

# OSTROWSKI TYPE INEQUALITIES OVER BALLS AND SHELLS VIA A TAYLOR-WIDDER FORMULA

GEORGE A. ANASTASSIOU

Department of Mathematical Sciences  
The University of Memphis  
Memphis, TN 38152, U.S.A.  
EMail: [ganastss@memphis.edu](mailto:ganastss@memphis.edu)

*Received:* 06 August, 2007

*Accepted:* 27 November, 2007

*Communicated by:* S.S. Dragomir

*2000 AMS Sub. Class.:* 26D10, 26D15.

*Key words:* Ostrowski inequality, multivariate inequality, ball and shell, Taylor-Widder formula, extended complete Tschebyshev system.

*Abstract:* The classical Ostrowski inequality for functions on intervals estimates the value of the function minus its average in terms of the maximum of its first derivative. This result is extended to higher order over shells and balls of  $\mathbb{R}^N$ ,  $N \geq 1$ , with respect to an *extended complete Tschebyshev system* and the *generalized radial derivatives of Widder type*. We treat radial and non-radial functions.



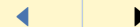
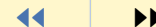
Ostrowski Type Inequalities via a  
Taylor-Widder Formula

George A. Anastassiou

vol. 8, iss. 4, art. 106, 2007

[Title Page](#)

[Contents](#)



Page 1 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

# Contents

<a href="#">1 Introduction</a>	<a href="#">3</a>
<a href="#">2 Background</a>	<a href="#">4</a>
<a href="#">3 Results on the Shell</a>	<a href="#">8</a>
<a href="#">4 Results on the Sphere</a>	<a href="#">14</a>
<a href="#">5 Addendum</a>	<a href="#">23</a>



---

Ostrowski Type Inequalities via a  
Taylor-Widder Formula

George A. Anastassiou

vol. 8, iss. 4, art. 106, 2007

---

[Title Page](#)

[Contents](#)



Page **2** of 25

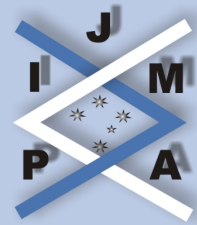
[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



## 1. Introduction

The classical Ostrowski inequality (of 1938, see [12]) is

$$\left| \frac{1}{b-a} \int_a^b f(y) dy - f(x) \right| \leq \left( \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty,$$

for  $f \in C^1([a, b])$ ,  $x \in [a, b]$ , and it is a sharp inequality. This was extended to  $\mathbb{R}^N$ ,  $N \geq 1$ , over balls and shells in [5], [6], [4]. Earlier this extension was done over boxes and rectangles, see [2, p. 507-520], and [3], see also [1]. The produced Ostrowski type inequalities, in the above mentioned references, were mostly sharp and they involved the first and higher order derivatives of the engaged function  $f$ .

Here we derive a set of very general higher order Ostrowski type inequalities over shells and balls with respect to an *extended complete Tschebyshev system* (see [10]) and *generalized derivatives of Widder type* (see [15]). The proofs are based on the *polar method* and the *general Taylor-Widder formula* (see [15], 1928). Our results generalize the higher order Ostrowski type inequalities established in the above mentioned references.

Ostrowski Type Inequalities via a  
Taylor-Widder Formula

George A. Anastassiou

vol. 8, iss. 4, art. 106, 2007

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 3 of 25

Go Back

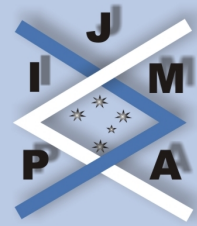
Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 4 of 25

Go Back

Full Screen

Close

## 2. Background

The following are taken from [15]. Let  $f, u_0, u_1, \dots, u_n \in C^{n+1}([a, b]), n \geq 0$ , and the Wronskians

$$W_i(x) := W[u_0(x), u_1(x), \dots, u_i(x)]$$

$$:= \begin{vmatrix} u_0(x) & u_1(x) & \dots & u_i(x) \\ u_0'(x) & u_1'(x) & \dots & u_i'(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_0^{(i)}(x) & u_1^{(i)}(x) & \dots & u_i^{(i)}(x) \end{vmatrix}, \quad i = 0, 1, \dots, n.$$

Assume  $W_i(x) > 0$  over  $[a, b]$ . Clearly then

$$\phi_0(x) := W_0(x) = u_0(x),$$

$$\phi_1(x) := \frac{W_1(x)}{(W_0(x))^2}, \dots,$$

$$\phi_i(x) := \frac{W_i(x)W_{i-2}(x)}{(W_{i-1}(x))^2}, \quad i = 2, 3, \dots, n$$

are positive on  $[a, b]$ .

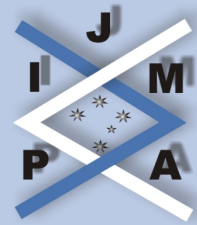
For  $i \geq 0$ , the linear differentiable operator of order  $i$ :

$$L_i f(x) := \frac{W[u_0(x), u_1(x), \dots, u_{i-1}(x), f(x)]}{W_{i-1}(x)}, \quad i = 1, \dots, n+1;$$

$$L_0 f(x) := f(x), \quad \forall x \in [a, b].$$

Then for  $i = 1, \dots, n+1$  we have

$$L_i f(x) = \phi_0(x)\phi_1(x) \cdots \phi_{i-1}(x) \frac{d}{dx} \frac{1}{\phi_{i-1}(x)} \frac{d}{dx} \frac{1}{\phi_{i-2}(x)} \frac{d}{dx} \cdots \frac{d}{dx} \frac{1}{\phi_1(x)} \frac{d}{dx} \frac{f(x)}{\phi_0(x)}.$$



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 5 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

Consider also

$$g_i(x, t) := \frac{1}{W_i(t)} \begin{vmatrix} u_0(t) & u_1(t) & \cdots & u_i(t) \\ u'_0(t) & u'_1(t) & \cdots & u'_i(t) \\ \cdots & \cdots & \cdots & \cdots \\ u_0^{(i-1)}(t) & u_1^{(i-1)}(t) & \cdots & u_i^{(i-1)}(t) \\ u_0(x) & u_1(x) & \cdots & u_i(x) \end{vmatrix},$$

$$i = 1, 2, \dots, n; g_0(x, t) := \frac{u_0(x)}{u_0(t)}, \quad \forall x, t \in [a, b].$$

