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GÂTEAUX DERIVATIVE AND ORTHOGONALITY IN C_p-CLASSES

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ABSTRACT. The general problem in this paper is minimizing the C_p- norm of suitable affine mappings from $B(H)$ to C_p, using convex and differential analysis (Gateaux derivative) as well as input from operator theory. The mappings considered generalize the so-called elementary operators and in particular the generalized derivations, which are of great interest by themselves. The main results obtained characterize global minima in terms of (Banach space) orthogonality, and constitute an interesting combination of infinite-dimensional differential analysis, operator theory and duality. Note that the results obtained generalize all results in the literature concerning operator which are orthogonal to the range of a derivation and the techniques used have not been done by other authors.

Key words and phrases: Elementary operators, Schatten p-classes, orthogonality, Gateaux derivative.

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1. INTRODUCTION

Let E be a complex Banach space. We first define orthogonality in E. We say that $b \in E$ is orthogonal to $a \in E$ if for all complex λ there holds

$$\|a + \lambda b\| \geq \|a\| . \tag{1.1}$$

This definition has a natural geometric interpretation. Namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$ is disjoint with the open ball $K(0, \|a\|)$, i.e., iff this complex line is a tangent one. Note that if b is orthogonal to a, then a need not be orthogonal to b. If E is a Hilbert space, then from (1.1) follows $\langle a, b\rangle = 0$, i.e., orthogonality in the usual sense. Next we define the von Neumann-Schatten classes C_p $(1 \leq p < \infty)$. Let $B(H)$ denote the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space H and let $T \in B(H)$ be compact, and let $s_1(T) \geq s_2(T) \geq \cdots \geq 0$ denote the singular values of T,

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i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator T is said to belong to the Schatten p-classes C_p if

$$\|T\|_p = \left[\sum_{i=1}^{\infty} s_i(T)^p\right]^{\frac{1}{p}} = [tr|T|^p]^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,$$

where tr denotes the trace functional. Hence C_1 is the trace class, C_2 is the Hilbert-Schmidt class, and C_∞ corresponds to the class of compact operators with

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|$$

denoting the usual operator norm. For the general theory of the Schatten p-classes the reader is referred to [16]. Recall (see [16]) that the norm $\|\cdot\|$ of the $B-$space V is said to be Gâteaux differentiable at non-zero elements $x \in V$ if there exists a unique support functional (in the dual space V^*) such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$, satisfying

$$\lim_{\mathbb{R}\ni t\to 0} \frac{\|x+ty\| - \|x\|}{t} = \operatorname{Re} D_x(y),$$

for all $y \in V$. Here $\mathbb{R}$ denotes the set of all reals and Re denotes the real part. The Gâteaux differentiability of the norm at x implies that x is a smooth point of a sphere of radius $\|x\|$.

It is well known (see [6] and the references therein) that for $1 < p < \infty$, C_p is a uniformly convex Banach space. Therefore every non-zero $T \in C_p$ is a smooth point and in this case the support functional of T is given by

$$D_T(X) = tr\left[\frac{|T|^{p-1} UX^*}{\|T\|_p^{p-1}}\right], \tag{1.2}$$

for all $X \in C_p$, where $T = U|T|$ is the polar decomposition of T. The first result concerning the orthogonality in a Banach space was given by Anderson [1] showing that if A is a normal operator on a Hilbert space H, then $AS = SA$ implies that for any bounded linear operator X there holds

$$\|S + AX - XA\| \ge \|S\|. \tag{1.3}$$

This means that the range of the derivation $\delta_A : B(H) \to B(H)$ defined by $\delta_A(X) = AX - XA$ is orthogonal to its kernel. This result has been generalized in two directions: by extending the class of elementary mappings

$$E_{A,B} : B(H) \to B(H); \qquad E_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X$$

and

$$\widetilde{E}_{A,B} : B(H) \to B(H); \qquad \widetilde{E}_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i,$$

where $(A_1, A_2, \ldots, A_n)$ and $(B_1, B_2, \ldots, B_n)$ are $n-$ tuples of bounded operators on H, and by extending the inequality (1.3) to C_p-classes with $1 < p < \infty$ see [3], [6], [9]. The Gâteaux derivative concept was used in [3, 5, 6, 7, 15] and [8], in order to characterize those operators which are orthogonal to the range of a derivation. The main purpose of this note is to characterize the global minimum of the map

$$X \mapsto \|S + \phi(X)\|_{C_p}, \quad \phi \text{ is a linear map in } B(H),$$

in C_p by using the Gateaux derivative. These results are then applied to characterize the operators $S \in C_p$ which are orthogonal to the range of elementary operators. It is very interesting to point out that our Theorem 2.3 and its Corollary 2.6 generalize Theorem 1 in [6] , Lemma 2 in [3] and Theorem 2.1 in [18].

