\left\| \widehat{f}|_M \right\|_{L^q(M)} \le c\, \|f\|_{L^p(\mathbb{R}^3)} \tag{2.3}$$

for $f \in S(\mathbb{R}^3)$.

Following the proof of Theorem 2.1 in [2] we can check that if in the last remark we take $J = \left[2^{-k}, 2^{-k+1}\right]$, $k \in \mathbb{N}$ in the case that $\left| \frac{\partial^2\psi}{\partial x_1^2}(x_1, x_2) \right| \ge c > 0$ uniformly on $I \times J$ with c independent of k, or $I = \left[2^{-k}, 2^{-k+1}\right]$, $k \in \mathbb{N}$ in the other case, then we can replace (2.3) by

$$\left\| \widehat{f}|_M \right\|_{L^q(M)} \le c' 2^{-k\left(\frac{1}{p} + \frac{1}{q} - 1\right)} \|f\|_{L^p(\mathbb{R}^3)} \tag{2.4}$$

with c' independent of k.

3. THE CASES $\varphi(x_1, x_2) = A|x_1|^a + B|x_2|^b$

In this cases we characterize, up to endpoints, the pairs $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$ with $\frac{3}{4} < \frac{1}{p} \leq 1$. We also obtain some border segments. If either $A = 0$ or $B = 0$, φ becomes homogeneous and these cases are treated in [4]. For the remainder situation we obtain the following

Theorem 3.1. *Let $a, b, A, B \in \mathbb{R}$ with $2 \leq a \leq b$, $A \neq 0$, $B \neq 0$, let $\varphi(x_1, x_2) = A|x_1|^a + B|x_2|^b$ and let E be the type set associated to φ. If $\frac{3}{4} < \frac{1}{p} \leq 1$ and $-\frac{a+b+ab}{a+b}\frac{1}{p} + \frac{a+b+ab}{a+b} < \frac{1}{q} \leq 1$ then $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$.*

Proof. Suppose $\frac{3}{4} < \frac{1}{p} \leq 1$ and $-\frac{a+b+ab}{a+b}\frac{1}{p} + \frac{a+b+ab}{a+b} < \frac{1}{q} \leq 1$. By Remark 2 it is enough to prove (2.2). Now, A_0 is contained in the union of the rectangles $Q = [-1,1] \times \left[\frac{1}{2}, 1\right]$, $Q' = \left[\frac{1}{2}, 1\right] \times [-1,1]$, and its symmetrics with respect to the x_1 and x_2 axes. Now we will study $\left\|\mathcal{R}^Q\right\|_{L^p(\mathbb{R}^3), L^q(\Sigma^Q)}$. We decompose $Q = \bigcup\limits_{k\in N} Q_k$ with

$$Q_k = \left(\left[-2^{-k+1}, -2^{-k}\right] \cup \left[2^{-k}, 2^{-k+1}\right]\right) \times \left[\frac{1}{2}, 1\right].$$

Now, as in Theorem 1, (3.2), in [3] we have

$$\left|\widehat{\sigma^{Q_k}}(\xi)\right| \leq A 2^{k\frac{a-2}{2}} \left(1 + |\xi_3|\right)^{-1}$$

and then Remark 3 implies

$$\left\|\mathcal{R}^{Q_k}\right\|_{L^{\frac{4}{3}}(\mathbb{R}^3), L^2\left(\Sigma^{Q_k}\right)} \leq c 2^{k\frac{a-2}{8}}. \tag{3.1}$$

Also, since $\left|\frac{\partial^2 \varphi}{\partial x_2^2}(x_1, x_2)\right| \geq c > 0$ uniformly on Q_k, from (2.4) we obtain

$$\left\|\mathcal{R}^{Q_k}\right\|_{L^p(\mathbb{R}^3), L^q\left(\Sigma^{Q_k}\right)} \leq c' 2^{-k\left(\frac{1}{p} + \frac{1}{q} - 1\right)}$$

for $\frac{1}{q} = 3\left(1 - \frac{1}{p}\right)$ and $\frac{3}{4} < \frac{1}{p} \leq 1$. Applying the Riesz interpolation theorem and then performing the sum on $k \in \mathbb{N}$ we obtain

$$\left\|\mathcal{R}^{Q}\right\|_{L^p(\mathbb{R}^3), L^q(\Sigma^{Q})} < \infty,$$

for $\frac{2+3a}{2+a}\left(1 - \frac{1}{p}\right) < \frac{1}{q} \leq 1$ and $\frac{3}{4} < \frac{1}{p} \leq 1$. In a similar way we get that