Note that  $g_i(x, t)$  as a function of  $x$  is a linear combination of  $u_0(x), u_1(x), \dots, u_i(x)$  and it holds

$$\begin{aligned} g_i(x, t) &= \frac{\phi_0(x)}{\phi_0(t) \cdots \phi_i(t)} \int_t^x \phi_1(x) \int_t^{x_1} \cdots \int_t^{x_{i-2}} \phi_{i-1}(x_{i-1}) \int_t^{x_{i-1}} \phi_i(x_i) dx_i dx_{i-1} \cdots dx_1 \\ &= \frac{1}{\phi_0(t) \cdots \phi_i(t)} \int_t^x \phi_0(s) \cdots \phi_i(s) g_{i-1}(x, s) ds, \quad i = 1, 2, \dots, n. \end{aligned}$$

**Example 2.1 ([15]).** The sets

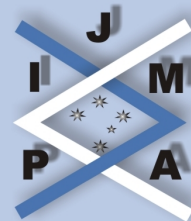
$$\{1, x, x^2, \dots, x^n\}, \quad \{1, \sin x, -\cos x, -\sin 2x, \cos 2x, \dots, (-1)^{n-1} \sin nx, (-1)^n \cos nx\}$$

fulfill the above theory.

We mention

**Theorem 2.1 (Karlin and Studden (1966), see [10, p. 376]).** Let  $u_0, u_1, \dots, u_n \in C^n([a, b])$ ,  $n \geq 0$ . Then  $\{u_i\}_{i=0}^n$  is an extended complete Tschebyshev system on  $[a, b]$  iff  $W_i(x) > 0$  on  $[a, b]$ ,  $i = 0, 1, \dots, n$ .

We also mention



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 6 of 25

Go Back

Full Screen

Close

**Theorem 2.2 (D. Widder, [15, p. 138]).** Let the functions

$$f(x), u_0(x), u_1(x), \dots, u_n(x) \in C^{n+1}([a, b]),$$

and the Wronskians  $W_0(x), W_1(x), \dots, W_n(x) > 0$  on  $[a, b]$ ,  $x \in [a, b]$ . Then for  $t \in [a, b]$  we have

$$f(x) = f(t) \frac{u_0(x)}{u_0(t)} + L_1 f(t) g_1(x, t) + \dots + L_n f(t) g_n(x, t) + R_n(x),$$

where

$$R_n(x) := \int_t^x g_n(x, s) L_{n+1} f(s) ds.$$

For example, one could take  $u_0(x) = c > 0$ . If  $u_i(x) = x^i$ ,  $i = 0, 1, \dots, n$ , defined on  $[a, b]$ , then

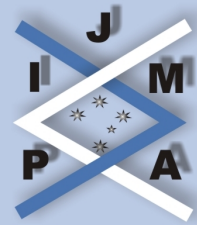
$$L_i f(t) = f^{(i)}(t) \quad \text{and} \quad g_i(x, t) = \frac{(x-t)^i}{i!}, \quad t \in [a, b].$$

So under the assumptions of Theorem 2.2, we have

$$(2.1) \quad f(x) = f(y) \frac{u_0(x)}{u_0(y)} + \sum_{i=1}^n L_i f(y) g_i(x, y) + \int_y^x g_n(x, t) L_{n+1} f(t) dt, \quad \forall x, y \in [a, b].$$

If  $u_0(x) = c > 0$ , then

$$(2.2) \quad f(x) = f(y) + \sum_{i=1}^n L_i f(y) g_i(x, y) + \int_y^x g_n(x, t) L_{n+1} f(t) dt, \quad \forall x, y \in [a, b].$$



Title Page

Contents



Page 7 of 25

Go Back

Full Screen

Close

We call  $L_i$  the *generalized Widder-type derivative*.

We need

*Notation 1.* Let  $A$  be a *spherical shell*  $\subseteq \mathbb{R}^N$ ,  $N \geq 1$ , i.e.  $A := B(0, R_2) - \overline{B(0, R_1)}$ ,  $0 < R_1 < R_2$ .

Here the ball  $B(0, R) := \{x \in \mathbb{R}^N : |x| < R\}$ ,  $R > 0$ , where  $|\cdot|$  is the Euclidean norm, also  $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$  is the unit sphere in  $\mathbb{R}^N$  with surface area  $\omega_N := \frac{2\pi^{N/2}}{\Gamma(N/2)}$ . For  $x \in \mathbb{R}^N - \{0\}$  one can write uniquely  $x = r\omega$ , where  $r > 0$ ,  $\omega \in S^{N-1}$ .

Let  $f \in C^{n+1}(\overline{A})$ ,  $n \geq 0$ . If  $f$  is radial, i.e.,  $f(x) = g(r)$ , where  $r = |x|$ ,  $R_1 \leq r \leq R_2$ , then  $g \in C^{n+1}([R_1, R_2])$ .

For radial  $f$  define

$$(2.3) \quad \theta_i f(x) := L_i g(r), \quad \text{all } i = 1, \dots, n+1, \quad \forall x \in \overline{A}.$$

Here

$$g^{(i)}(r) = \frac{\partial^i f(x)}{\partial r^i}, \quad i = 1, \dots, n+1.$$

For  $F \in C(\overline{A})$  we have

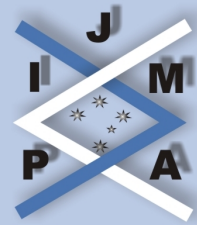
$$(2.4) \quad \int_A F(x) dx = \int_{S^{N-1}} \left( \int_{R_1}^{R_2} F(r\omega) r^{N-1} dr \right) d\omega.$$

We notice that

$$(2.5) \quad \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} s^{N-1} ds = 1,$$

and

$$\text{Vol}(A) = \frac{\omega_N (R_2^N - R_1^N)}{N}.$$



Title Page

Contents



Page 8 of 25

Go Back

Full Screen

Close

### 3. Results on the Shell

*Remark 1.* Here, let  $u_0, u_1, \dots, u_n \in C^{n+1}([R_1, R_2])$ , and  $W_0, W_1, \dots, W_n > 0$  on  $[R_1, R_2]$ ,  $0 < R_1 < R_2$ ,  $n \geq 0$  integer, with  $u_0(r) = c > 0$ .