2. MAIN RESULTS

Let $\phi : B(H) \to B(H)$ be a linear map, that is, $\phi(\alpha X + \beta Y) = \alpha\phi(X) + \beta\phi(Y)$, for all $\alpha, \beta \in \mathbb{C}$ and all $X, Y \in B(H)$, and let $S \in C_p$ $(1 < p < \infty)$. Put

$$\mathcal{U} = \{X \in B(H) : \phi(X) \in C_p\}.$$

Let $\psi : \mathcal{U} \to C_p$ be defined by

$$\psi(X) = S + \phi(X).$$

Define the function $F_\psi : \mathcal{U} \to \mathbb{R}^+$ by $F_\psi(X) = \|\psi(X)\|_{C_p}$. Now we are ready to prove our first result in C_p-classes $(1 < p < \infty)$. It gives a necessary and sufficient optimality condition for minimizing F_ψ.

Let X be a Banach space, ϕ a linear map $X \to X$, and $\psi(x) = \phi(x) + s$ for some element $s \in X$. Use the notation

$$D_x(y) = \lim_{t \to 0^+} \frac{1}{t}(\|x + ty\| - \|x\|).$$

It is obvious that D_x is sub-additive and $D_x(y) \le \|y\|$, also $D_x(x) = \|x\|$ and $D_x(-x) = -\|x\|$.

Theorem 2.1. *The map $F_\psi = \|\psi(x)\|$ has a global minimum at $x \in X$ if and only if*

$$D_{\psi(x)}(\phi(y)) \ge 0, \quad \forall y \in X. \tag{2.1}$$

Proof. Necessity is immediate from $\psi(x) + t\phi(y) = \psi(x + ty)$. Sufficency: assume the stated condition and choose y. Note that $\phi(y - x) = \psi(y) - \psi(x)$. For brevity we let $D_{\psi(x)} = L$. Then

$$\begin{aligned} \|\psi(x)\| &= -L(-\psi(x)) \\ &\le -L(-\psi(x)) + L(\psi(y) - \psi(x)) \quad \text{by hypothesis} \\ &\le L(\psi(y)) \quad \text{by sub-additivity} \\ &\le \|\psi(y)\|. \end{aligned}$$

□

Theorem 2.2 ([7]). *Let $X, Y \in C_p$. Then, there holds*

$$D_X(Y) = p\mathrm{Re}\left\{tr(|X|^{p-1}U^*Y)\right\},$$

where $X = U|X|$ is the polar decomposition of X.

The following corollary establishes a characterization of the Gateaux derivative of the norm in C_p-classes $(1 < p < \infty)$. Now we are going to characterize the global minimum of F_ψ on C_p $(1 < p < \infty)$, when ϕ is a linear map satisfying the following useful condition:

$$tr(X\phi(Y)) = tr(\phi^*(X)Y), \quad \forall\, X, Y \in C_p \tag{2.2}$$

where ϕ^* is an appropriate conjugate of the linear map ϕ. We state some examples of ϕ and ϕ^* which satisfy the above condition (2.2).

(1) The elementary operator $\widetilde{E}_{A,B} : \mathcal{I} \to \mathcal{I}$ defined by

$$\widetilde{E}_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i,$$

where $A_i, B_i \in B(H)$ $(1 \leq i \leq n)$ and $\mathcal{I}$ is a separable ideal of compact operators in $B(H)$ associated with some unitarily invariant norm. It is easy to show that the conjugate operator $E^*_{A,B} : \mathcal{I}^* \to \mathcal{I}^*$ of $E_{A,B}$ has the form

$$\widetilde{E}^*_{A,B}(X) = \sum_{i=1}^{n} B_i X A_i,$$

and that the operators $\widetilde{E}_{A,B}$ and $\widetilde{E}^*_{A,B}$ satisfy the condition (2.2).

(2) Using the previous example we can check that the conjugate operator $E^*_{A,B} : \mathcal{I}^* \to \mathcal{I}^*$ of the elementary operator $E_{A,B}$ defined by $E_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X$, has the form

$$E^*_{A,B}(X) = \sum_{i=1}^{n} B_i X A_i - X,$$

and that the operators $E_{A,B}$ and $E^*_{A,B}$ satisfy the condition (2.2).

Now, we are in position to prove the following theorem.