$$\left\|\mathcal{R}^{Q'}\right\|_{L^p(\mathbb{R}^3), L^q\left(\Sigma^{Q'}\right)} < \infty,$$

for $\frac{2+3b}{2+b}\left(1 - \frac{1}{p}\right) < \frac{1}{q} \leq 1$ and $\frac{3}{4} < \frac{1}{p} \leq 1$. The study for the symmetric rectangles is analogous. Thus

$$\left\|\mathcal{R}^{A_0}\right\|_{L^p(\mathbb{R}^3), L^q\left(\Sigma^{A_0}\right)} < \infty$$

for $\frac{3}{4} < \frac{1}{p} \leq 1$ and $-\frac{a+b+ab}{a+b}\frac{1}{p} + \frac{a+b+ab}{a+b} < \frac{1}{q} \leq 1$ and the theorem follows. □

Remark 5.

i) If $\frac{b+2}{8} < \frac{1}{q} \leq 1$ then $\left(\frac{3}{4}, \frac{1}{q}\right) \in E$.

ii) The point $\left(\frac{a+b+2ab}{2a+2b+2ab}, \frac{1}{2}\right) \in E$.

From (3.1) and the Hölder inequality we obtain that

$$\left\| \mathcal{R}^{Q_k} \right\|_{L^{\frac{4}{3}}(\mathbb{R}^3), L^q\left(\Sigma^{Q_k}\right)} \leq c 2^{k\left(\frac{a-2}{8} - \frac{2-q}{2q}\right)}$$

for $\frac{1}{2} \leq \frac{1}{q} \leq 1$. Then if $\frac{a+2}{8} < \frac{1}{q} \leq 1$ we perform the sum over $k \in \mathbb{N}$ to get

$$\left\| \mathcal{R}^{Q} \right\|_{L^{\frac{4}{3}}(\mathbb{R}^3), L^q(\Sigma^{Q})} < \infty,$$

for these q's. Analogously, if $\frac{b+2}{8} < \frac{1}{q} \leq 1$ we get

$$\left\| \mathcal{R}^{Q'} \right\|_{L^{\frac{4}{3}}(\mathbb{R}^3), L^q\left(\Sigma^{Q'}\right)} < \infty,$$

thus since $a \leq b$, if $\frac{b+2}{8} < \frac{1}{q} \leq 1$,

$$\left\| \mathcal{R}^{A_0} \right\|_{L^{\frac{4}{3}}(\mathbb{R}^3), L^q\left(\Sigma^{A_0}\right)} < \infty,$$

and $i)$ follows from Remark 2.

Assertion $ii)$ follows from Remark 3, since from Lemma 3 in [3] we have that

$$\left|\widehat{\sigma}\left(\xi\right)\right| \leq c\left(1 + \left|\xi_3\right|\right)^{-\frac{1}{a} - \frac{1}{b}}.$$

4. THE POLYNOMIAL CASES

In this section we deal with mixed homogeneous polynomial functions φ satisfying (1.1). The following result is sharp (up to the endpoints) for $\frac{3}{4} < \frac{1}{p} \leq 1$, as a consequence of Remark 1.

Theorem 4.1. *Let φ be a mixed homogeneous polynomial function satisfying (1.1). Suppose that the gaussian curvature of Σ does not vanish identically and that at each point of $\Sigma^{B-\{0\}}$ with vanishing curvature, at least one principal curvature is different from zero. If $(a, b) \neq (2, 4)$, $\frac{3}{4} < \frac{1}{p} \leq 1$ and $-\frac{a+b+ab}{a+b}\frac{1}{p} + \frac{a+b+ab}{a+b} < \frac{1}{q} \leq 1$ then $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$.*

Proof. We first study the operator $\mathcal{R}^{A_0}$. Let $(x_1^0, x_2^0) \in A_0$. If $Hess\varphi\left(x_1^0, x_2^0\right) \neq 0$ there exists a neighborhood U of (x_1^0, x_2^0) such that $Hess\varphi\left(x_1, x_2\right) \neq 0$ for $(x_1, x_2) \in U$. From the proposition in [8, pp. 386], it follows that

$$\left\| \mathcal{R}^{U} \right\|_{L^p(\mathbb{R}^3), L^q(\Sigma^{U})} < \infty \tag{4.1}$$