Let also  $f \in C^{n+1}(\bar{A})$ . We assume first that  $f$  is radial, i.e. there exists  $g$  such that  $f(x) = \underline{g}(r)$ ,  $r = |x|$ ,  $R_1 \leq r \leq R_2$ . Clearly  $g \in C^{m+1}([R_1, R_2])$ .

Let  $x \in \bar{A}$ . Then by using the polar method (2.4) we obtain

$$(3.1) \quad E(x) := \left| f(x) - \frac{\int_A f(y) dy}{Vol(A)} \right| = \left| f(x) - \frac{N \int_{S^{N-1}} \left( \int_{R_1}^{R_2} f(s\omega) s^{N-1} ds \right) d\omega}{\omega_N (R_2^N - R_1^N)} \right|$$

$$(3.2) \quad = \left| g(r) - \frac{N \int_{S^{N-1}} \left( \int_{R_1}^{R_2} g(s) s^{N-1} ds \right) d\omega}{\omega_N (R_2^N - R_1^N)} \right|$$

$$(3.3) \quad = \left| g(r) - \left( \frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s) s^{N-1} ds \right|$$

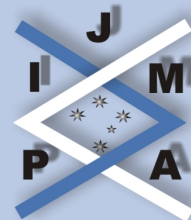
$$(3.4) \quad = \left| \left( \frac{N}{R_2^N - R_1^N} \right) \left( \int_{R_1}^{R_2} g(r) s^{N-1} ds - \int_{R_1}^{R_2} g(s) s^{N-1} ds \right) \right|$$

$$(3.5) \quad = \left( \frac{N}{R_2^N - R_1^N} \right) \left| \int_{R_1}^{R_2} (g(r) - g(s)) s^{N-1} ds \right| =: (*).$$

Let  $s, r \in [R_1, R_2]$ , then by generalized Taylor's formula (2.2) we get

$$(3.6) \quad g(s) - g(r) = \sum_{i=1}^n L_i g(r) g_i(s, r) + R_n(r, s),$$





[Title Page](#)

[Contents](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 9 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

where

$$(3.7) \quad R_n(r, s) := \int_r^s g_n(s, t) L_{n+1} g(t) dt.$$

But it holds

$$(3.8) \quad |R_n(r, s)| \leq \left| \int_r^s |g_n(s, t)| dt \right| \|L_{n+1} g\|_{\infty, [R_1, R_2]}.$$

By calling

$$(3.9) \quad N_n(r, s) := \left| \int_r^s |g_n(s, t)| dt \right|, \quad \forall s, r \in [R_1, R_2],$$

we get

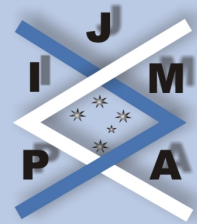
$$(3.10) \quad |R_n(r, s)| \leq N_n(r, s) \|L_{n+1} g\|_{\infty, [R_1, R_2]}, \quad \forall s, r \in [R_1, R_2].$$

Therefore by (3.5), (3.6), we have

$$(3.11) \quad (*) = \left( \frac{N}{R_2^N - R_1^N} \right) \left| \int_{R_1}^{R_2} \left[ \sum_{i=1}^n L_i g(r) g_i(s, r) + R_n(r, s) \right] s^{N-1} ds \right|$$

$$\leq \left( \frac{N}{R_2^N - R_1^N} \right) \left[ \sum_{i=1}^n \left| \int_{R_1}^{R_2} L_i g(r) g_i(s, r) s^{N-1} ds \right| \right. \\ \left. + \int_{R_1}^{R_2} |R_n(r, s)| s^{N-1} ds \right]$$

$$(3.12) \quad \text{by (3.10)} \leq \left( \frac{N}{R_2^N - R_1^N} \right) \left[ \sum_{i=1}^n |L_i g(r)| \left| \int_{R_1}^{R_2} g_i(s, r) s^{N-1} ds \right| \right. \\ \left. + (\|L_{n+1} g\|_{\infty, [R_1, R_2]}) \int_{R_1}^{R_2} N_n(r, s) s^{N-1} ds \right].$$



[Title Page](#)

[Contents](#)

◀ ▶

◀ ▶

Page 10 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

We have established the following result.

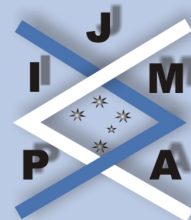
**Theorem 3.1.** Let  $u_0, u_1, \dots, u_n \in C^{n+1}([R_1, R_2])$ ,  $W_0, W_1, \dots, W_n > 0$  on  $[R_1, R_2]$ ,  $0 < R_1 < R_2$ ,  $n \geq 0$  integer, with  $u_0(r) = c > 0$ . Let  $f \in C^{n+1}(\bar{A})$  be radial, i.e. there exists  $g$  such that  $f(x) = g(r)$ ,  $r = |x|$ ,  $R_1 \leq r \leq R_2$ ,  $x \in \bar{A}$ .

Then

$$\begin{aligned}
 E(x) &:= \left| f(x) - \frac{\int_A f(y) dy}{Vol(A)} \right| \\
 &= \left| g(r) - \left( \frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s) s^{N-1} ds \right| \\
 &\leq \left( \frac{N}{R_2^N - R_1^N} \right) \left[ \sum_{i=1}^n |L_i g(r)| \left| \int_{R_1}^{R_2} g_i(s, r) s^{N-1} ds \right| \right. \\
 (3.13) \quad &\quad \left. + (\|L_{n+1} g\|_{\infty, [R_1, R_2]}) \int_{R_1}^{R_2} N_n(r, s) s^{N-1} ds \right] \\
 &= \left( \frac{N}{R_2^N - R_1^N} \right) \left[ \sum_{i=1}^n |\theta_i f(x)| \left| \int_{R_1}^{R_2} g_i(s, |x|) s^{N-1} ds \right| \right. \\
 &\quad \left. + (\|\theta_{n+1} f\|_{\infty, \bar{A}}) \int_{R_1}^{R_2} N_n(|x|, s) s^{N-1} ds \right].
 \end{aligned}$$