for $\frac{1}{q} = 2\left(1 - \frac{1}{p}\right)$ and $\frac{3}{4} \leq \frac{1}{p} \leq 1$. Suppose now that $Hess\varphi\left(x_1^0, x_2^0\right) = 0$ and that either $\frac{\partial^2\varphi}{\partial x_1^2}\left(x_1^0, x_2^0\right) \neq 0$ or $\frac{\partial^2\varphi}{\partial x_2^2}\left(x_1^0, x_2^0\right) \neq 0$. Then there exists a neighborhood $V = I \times J$ of (x_1^0, x_2^0) such that either $\left|\frac{\partial^2\varphi}{\partial x_1^2}\left(x_1, x_2\right)\right| \geq c > 0$ or $\left|\frac{\partial^2\varphi}{\partial x_2^2}\left(x_1, x_2\right)\right| \geq c > 0$ uniformly on V. So from Remark 4 we obtain that

$$\left\| \mathcal{R}^{V} \right\|_{L^p(\mathbb{R}^3), L^q(\Sigma^{V})} < \infty \tag{4.2}$$

for $\frac{1}{q} = 3\left(1 - \frac{1}{p}\right)$ and $\frac{3}{4} < \frac{1}{p} \leq 1$. From (4.1), (4.2) and Hölder´s inequality, it follows that

$$\left\| \mathcal{R}^{A_0} \right\|_{L^p(\mathbb{R}^3), L^q\left(\Sigma^{A_0}\right)} < \infty \tag{4.3}$$

for $\frac{1}{q} \geq 3\left(1 - \frac{1}{p}\right)$ and $\frac{3}{4} < \frac{1}{p} \leq 1$. So, if $\frac{a+b+ab}{a+b} \geq 3$, the theorem follows from Remark 2. The only cases left are $(a, b) = (3, 4)$, $(a, b) = (3, 5)$, $(a, b) = (4, 5)$ and $(a, b) = (2, b)$, $b > 2$. If $(a, b) = (3, 4)$ and φ has a monomial of the form $a_{i,j}x^iy^j$, with $a_{ij} \neq 0$, then $\frac{i}{3} + \frac{j}{4} = 1$ so $4i + 3j = 12$ and so either $(i, j) = (0, 4)$ or $(i, j) = (3, 0)$. So $\varphi\left(x_1, x_2\right) = a_{3,0}x_1^3 + a_{0,4}x_2^4$.

The hypothesis about the derivatives of φ imply that $a_{3,0} \neq 0$ and $a_{0,4} \neq 0$ and the theorem follows using Theorem 3.1 in each quadrant. The cases $(a, b) = (3, 5)$, or $(a, b) = (4, 5)$ are completely analogous.

Now we deal with the cases $(a, b) = (2, b)$, $b > 2$. We note that

$$\varphi(x_1, x_2) = Ax_1^2 + Bx_1x_2^{\frac{b}{2}} + Cx_2^b \tag{4.4}$$

where $B = 0$ for b odd. The hypothesis about φ implies $A \neq 0$. For b odd, $\varphi(x_1, x_2) = Ax_1^2 + Cx_2^b$ and since $C \neq 0$ (on the contrary $Hess\varphi(x_1, x_2) \equiv 0$), the theorem follows using Theorem 3.1 as before. Now we consider b even and φ given by (4.4). If $B = 0$ the theorem follows as above, so we suppose $B \neq 0$.

$$Hess\varphi(x_1, x_2) = -\frac{x_2^{\frac{b}{2}-2}}{4}\left(\left(B^2b^2 + 8ACb - 8ACb^2\right)x_2^{\frac{b}{2}} - 2(b-2)ABbx_1\right). \tag{4.5}$$

So if $Hess\varphi(x_1^0, x_2^0) = 0$ then either $x_2^0 = 0$ or

$$\left(B^2b^2 + 8ACb - 8ACb^2\right)\left(x_2^0\right)^{\frac{b}{2}} - 2(b-2)ABbx_1^0 = 0.$$

In the first case we have $b > 4$. We take a neighborhood $W_1 = I \times \left[-2^{-k_0}, 2^{-k_0}\right] \subset A_0$, $k_0 \in \mathbb{N}$, of the point $(x_1^0, 0)$ such that $Hess\varphi$ vanishes, on W_1, only along the x_1 axes. For $k \in \mathbb{N}$, $k > k_0$, we take $U_k = I \times J_k$ where $J_k = [-2^{-k+1}, -2^{-k}] \cup [2^{-k}, 2^{-k+1}]$. So $W_1 = \overline{\cup U_k}$. For $(x_1, x_2) \in U_k$, it follows from (4.5) that

$$|Hess\varphi(x_1, x_2)| \geq c2^{-k\left(\frac{b}{2}-2\right)},$$

so for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$,

$$\left|\widehat{\sigma^{U_k}}(\xi)\right| \leq c2^{k\frac{b-4}{4}}(1 + |\xi_3|)^{-1}$$

and from Remark 3 we get

$$\left\|\mathcal{R}^{U_k}\right\|_{L^{\frac{4}{3}}(\mathbb{R}^3), L^2\left(\Sigma^{U_k}\right)} \leq c2^{k\frac{b-4}{16}}. \tag{4.6}$$

Also, since $\left|\frac{\partial^2\varphi}{\partial x_1^2}(x_1, x_2)\right| \geq c > 0$ uniformly on U_k, as in (2.4) we obtain

$$\left\|\mathcal{R}^{U_k}\right\|_{L^p(\mathbb{R}^3), L^q\left(\Sigma^{U_k}\right)} \leq c2^{-k\left(2-\frac{2}{p}\right)} \tag{4.7}$$

for $\frac{3}{4} < \frac{1}{p} \leq 1$ and $\frac{1}{q} = 3\left(1 - \frac{1}{p}\right)$. From (4.6), (4.7) and the Riesz Thorin theorem we obtain

$$\left\|\mathcal{R}^{U_k}\right\|_{L^{p_t}(\mathbb{R}^3), L^{q_t}\left(\Sigma^{U_k}\right)} \leq c2^{k\left(t\frac{b-4}{16} - (1-t)\left(2-\frac{2}{p}\right)\right)} \tag{4.8}$$

for $\frac{1}{q_t} = t\frac{1}{2} + (1-t)\,3\left(1 - \frac{1}{p}\right)$ and $\frac{1}{p_t} = t\frac{3}{4} + (1-t)\frac{1}{p}$.

A simple computation shows that if $\frac{1}{p} = \frac{3}{4}$ then the exponent in (4.8) is negative for $t < t_0 = \frac{8}{4+b}$ and that

$$\frac{1}{q_{t_0}} - \frac{2+3b}{4(2+b)} < 0,$$

so for $\frac{1}{p} > \frac{3}{4}$ and $t < t_0$, both near enough, the exponent is still negative and

$$\frac{1}{q_t} - \frac{2+3b}{2+b}\left(1 - \frac{1}{p_t}\right) < 0,$$

thus

$$\left\|\mathcal{R}^{W_1}\right\|_{L^p(\mathbb{R}^3),L^q\left(\Sigma^{W_1}\right)} < \infty \tag{4.9}$$

for $\frac{3}{4} < \frac{1}{p}$ near enough and $\frac{1}{q} = \frac{2+3b}{2+b}\left(1-\frac{1}{p}\right)$. Finally, if

$$\left(B^2b^2 + 8ACb - 8ACb^2\right)\left(x_2^0\right)^{\frac{b}{2}} - 2(b-2)ABbx_1^0 = 0$$

then we study the order of $Hess\varphi\left(x_1, x_2^0\right)$ for $2^{-k-1} \le |x_1 - x_1^0| \le 2^{-k}$, $k \in \mathbb{N}$.

$$\begin{aligned}\left|\frac{\left(x_2^0\right)^{\frac{b}{2}-2}}{4}\left(\left(B^2b^2 + 8ACb - 8ACb^2\right)\left(x_2^0\right)^{\frac{b}{2}} - 2(b-2)ABbx_1\right)\right| \\ = \left|\frac{\left(x_2^0\right)^{\frac{b}{2}-2}}{2}(b-2)ABb\left(x_1 - x_1^0\right)\right| \ge c2^{-k}.\end{aligned} \tag{4.10}$$

We take the following neighborhood of $\left(x_1^0, x_2^0\right)$, $W_2 = \overline{\cup_{k\in\mathbb{N}} V_k}$, with

$$V_k = \left\{\left(r^{\frac{1}{2}}x_1, r^{\frac{1}{b}}x_2^0\right) : 2^{-k-1} \le \left|x_1 - x_1^0\right| \le 2^{-k},\ \frac{1}{2} \le r \le 2\right\}.$$