**Corollary 3.2.** Let the conditions of Theorem 3.1 hold. Assume further that  $L_i g(r_0) = 0$ ,  $i = 1, \dots, n$ , for a fixed  $r_0 \in [R_1, R_2]$ . For all  $x_0 = r_0 \omega \in \bar{A}$ ,  $\omega \in S^{N-1}$ , we have

$$E(x_0) = \left| f(x_0) - \frac{\int_A f(y) dy}{Vol(A)} \right|$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 11 of 25

Go Back

Full Screen

Close

$$\begin{aligned}
 &= \left| g(r_0) - \left( \frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s) s^{N-1} ds \right| \\
 &\leq \left( \frac{N}{R_2^N - R_1^N} \right) (\|L_{n+1}g\|_{\infty, [R_1, R_2]}) \left( \int_{R_1}^{R_2} N_n(r_0, s) s^{N-1} ds \right) \\
 (3.14) \quad &= \left( \frac{N}{R_2^N - R_1^N} \right) (\|\theta_{n+1}f\|_{\infty, \bar{A}}) \left( \int_{R_1}^{R_2} N_n(|x_0|, s) s^{N-1} ds \right) ;
 \end{aligned}$$

Interesting cases also arise when  $r_0 = R_1$  or  $R_2$ .

We continue Remark 1 with

*Remark 2.* Now let  $f \in C^{n+1}(\bar{A})$ ,  $n \geq 0$ ,  $x \in \bar{A}$ ,  $x = r\omega$ ,  $r > 0$ . Clearly for fixed  $\omega \in S^{N-1}$ , since the function  $f(r\omega)$ ,  $r \in [R_1, R_2]$  is radial, it also belongs to  $C^{n+1}([R_1, R_2])$ .

By applying the internal inequality (3.13) we get

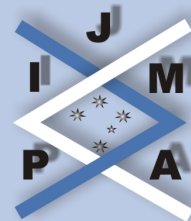
$$\begin{aligned}
 (3.15) \quad &\left| f(r\omega) - \left( \frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} f(s\omega) s^{N-1} ds \right| \\
 &\leq \left( \frac{N}{R_2^N - R_1^N} \right) \left[ \sum_{i=1}^n |(L_i f(\cdot\omega))(r)| \left| \int_{R_1}^{R_2} g_i(s, r) s^{N-1} ds \right| \right. \\
 &\quad \left. + (\|L_{n+1}f(\cdot\omega)\|_{\infty, [R_1, R_2]}) \int_{R_1}^{R_2} N_n(r, s) s^{N-1} ds \right].
 \end{aligned}$$

For non-radial  $f$  we define again

$$(3.16) \quad \theta_i f(x) = \theta_i f(r\omega) := (L_i f(\cdot\omega))(r), \quad \text{all } i = 1, \dots, n+1, \quad \forall x \in \bar{A}.$$

Here the involved

$$\frac{\partial^i f(r\omega)}{\partial r^i} = \frac{\partial^i f(x)}{\partial r^i}, \quad i = 1, \dots, n+1,$$



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 12 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

are the radial derivatives. In a sense  $\theta_i$  is a generalized radial derivative of Widder-type.

Hence

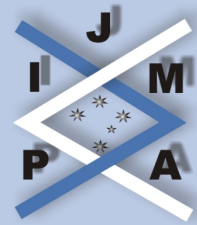
$$(3.17) \quad R.H.S.(3.15) \leq \left( \frac{N}{R_2^N - R_1^N} \right) \left[ \sum_{i=1}^n |\theta_i f(r\omega)| \left| \int_{R_1}^{R_2} g_i(s, r) s^{N-1} ds \right| + \|\theta_{n+1} f\|_{\infty, \bar{A}} \int_{R_1}^{R_2} N_n(|x|, s) s^{N-1} ds \right].$$

Therefore, by (3.15) and (3.17) we have

$$(3.18) \quad \left| \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{N/2}} \int_{S^{N-1}} f(r\omega) d\omega - \frac{1}{Vol(\bar{A})} \int_{\bar{A}} f(y) dy \right| \leq \left( \frac{N}{R_2^N - R_1^N} \right) \left[ \frac{\Gamma(N/2)}{2\pi^{N/2}} \left\{ \sum_{i=1}^n \left( \int_{S^{N-1}} |\theta_i f(r\omega)| d\omega \right) \left| \int_{R_1}^{R_2} g_i(s, r) s^{N-1} ds \right| + \|\theta_{n+1} f\|_{\infty, \bar{A}} \int_{R_1}^{R_2} N_n(|x|, s) s^{N-1} ds \right\} \right].$$

We have established the following.

**Theorem 3.3.** Let  $u_0, u_1, \dots, u_n \in C^{n+1}([R_1, R_2])$ ;  $W_0, W_1, \dots, W_n > 0$  on  $[R_1, R_2]$ ,  $0 < R_1 < R_2$ ,  $n \geq 0$  an integer, with  $u_0(r) = c > 0$ . Let  $f \in C^{n+1}(\bar{A})$ ,  $x \in \bar{A}$ ;  $x = r\omega$ ,  $r \in [R_1, R_2]$ ,  $\omega \in S^{N-1}$ .