From the homogeneity of φ and (4.10) we obtain

$$\left|Hess\varphi\left(r^{\frac{1}{2}}x_1, r^{\frac{1}{b}}x_2^0\right)\right| = r^{1-\frac{2}{b}}\left|Hess\varphi\left(x_1, x_2^0\right)\right| \ge c2^{-k},$$

then from Proposition 6 in [8, p. 344], we get for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$

$$\left|\widehat{\sigma^{V_k}}(\xi)\right| \le c2^{\frac{k}{2}}\left(1 + |\xi_3|\right)^{-1},$$

so from Remark 3

$$\left\|\mathcal{R}^{V_k}\right\|_{L^{\frac{4}{3}}(\mathbb{R}^3),L^2\left(\Sigma^{V_k}\right)} \le c2^{\frac{k}{8}}$$

and by Hölder's inequality, for $q < 2$ we have

$$\left\|\mathcal{R}^{V_k}\right\|_{L^{\frac{4}{3}}(\mathbb{R}^3),L^q\left(\Sigma^{V_k}\right)} \le c2^{k\left(\frac{1}{8}-\frac{2-q}{2q}\right)}.$$

This exponent is negative for $\frac{1}{q} > \frac{5}{8}$ and so we sum on k to obtain

$$\left\|\mathcal{R}^{W_2}\right\|_{L^{\frac{4}{3}}(\mathbb{R}^3),L^q\left(\Sigma^{W_2}\right)} < \infty \tag{4.11}$$

for $\frac{5}{8} < \frac{1}{q} \le 1$. Since $b \ge 6$, $\frac{5}{8} \le \frac{2+3b}{4(2+b)}$ and then from (4.1), (4.9) and (4.11), we get

$$\left\|\mathcal{R}^{A_0}\right\|_{L^p(\mathbb{R}^3),L^q\left(\Sigma^{A_0}\right)} < \infty,$$

for $\frac{3}{4} < \frac{1}{p}$ near enough and $\frac{1}{q} > \frac{2+3b}{2+b}\left(1-\frac{1}{p}\right)$ and the theorem follows from standard considerations involving Hölder's inequality, the Riesz Thorin theorem and from Remark 2. □

Remark 6. In the case $(a,b) = (2,b)$, $b > 2$, we have (4.11). In a similar way we get, from (4.6) and Hölder's inequality,

$$\left\|\mathcal{R}^{W_1}\right\|_{L^{\frac{4}{3}}(\mathbb{R}^3),L^q\left(\Sigma^{W_1}\right)} < \infty$$

for $\frac{b+4}{16} < \frac{1}{q} \le 1$. So

$$\left\|\mathcal{R}\right\|_{L^{\frac{4}{3}}(\mathbb{R}^3),L^q(\Sigma)} < \infty$$

for $\max\left\{\frac{5}{8}, \frac{b+4}{16}, \frac{2+3b}{8+4b}\right\} < \frac{1}{q} \leq 1$. We observe that if $b = 6$ then $\frac{5}{8} = \frac{b+4}{16} = \frac{2+3b}{8+4b}$, thus from Remark 1 we see that, in this case, this condition for $\frac{1}{q}$ is sharp, up to the end point.

Now we will show some examples of functions φ not satisfying the hypothesis of the previous theorem, for which we obtain that the portion of the type set E in the region $\frac{3}{4} < \frac{1}{p} \leq 1$ is smaller than the region

$$E_{a,b} = \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) : \frac{3}{4} < \frac{1}{p} \leq 1, \frac{a+b+ab}{a+b}\left(1 - \frac{1}{p}\right) < \frac{1}{q} \leq 1 \right\}$$

stated in Theorem 4.1.

We consider $\varphi(x_1, x_2) = x_1^2$, which is a mixed homogeneous function satisfying (1.1) for any $b > 2$. In this case $\varphi_{x_1x_1} \equiv 2$ but $Hess\varphi \equiv 0$. From Remark 2.8 in [4] and Remark 4 we obtain that the corresponding type set is the region $\frac{1}{q} \geq 3\left(1 - \frac{1}{p}\right)$, $\frac{3}{4} < \frac{1}{p} \leq 1$ which is smaller than the region $E_{a,b}$.

We consider now a mixed homogeneous function φ satisfying (1.1), of the form

$$\varphi(x_1, x_2) = x_2^l P(x_1, x_2), \tag{4.12}$$

with $P(x_1, 0) \neq 0$ for $x_1 \neq 0$. Since $a < b$ it can be checked that $l \geq 2$ and that for $l > 2$, $\varphi_{x_1x_1}(x_1, 0) = \varphi_{x_2x_2}(x_1, 0) = 0$. Moreover

$$Hess\varphi = x_2^{2l-2}\left(P_{x_1x_1}\left(l(l-1)P + 2lx_2P_{x_2} + x_2^2P_{x_2x_2}\right) - \left(lP_{x_1} + x_2P_{x_1x_2}\right)^2\right), \tag{4.13}$$

which vanishes at $(x_1, 0)$. A computation shows that the second factor is different from zero at a point of the form $(x_1, 0)$. So $Hess\varphi$ does not vanish identically.

Proposition 4.2. *Let φ be a mixed homogeneous function satisfying (1.1) and (4.12). If $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$ then $\frac{1}{q} \geq (l+1)\left(1 - \frac{1}{p}\right)$.*

Proof. Let $f\varepsilon = \chi_{K_\varepsilon}$ the characteristic function of the set $K_\varepsilon = \left[0, \frac{1}{3}\right] \times \left[0, \frac{\varepsilon^{-1}}{3}\right] \times \left[0, \frac{\varepsilon^{-l}}{3M}\right]$, with $M = \max_{(x_1,x_2)\in[0,1]\times[0,1]} P(x_1, x_2)$. If $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$ then

$$\|\mathcal{R}f_\varepsilon\|_{L^q(\Sigma)} \leq c\|f_\varepsilon\|_{L^p(\mathbb{R}^3)} = c\varepsilon^{-\frac{1+l}{p}}. \tag{4.14}$$

By the other side,

$$\|\mathcal{R}f_\varepsilon\|_{L^q(\Sigma)} \geq \left(\int_{W\varepsilon} \left|\widehat{f_\varepsilon}(x_1, x_2, \varphi(x_1, x_2))\right|^q dx_1dx_2\right)^{\frac{1}{q}}$$

where $W_\varepsilon = \left[\frac{1}{2}, 1\right] \times [0, \varepsilon]$. Now, for $(x_1, x_2) \in W_\varepsilon$ and $(y_1, y_2, y_3) \in K_\varepsilon$,

$$|x_1y_1 + x_2y_2 + \varphi(x_1, x_2)y_3| \leq 1$$

so

$$\begin{aligned}
&\left|\widehat{f_\varepsilon}(x_1, x_2, \varphi(x_1, x_2))\right| \\
&\qquad = \left|\int_{K_\varepsilon} e^{-i(x_1y_1+x_2y_2+\varphi(x_1,x_2)y_3)}dy_1dy_2dy_3\right| \\
&\qquad \geq \int_{K_\varepsilon} \cos(x_1y_1 + x_2y_2 + \varphi(x_1, x_2)y_3)\,dy_1dy_2dy_3 \geq c\varepsilon^{-1-l}.
\end{aligned}$$

Thus

$$\|\mathcal{R}f_\varepsilon\|_{L^q(\Sigma)} \geq c\varepsilon^{-1-l+\frac{1}{q}}. \tag{4.15}$$

The proposition follows from (4.14) and (4.15). □

We note that in the case that $(a+b)\,l > ab$ (for example $\varphi(x_1,x_2) = x_2^4\left(x_1^2+x_2^4\right)$) the portion of the type set corresponding to $\frac{3}{4} < \frac{1}{p} \leq 1$ will be smaller than the region $E_{a,b}$.

Also, $\varphi(x_1,x_2) = x_2^2\left(x_1+x_2^2\right)$ is an example where $a=2$, $b=4$, $Hess\varphi(x_1,x_2) = -4x_2^2$ and if $x_2 = 0$ and $x_1 \neq 0$, $\varphi_{x_2x_2}(x_1,x_2) = 2x_1 \neq 0$. Again, since $12 = (a+b)\,l > ab = 8$, we get that the portion of the type set corresponding to $\frac{3}{4} < \frac{1}{p} \leq 1$ will be smaller than the region $E_{a,b}$.