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 13 of 25

Go Back

Full Screen

Close

Then

$$(3.19) \quad E(x) = \left| f(x) - \frac{\int_A f(y) dy}{Vol(A)} \right| \leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right) \int_{s^{N-1}} f(r\omega) d\omega}{2\pi^{N/2}} \right|$$

$$+ \left( \frac{N}{R_2^N - R_1^N} \right) \left[ \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{N/2}} \left\{ \sum_{i=1}^n \left( \int_{s^{N-1}} |\theta_i f(r\omega)| d\omega \right) \left| \int_{R_1}^{R_2} g_i(s, r) s^{N-1} ds \right| \right\} \right.$$

$$\left. + \|\theta_{n+1} f\|_{\infty, \bar{A}} \int_{R_1}^{R_2} N_n(|x|, s) s^{N-1} ds \right], \quad \forall x \in \bar{A}.$$

**Corollary 3.4.** *Let the conditions of Theorem 3.3 hold. Assume that*

$$\theta_i f(r_0 \omega) = 0, \quad i = 1, \dots, n, \quad \text{for a fixed } r_0 \in [R_1, R_2], \quad \forall \omega \in S^{N-1};$$

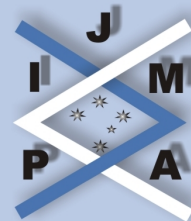
*also consider all*  $x_0 = r_0 \omega \in \bar{A}$  *for any*  $\omega \in S^{N-1}$ . *Then*

$$(3.20) \quad E(x_0) = \left| f(x_0) - \frac{\int_A f(y) dy}{Vol(A)} \right|$$

$$\leq \left| f(x_0) - \frac{\Gamma\left(\frac{N}{2}\right) \int_{S^{N-1}} f(r_0 \omega) d\omega}{2\pi^{N/2}} \right|$$

$$+ \left( \frac{N}{R_2^N - R_1^N} \right) \|\theta_{n+1} f\|_{\infty, \bar{A}} \left( \int_{R_1}^{R_2} N_n(|x_0|, s) s^{N-1} ds \right).$$

When  $r_0 = R_1$  or  $R_2$  is of special interest.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 14 of 25

Go Back

Full Screen

Close

## 4. Results on the Sphere

*Notation 2.* Let  $N \geq 1$ ,  $B(0, R) := \{x \in \mathbb{R}^N : |x| < R\}$  be the ball in  $\mathbb{R}^N$  centered at the origin and of radius  $R > 0$ . Note that  $Vol(B(0, R)) = \frac{\omega_N R^N}{N}$ .

Let  $f$  from  $\overline{B(0, R)}$  into  $\mathbb{R}$  and consider  $f$  to be radial, i.e.  $f(x) = g(r)$ , where  $r = |x|$ ,  $0 \leq r \leq R$ . We assume  $g \in C^{n+1}([0, R])$ ,  $n \geq 0$ . Clearly then  $f \in C(\overline{B(0, R)})$ .

For  $F \in C(\overline{B(0, R)})$  we have

$$(4.1) \quad \int_{B(0, R)} F(x) dx = \int_{S^{N-1}} \left( \int_0^R F(r\omega) r^{N-1} dr \right) d\omega.$$

We notice that

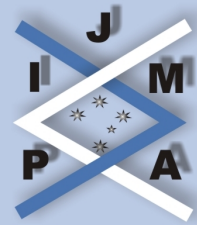
$$(4.2) \quad \frac{N}{R^N} \int_0^R s^{N-1} ds = 1.$$

The operator  $\theta_i$  in the radial case, is as defined in (2.3), now  $\forall x \in \overline{B(0, R)}$ . In the non-radial case, for  $f \in C^{n+1}(\overline{B(0, R)})$ ,  $\theta_i$  is defined as in (3.16),  $\forall x \in \overline{B(0, R)} - \{0\}$ .

We make the following remark.

*Remark 3.* Here, let  $u_0, u_1, \dots, u_n \in C^{n+1}([0, R])$  and  $W_0, W_1, \dots, W_n > 0$  on  $[0, R]$ ,  $R > 0$ ,  $n \geq 0$  an integer, with  $u_0(r) = c > 0$ .

We again first assume that  $f$  is radial on  $\overline{B(0, R)}$ , i.e. there exists  $g$  such that  $f(x) = g(r)$ ,  $r = |x|$ ,  $0 \leq r \leq R$ . Assume further that  $g \in C^{n+1}([0, R])$ . Let



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 15 of 25

Go Back

Full Screen

Close

$x \in \overline{B(0, R)}$ . Then by using the polar method (4.1) we obtain

$$\begin{aligned} (4.3) \quad E(x) &:= \left| f(x) - \frac{\int_{B(0,R)} f(y) dy}{Vol(B(0, R))} \right| \\ &= \left| g(r) - \frac{N \int_{S^{N-1}} \left( \int_0^R g(s) s^{n-1} ds \right) d\omega}{\omega_N R^N} \right| \\ &= \left| g(r) - \frac{N}{R^N} \int_0^R g(s) s^{n-1} ds \right| \\ (4.4) \quad &\stackrel{\text{by (4.2)}}{=} \frac{N}{R^N} \left| \int_0^R (g(r) - g(s)) s^{N-1} ds \right| =: (*). \end{aligned}$$

Let  $s, r \in [0, R]$ , then by the generalized Taylor's formula (2.2) we get

$$(4.5) \quad g(s) - g(r) = \sum_{i=1}^n L_i g(r) g_i(s, r) + R_n(r, s),$$

where

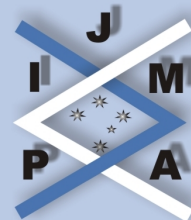
$$(4.6) \quad R_n(r, s) := \int_r^s g_n(s, t) L_{n+1} g(t) dt.$$

But it holds that

$$(4.7) \quad |R_n(r, s)| \leq \left| \int_r^s |g_n(s, t)| dt \right| \|L_{n+1} g\|_{\infty, [0, R]}.$$

By calling

$$(4.8) \quad N_n(r, s) := \left| \int_r^s |g_n(s, t)| dt \right|, \quad \forall s, r \in [0, R],$$



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 16 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

we get

$$(4.9) \quad |R_n(r, s)| \leq N_n(r, s) \|L_{n+1}g\|_{\infty, [0, R]}, \quad \forall s, r \in [0, R].$$

Therefore by (4.4) and (4.5), we have

$$(4.10) \quad \begin{aligned} (*) &= \frac{N}{R^N} \left| \int_0^R \left[ \sum_{i=1}^n L_i g(r) g_i(s, r) + R_n(r, s) \right] s^{N-1} ds \right| \\ &\leq \frac{N}{R^N} \left[ \sum_{i=1}^n \left| \int_0^R L_i g(r) g_i(s, r) s^{N-1} ds \right| + \int_0^R |R_n(r, s)| s^{N-1} ds \right] \end{aligned}$$

$$(4.11) \quad \begin{aligned} &\text{by (4.9)} \leq \frac{N}{R^N} \left[ \sum_{i=1}^n |L_i g(r)| \left| \int_0^R g_i(s, r) s^{N-1} ds \right| \right. \\ &\quad \left. + (\|L_{n+1}g\|_{\infty, [0, R]}) \int_0^R N_n(r, s) s^{N-1} ds \right]. \end{aligned}$$

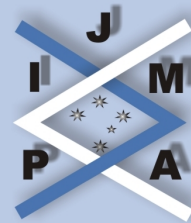
We have established the next result.

**Theorem 4.1.** *Let  $u_0, u_1, \dots, u_n \in C^{n+1}([0, R])$ ,  $W_0, W_1, \dots, W_n > 0$  on  $[0, R]$ ,  $R > 0$ ,  $n \geq 0$  an integer, with  $u_0(r) = c > 0$ . Let  $f$  from  $\overline{B(0, R)}$  into  $\mathbb{R}$  be radial, i.e. there exists  $g$  such that  $f(x) = g(r)$ ,  $r = |x|$ ,  $0 \leq r \leq R$ ,  $\forall x \in \overline{B(0, R)}$ ; further assume that  $g \in C^{n+1}([0, R])$ .*

Then

$$\begin{aligned} E(x) &:= \left| f(x) - \frac{\int_{B(0, R)} f(y) dy}{Vol(B(0, R))} \right| \\ &= \left| g(r) - \frac{N}{R^N} \int_0^R g(s) s^{N-1} ds \right| \end{aligned}$$





Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 17 of 25

Go Back

Full Screen

Close

$$\begin{aligned}
 &\leq \frac{N}{R^N} \left[ \sum_{i=1}^n |L_i g(r)| \left| \int_0^R g_i(s, r) s^{N-1} ds \right| \right. \\
 &\quad \left. + (\|L_{n+1} g\|_{\infty, [0, R]}) \int_0^R N_n(r, s) s^{N-1} ds \right] \\
 &= \frac{N}{R^N} \left[ \sum_{i=1}^n |\theta_i f(x)| \left| \int_0^R g_i(s, |x|) s^{N-1} ds \right| \right. \\
 (4.12) \quad &\quad \left. + (\|\theta_{n+1} f\|_{\infty, \overline{B(0, R)}}) \int_0^R N_n(|x|, s) s^{N-1} ds \right].
 \end{aligned}$$

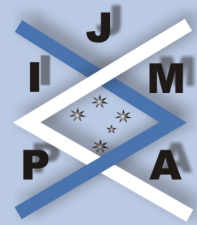
The following corollary holds.

**Corollary 4.2.** *Let the conditions of Theorem 4.1 hold. Assume further that  $L_i g(r_0) = 0$ ,  $i = 1, \dots, n$ , for a fixed  $r_0 \in [0, R]$ ; consider all  $x_0 = r_0 \omega \in \overline{B(0, R)}$  and any  $\omega \in S^{N-1}$ .*

Then

$$\begin{aligned}
 E(x_0) &:= \left| f(x_0) - \frac{\int_{B(0, R)} f(y) dy}{Vol(B(0, R))} \right| \\
 &= \left| g(r_0) - \frac{N}{R^N} \int_0^R g(s) s^{n-1} ds \right| \\
 &\leq \frac{N}{R^N} (\|L_{n+1} g\|_{\infty, [0, R]}) \left( \int_0^R N_n(r_0, s) s^{N-1} ds \right) \\
 (4.13) \quad &= \frac{N}{R^N} (\|\theta_{n+1} f\|_{\infty, \overline{B(0, R)}}) \left( \int_0^R N_n(|x_0|, s) s^{N-1} ds \right).
 \end{aligned}$$

Interesting cases especially arise when  $r_0 = 0$  or  $R$ .



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 18 of 25

Go Back

Full Screen

Close

We continue Remark 3 with

*Remark 4.* Let  $f$  be non-radial. Here assume that  $f \in C^{n+1}(\overline{B(0, R)})$ . Consider  $x \in \overline{B(0, R)} - \{0\}$ , which is written uniquely as  $x = r\omega$ ,  $r \in (0, R]$ ,  $\omega \in S^{N-1}$ . Then by using again the polar method (4.1) we obtain

$$(4.14) \quad \left| \frac{\int_{S^{N-1}} f(r\omega) d\omega}{\omega_N} - \frac{\int_{B(0, R)} f(y) dy}{Vol(B(0, R))} \right|$$

$$(4.15) \quad = \left| \frac{\int_{S^{N-1}} f(r\omega) d\omega}{\omega_N} - \frac{N \int_{S^{N-1}} \left( \int_0^R f(s\omega) s^{N-1} ds \right) d\omega}{\omega_N R^N} \right|$$

$$(4.16) \quad \begin{aligned} & \stackrel{\text{by (4.2)}}{=} \left| \frac{N}{\omega_N R^N} \left( \int_{S^{N-1}} \left( \int_0^R f(r\omega) s^{N-1} ds \right) d\omega \right) \right. \\ & \quad \left. - \frac{N}{\omega_N R^N} \left( \int_{S^{N-1}} \left( \int_0^R f(s\omega) s^{N-1} ds \right) d\omega \right) \right| \\ & = \frac{N}{\omega_N R^N} \left| \int_{S^{N-1}} \left( \int_0^R (f(r\omega) - f(s\omega)) s^{N-1} ds \right) d\omega \right| =: (*). \end{aligned}$$

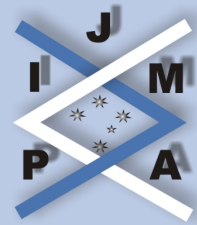
Clearly here  $f(\cdot\omega) \in C^{n+1}((0, R])$ .