Proposition 4.3. *Let φ be a mixed homogeneous function satisfying (1.1) and (4.12) with $l \geq \frac{b}{2}$. If $\frac{3}{4} \leq \frac{1}{p} \leq 1$ and $\frac{1}{q} > (l+1)\left(1-\frac{1}{p}\right)$, then*

$$\left\|\mathcal{R}^{A_0}\right\|_{L^p(\mathbb{R}^3),L^q\left(\Sigma^{A_0}\right)} \leq c.$$

Proof. Let $\left(x_1^0,x_2^0\right) \in A_0$, if $Hess\varphi\left(x_1^0,x_2^0\right) \neq 0$, as in the proof of Theorem 4.1 we find a neighborhood U of $\left(x_1^0,x_2^0\right)$ such that (4.1) holds. If $Hess\varphi\left(x_1^0,x_2^0\right) = 0$, by (4.13), either $x_2^0 = 0$ or the polynomial Q given by $P_{x_1x_1}\left(l\,(l-1)\,P + 2lx_2P_{x_2} + x_2^2P_{x_2x_2}\right) - \left(lP_{x_1} + x_2P_{x_1x_2}\right)^2$ vanishes at $\left(x_1^0,x_2^0\right)$. In the first case, using the fact that $P(x_1,0) \neq 0$ for $x_1 \neq 0$, we get that

$$\left(P_{x_1x_1}l\,(l-1)\,P - l^2P_{x_1}^2\right)\left(x_1^0,0\right) \neq 0.$$

We take a neighborhood W_1 of the point $\left(x_1^0,0\right)$ and U_k as in the proof of Theorem 4.1. So for $(x_1,x_2) \in U_k$,

$$\left|Hess\varphi(x_1,x_2)\right| \geq c2^{-k(2l-2)}$$

and so

$$\left|\widehat{\sigma^{U_k}}(\xi_1,\xi_2,\xi_3)\right| \leq \frac{2^{k(l-1)}}{1+|\xi_3|}.$$

By the other side,

$$\left|\widehat{\sigma^{U_k}}(\xi_1,\xi_2,\xi_3)\right| \leq 2^{-k}$$

so for $0 \leq \tau \leq 1$,

$$\left|\widehat{\sigma^{U_k}}(\xi_1,\xi_2,\xi_3)\right| \leq \frac{2^{k(\tau l-1)}}{(1+|\xi_3|)^{\tau}}$$

and by Remark 3

$$\left\|\mathcal{R}^{U_k}\right\|_{L^p(\mathbb{R}^3),L^2\left(\Sigma^{U_k}\right)} \leq c_\tau 2^{\frac{k(\tau l-1)}{2(1+\tau)}}$$

for $p = \frac{2(1+\tau)}{2+\tau}$ and so Hölder's inequality implies, for $1 \leq q < 2$,

$$\left\|\mathcal{R}^{U_k}\right\|_{L^p(\mathbb{R}^3),L^q\left(\Sigma^{U_k}\right)} \leq c_\tau 2^{k\left(\frac{\tau l-1}{2(1+\tau)}-\frac{2-q}{2q}\right)}$$

and a computation shows that this exponent is negative for $\frac{1}{q} > (l+1)\left(1-\frac{1}{p}\right)$. Thus

$$\left\|\mathcal{R}^{W_1}\right\|_{L^p(\mathbb{R}^3),L^q\left(\Sigma^{W_1}\right)} < \infty \tag{4.16}$$

for $\frac{3}{4} \leq \frac{1}{p} \leq 1$ and $(l+1)\left(1-\frac{1}{p}\right) < \frac{1}{q} \leq 1$. Now we suppose $Q\left(x_1^0,x_2^0\right) = 0$. We observe that

$$\deg Q \leq 2\deg P - 2 \leq 2\,(b-l) - 2 \leq 2l-2$$

and so $Hess\varphi\left(x_1,x_2^0\right)$ vanishes at x_1^0 with order at most $2l-2$. Then defining W_2 and V_k as in the proof of Theorem 4.1, we have

$$\left|Hess\varphi\left(x_1,x_2^0\right)\right| \geq 2^{-k(2l-2)}$$

and as in the previous case we obtain

$$\left\|\mathcal{R}^{W_2}\right\|_{L^p(\mathbb{R}^3),L^q\left(\Sigma^{W_2}\right)} < \infty \tag{4.17}$$

for $\frac{3}{4} \leq \frac{1}{p} \leq 1$ and $\frac{1}{q} > (l+1)\left(1-\frac{1}{p}\right)$. The proposition follows from (4.16), (4.17) and (4.1). □

From Proposition 4.3 and Remark 2 we obtain the following result, sharp up to the end points, for $\frac{3}{4} \leq \frac{1}{p} \leq 1$.