Let  $\rho \in (0, R]$ , then by (2.2) we get

$$(4.17) \quad f(\rho\omega) - f(r\omega) = \sum_{i=1}^n ((L_i(f(\cdot\omega)))(r)) g_i(\rho, r) + R_n(r, \rho),$$

where

$$(4.18) \quad R_n(r, \rho) := \int_r^\rho g_n(\rho, t) (L_{n+1}(f(\cdot\omega)))(t) dt.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 19 of 25

Go Back

Full Screen

Close

That is

$$(4.19) \quad f(\rho\omega) - f(r\omega) = \sum_{i=1}^n (\theta_i f(r\omega)) g_i(\rho, r) + R_n(r, \rho),$$

with

$$(4.20) \quad R_n(r, \rho) = \int_r^\rho g_n(\rho, t) \theta_{n+1} f(t\omega) dt.$$

We further assume that

$$(4.21) \quad \theta := \|\theta_{n+1} f\|_{\infty, \overline{B(0, R)} - \{0\}} < +\infty.$$

Therefore we obtain

$$(4.22) \quad |R_n(r, \rho)| \leq \left| \int_r^\rho |g_n(\rho, t)| dt \right| \theta.$$

By calling

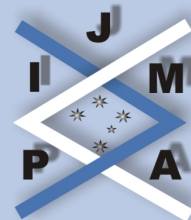
$$(4.23) \quad N_n(r, \rho) := \left| \int_r^\rho |g_n(\rho, t)| dt \right|, \quad \forall \rho, r \in [0, R],$$

we get

$$(4.24) \quad |R_n(r, \rho)| \leq N_n(r, \rho) \theta.$$

Hence by (4.19) we obtain

$$(4.25) \quad \begin{aligned} |f(\rho\omega) - f(r\omega)| &\leq \sum_{i=1}^n |\theta_i f(r\omega)| |g_i(\rho, r)| + |R_n(r, \rho)| \\ &\leq \sum_{i=1}^n |\theta_i f(r\omega)| |g_i(\rho, r)| + N_n(r, \rho) \theta. \end{aligned}$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 20 of 25

Go Back

Full Screen

Close

By the continuity of  $f$  and  $g_i$ ,  $i = 1, \dots, n$ , and by taking the limit as  $\rho \rightarrow 0$  in the external inequality (4.25), we obtain

$$(4.26) \quad |f(0) - f(r\omega)| \leq \sum_{i=1}^n |\theta_i f(r\omega)| |g_i(0, r)| + N_n(r, 0)\theta.$$

Notice here that  $g_n(\rho, t)$  is jointly continuous in  $(\rho, t) \in [0, R]^2$ , hence  $N_n(r, \rho)$  is continuous in  $\rho \in [0, R]$ .

That is,  $\forall s \in [0, R]$  we get

$$(4.27) \quad |f(s\omega) - f(r\omega)| \leq \sum_{i=1}^n |\theta_i f(r\omega)| |g_i(s, r)| + N_n(r, s)\theta.$$

Consequently by (4.16) and (4.27) we find

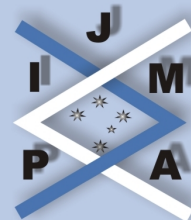
$$(4.28) \quad (*) \leq \frac{N}{\omega_N R^N} \int_{S^{N-1}} \left( \int_0^R |f(s\omega) - f(r\omega)| s^{N-1} ds \right) d\omega$$

$$(4.29) \quad \leq \frac{N}{\omega_N R^N} \left[ \sum_{i=1}^n \left( \int_{S^{N-1}} \left( \int_0^R |\theta_i f(r\omega)| |g_i(s, r)| s^{N-1} ds \right) d\omega \right) \right.$$

$$\left. + \theta \int_{S^{N-1}} \left( \int_0^R N_n(r, s) s^{N-1} ds \right) d\omega \right]$$

$$= \frac{\sum_{i=1}^n \left( \int_{S^{N-1}} \left( \int_0^R |\theta_i f(r\omega)| |g_i(s, r)| s^{N-1} ds \right) d\omega \right)}{\text{Vol}(B(0, R))}$$

$$(4.30) \quad + \frac{\theta N \left( \int_0^R N_n(r, s) s^{N-1} ds \right)}{R^N}$$



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 21 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

$$(4.31) \quad = \sum_{i=1}^N \left( \frac{\left( \int_{S^{N-1}} |\theta_i f(r\omega)| d\omega \right) \left( \int_0^R |g_i(s, r)| s^{N-1} ds \right)}{\text{Vol}(B(0, R))} \right) + \frac{\theta N \left( \int_0^R N_n(r, s) s^{N-1} ds \right)}{R^N}.$$

That is

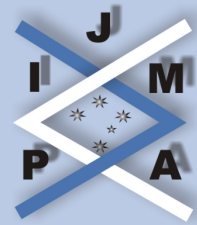
$$(4.32) \quad \Delta := \left| \frac{\int_{S^{N-1}} f(r\omega) d\omega}{\omega_N} - \frac{\int_{B(0, R)} f(y) dy}{\text{Vol}(B(0, R))} \right| \leq \sum_{i=1}^n \left[ \frac{\left( \int_{S^{N-1}} |\theta_i f(r\omega)| d\omega \right) \left( \int_0^R |g_i(s, r)| s^{N-1} ds \right)}{\text{Vol}(V(0, R))} \right] + \frac{\theta N \left( \int_0^R N_n(r, s) s^{N-1} ds \right)}{R^N}.$$

The following theorem holds.

**Theorem 4.3.** Let  $u_0, u_1, \dots, u_n \in C^{n+1}([0, R])$ ,  $W_0, W_1, \dots, W_n > 0$  on  $[0, R]$ ,  $R > 0$ ,  $n \geq 0$  an integer, with  $u_0(r) = c > 0$ . Let  $f \in C^{n+1}(\overline{B(0, R)})$ . Assume that

$$(4.33) \quad \theta := \|\theta_{n+1} f\|_{\infty, \overline{B(0, R)} - \{0\}} < +\infty.$$

Let  $x \in \overline{B(0, R)} - \{0\}$ , which is written uniquely as  $x = r\omega$ ,  $r \in (0, R]$ ,  $\omega \in S^{N-1}$ .



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 22 of 25

Go Back

Full Screen

Close

Then

$$\begin{aligned}
 E(x) &:= \left| f(x) - \frac{\int_{B(0,R)} f(y) dy}{Vol(B(0,R))} \right| \\
 &\leq \left| f(x) - \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{S^{N-1}} f(r\omega') d\omega' \right| \\
 &\quad + \sum_{i=1}^n \left[ \frac{\left( \int_{S^{N-1}} |\theta_i f(r\omega')| d\omega' \right) \left( \int_0^R |g_i(s,r)| s^{N-1} ds \right)}{Vol(B(0,R))} \right] \\
 &\quad + \frac{\theta N \left( \int_0^R N_n(r,s) s^{N-1} ds \right)}{R^N}.
 \end{aligned}
 \tag{4.34}$$

We finally give the following corollary.