Theorem 4.4. *Let φ be a mixed homogeneous function satisfying (1.1) and (4.12) with $l \geq \frac{b}{2}$. If $m = \max\left\{l+1, \frac{a+b+ab}{a+b}\right\}$, $\frac{3}{4} \leq \frac{1}{p} \leq 1$ and $\frac{1}{q} > m\left(1-\frac{1}{p}\right)$, then $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$.*

4.1. **Sharp $L^p - L^2$ Estimates.** In [4] we obtain sharp $L^p - L^2$ estimates for the restriction of the Fourier transform to homogeneous polynomial surfaces in $\mathbb{R}^3$. The principal tools we used there were two Littlewood Paley decompositions. Adapting this proof to the setting of non isotropic dilations we obtain the following results.

Lemma 4.5. *Let $\frac{a+b+2ab}{2a+2b+2ab} \leq \frac{1}{p} \leq 1$. If*

$$\left\|\mathcal{R}^{A_0}\right\|_{L^p(\mathbb{R}^3),L^2\left(\Sigma^{A_0}\right)} < \infty$$

then $\left(\frac{1}{p}, \frac{1}{2}\right) \in E$.

Proof. From (2.1), the lemma follows from a process analogous to the proof of Lemma 4.3 in [4]. □

Theorem 4.6.

$i)$ *If φ is a mixed homogeneous polynomial function satisfying the hypothesis of Theorem 4.1 then $\left(\frac{a+b+2ab}{2a+2b+2ab}, \frac{1}{2}\right) \in E$.*

$ii)$ *Let $\frac{1}{p_0} = \max\left\{\frac{a+b+2ab}{2a+2b+2ab}, \frac{2l+1}{2l+2}\right\}$. If φ is a mixed homogeneous polynomial function satisfying the hypothesis of Theorem 4.4 then $\left(\frac{1}{p_0}, \frac{1}{2}\right) \in E$.*

Proof. $i)$ If $\frac{a+b+ab}{a+b} \geq 3$, $i)$ follows from (4.3) and Lemma 4.5. The cases $(a,b) = (3,4)$, $(a,b) = (3,5)$ and $(a,b) = (4,5)$ are solved in Remark 5, part $ii)$. The cases $(a,b) = (2,b)$ with b odd or $B = 0$ are also included in Remark 5, part $ii)$. For the remainder cases $(2,b)$, we observe that, if $b > 6$, from the proof of Theorem 4.1 we obtain

$$\left\|\mathcal{R}^{A_0}\right\|_{L^p(\mathbb{R}^3),L^2\left(\Sigma^{A_0}\right)} < \infty, \tag{4.18}$$

for $\frac{1}{p} = \frac{a+b+2ab}{2a+2b+2ab}$, so $i)$ follows from Lemma 4.5. For $b = 6$, as before we get

$$\left\|\mathcal{R}^{W_1}\right\|_{L^p(\mathbb{R}^3),L^2\left(\Sigma^{W_1}\right)} < \infty,$$

and

$$\left\|\mathcal{R}^{V_k}\right\|_{L^p(\mathbb{R}^3),L^2\left(\Sigma^{V_k}\right)} < \infty$$

for $k \in \mathbb{N}$, $\frac{1}{p} = \frac{a+b+2ab}{2a+2b+2ab}$. In a similar way to Lemma 4.3 of [4], we use a uni-dimensional Littlewood Paley decomposition to obtain

$$\left\|\mathcal{R}^{W_2}\right\|_{L^p(\mathbb{R}^3),L^2\left(\Sigma^{W_2}\right)} < \infty$$

and then we have (4.18) for $\frac{1}{p} = \frac{a+b+2ab}{2a+2b+2ab}$. So $i)$ follows from Lemma 4.5.

$ii)$ From the proof of Proposition 4.3, we use a uni-dimensional Littlewood Paley decomposition to obtain (4.18) for $\frac{1}{p} = \max\left\{\frac{a+b+2ab}{2a+2b+2ab}, \frac{2l+1}{2l+2}\right\}$, and $ii)$ follows from Lemma 4.5. □

Remark 7. In [7] the authors obtain sharp estimates for the Fourier transform of measures σ associated to surfaces Σ like ours, when φ is a polynomial function satisfiyng (1.1) and the condition that φ and $Hess\varphi$ do not vanish simultaneously on $B - \{(0,0)\}$. In these cases, part $i)$ of the above theorem follows from Remark 3. We observe that our hypotheses are less restrictive, for example $\varphi(x_1, x_2) = x_1^4 x_2^2 + x_2^{10}$ satisfies the hypothesis of part $i)$ of the above theorem but φ and $Hess\varphi$ vanish at any (x_1, x_2) with $x_2 = 0$.

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