**Corollary 4.4.** *Let the conditions of Theorem 4.3 hold. Assume further that  $\theta_i f(r_0\omega') = 0, \forall \omega'^{N-1}$ , for some  $r_0 \in (0, R]$ , for all  $i = 1, \dots, n$ . Consider all  $x_0 = r_0\omega \in B(0, R) - \{0\}$ , and any  $\omega \in S^{N-1}$ .*

Then

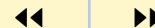
$$\begin{aligned}
 E(x_0) &:= \left| f(x_0) - \frac{\int_{B(0,r)} f(y) dy}{Vol(B(0,R))} \right| \\
 &\leq \left| f(x_0) - \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{S^{N-1}} f(|x_0|\omega') d\omega' \right| \\
 &\quad + \frac{\theta N \left( \int_0^R N_n(|x_0|, s) s^{N-1} ds \right)}{R^N}.
 \end{aligned}
 \tag{4.35}$$

An interesting case is when  $r_0 = R$ .



Title Page

Contents



Page 23 of 25

Go Back

Full Screen

Close

## 5. Addendum

We give

**Proposition 5.1.** Let  $f \in C^1(\overline{B(0, R)})$  such that  $f$  is radial, i.e.  $f(x) = g(r)$ ,  $r = |x|$ ,  $0 \leq r \leq R$ ,  $\forall x \in \overline{B(0, R)}$ . Then

(i)  $\exists g' \in C((0, R])$ ,

(ii)  $\exists g'(0) = 0$ .

*Proof.* (i) is obvious.

(ii) Let  $u$  be a unit vector,  $h > 0$ , then

$$\frac{g(h) - g(0)}{h} = \frac{f(hu) - f(0)}{h}$$

$$\begin{aligned} \text{(by first order multivariate Taylor's formula)} &= \frac{\nabla f(0) \cdot hu + o(h)}{h} \\ &= \frac{\nabla f(0) \cdot hu}{h} + \frac{o(h)}{h} = \nabla f(0) \cdot u + \frac{o(h)}{h} \\ &\xrightarrow{h \rightarrow 0} \nabla f(0) \cdot u, \text{ by } \frac{o(h)}{h} \rightarrow 0. \end{aligned}$$

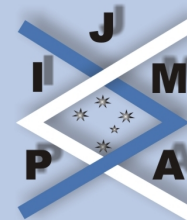
The last is true for any unit vector  $u$ . Thus  $\nabla f(0) = 0$ , proving the claim.  $\square$

By Proposition 5.1, we see that, it may well be that  $g'$  is discontinuous at zero, if only  $f \in C^1(\overline{B(0, R)})$ .

Therefore the assumption that  $g \in C^{n+1}([0, R])$  in Theorem 4.1 seems to be the best.

## References

- [1] G.A. ANASTASSIOU, Ostrowski type inequalities, *Proc. AMS*, **123** (1995), 3775–3781.
- [2] G.A. ANASTASSIOU, *Quantitative Approximations*, Chapman & Hall/CRC, Boca Raton, New York, 2001.
- [3] G.A. ANASTASSIOU, Multivariate Ostrowski type inequalities, *Acta Math. Hungarica*, **76**(4) (1997), 267–278.
- [4] G.A. ANASTASSIOU, Ostrowski type inequalities over spherical shells, accepted in *Serdica*.
- [5] G.A. ANASTASSIOU AND J.A. GOLDSTEIN, Ostrowski type inequalities over Euclidean domains, *Rend. Lincei Mat. Appl.*, **18** (2007), 305–310.
- [6] G.A. ANASTASSIOU AND J.A. GOLDSTEIN, Higher order Ostrowski type inequalities over Euclidean domains, *J. Math. Anal. and Applics.*, **337** (2008), 962–968.
- [7] N.S. BARNETT AND S.S. DRAGOMIR, An Ostrowski type inequality for double integrals and applications for cubature formulae, *Soochow J. Math.*, **27**(1) (2001), 1–10.
- [8] P. CERONE, Approximate multidimensional integration through dimension reduction via the Ostrowski functional, *Nonlinear Funct. Anal. Appl.*, **8**(3) (2003), 313–333.
- [9] S.S. DRAGOMIR, N.S. BARNETT AND P. CERONE, An  $n$ -dimensional version of Ostrowski's inequality for mappings of the Hölder type, *Kyungpook Math. J.*, **40**(1) (2000), 65–75.



---

Ostrowski Type Inequalities via a  
Taylor-Widder Formula

George A. Anastassiou

vol. 8, iss. 4, art. 106, 2007

---

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 24 of 25

Go Back

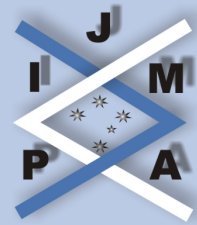
Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756





- [10] S. KARLIN AND W.J. STUDDEN, *Tchebycheff Systems: with Applications in Analysis and Statistics*, Interscience, New York (1966).
- [11] M. MATIC, J. PEČARIĆ AND N. UJEVIĆ, Weighted version of multivariate Ostrowski type inequalities, *Rocky Mountain J. Math.*, **31**(2) (2001), 511–538.
- [12] A. OSTROWSKI, Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helv.*, **10** (1938), 226–227.
- [13] B.G. PACHPATTE, On an inequality of Ostrowski type in three independent variables, *J. Math. Anal. Appl.*, **249**(2) (2000), 583–591.
- [14] B.G. PACHPATTE, An inequality of Ostrowski type in  $n$  independent variables, *Facta Univ. Ser. Math. Inform.*, **16** (2001), 21–24.
- [15] D.V. WIDDER, A generalization of Taylor's series, *Transactions of AMS*, **30**(1) (1928), 126–154.

---

Ostrowski Type Inequalities via a  
Taylor-Widder Formula

George A. Anastassiou

vol. 8, iss. 4, art. 106, 2007

---

Title Page

Contents



Page 25 of 25

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756