Theorem 2.3. *Let $V \in C_p$, and let $\psi(V)$ have the polar decomposition $\psi(V) = U\,|\psi(V)|$. Then F_ψ has a global minimum on C_p at V if and only if $|\psi(V)|U^* \in \ker \phi^*$.*

Proof. Assume that F_ψ has a global minimum on C_p at V. Then

$$D_{\psi(V)}(\phi(Y)) \geq 0, \tag{2.3}$$

for all $Y \in C_p$. That is,

$$p\mathrm{Re}\left\{tr(|\psi(V)|^{p-1}U^*\phi(Y))\right\} \geq 0, \quad \forall Y \in C_p.$$

This implies that

$$\mathrm{Re}\{tr(|\psi(V)|^{p-1}U^*\phi(Y))\} \geq 0, \quad \forall Y \in C_p. \tag{2.4}$$

Let $f \otimes g$, be the rank one operator defined by $x \mapsto \langle x, f\rangle\, g$ where f, g are arbitrary vectors in the Hilbert space H. Take $Y = f \otimes g$, since the map ϕ satisfies (2.2) one has

$$tr(|\psi(V)|^{p-1}U^*\phi(Y)) = tr(\phi^*(|\psi(V)|^{p-1}U^*)Y).$$

Then (2.4) is equivalent to $\mathrm{Re}\{tr(\phi^*(|\psi(V)|^{p-1}U^*)Y)\} \geq 0$, for all $Y \in C_p$, or equivalently

$$\mathrm{Re}\left\langle \phi^*(|\psi(V)|^{p-1}U^*)g, f\right\rangle \geq 0, \quad \forall f, g \in H.$$

If we choose $f = g$ such that $\|f\| = 1$, we get

$$\mathrm{Re}\left\langle \phi^*(|\psi(V)|^{p-1}U^*)f, f\right\rangle \geq 0. \tag{2.5}$$

Note that the set

$$\{\left\langle \phi^*(|\psi(V)|^{p-1}U^*)f, f\right\rangle :\ \|f\| = 1\}$$

is the numerical range of $\phi^*(|\psi(V)|^{p-1}U^*)$ on $\mathcal{U}$ which is a convex set and its closure is a closed convex set. By (2.5) it must contain one value of positive real part, under all rotation around the origin, it must contain the origin, and we get a vector $f \in H$ such that $\langle \phi^*(|\psi(V)|^{p-1}U^*)f, f\rangle < \epsilon$ where ϵ is positive. Since ϵ is arbitrary, we get $\langle \phi^*(|\psi(V)|^{p-1}U^*)f, f\rangle = 0$. Thus $\phi^*(|\psi(V)|^{p-1}U^*) = 0$, i.e., $|\psi(V)|^{p-1}U^* \in \ker \phi^*$.

Conversely, if $\psi(V)|^{p-1}U^* \in \ker \phi^*$, then $|\psi(V)|^{p-1}U^* \in \ker \phi^*$. It is easily seen (using the same arguments above) that

$$\mathrm{Re}\left\{tr(|\psi(V)|^{p-1}U^*\phi(Y))\right\} \geq 0, \quad \forall Y \in C_p.$$

By this we get (2.3). □

We state our first corollary of Theorem 2.3. Let $\phi = \delta_{A,B}$, where $\delta_{A,B} : B(H) \to B(H)$ is the generalized derivation defined by $\delta_{A,B}(X) = AX - XB$.

Corollary 2.4. *Let $V \in C_p$, and let $\psi(V)$ have the polar decomposition $\psi(V) = U\,|\psi(V)|$. Then F_ψ has a global minimum on C_p at V, if and only if $|\psi(V)|^{p-1}U^* \in \ker \delta_{B,A}$.*

Proof. It is a direct consequence of Theorem 2.3. □

This result may be reformulated in the following form where the global minimum V does not appear. It characterizes the operators S in C_p which are orthogonal to the range of the derivation $\delta_{A,B}$.

Theorem 2.5. *Let $S \in C_p$, and let $\psi(S)$ have the polar decomposition $\psi(S) = U\,|\psi(S)|$. Then*

$$\|\psi(X)\|_{C_p} \geq \|\psi(S)\|_{C_p},$$

for all $X \in C_p$ if and only if $|\psi(S)|^{p-1}U^* \in \ker \delta_{B,A}$.

As a corollary of this theorem we have

Corollary 2.6. *Let $S \in C_p \cap \ker \delta_{A,B}$ have the polar decomposition $S = U\,|S|$. Then the two following assertions are equivalent:*

(1)
$$\|S + (AX - XB)\|_{C_p} \geq \|S\|_{C_p}, \text{ for all } X \in C_p.$$

(2) $|S|^{p-1}U^* \in \ker \delta_{B,A}$.

Remark 2.7. We point out that, thanks to our general results given previously with more general linear maps ϕ, Theorem 2.5 and its Corollary 2.6 are true for the nuclear operator $\Delta_{A,B}(X) = AXB - X$ and other more general classes of operators than $\delta_{A,B}$ such as the elementary operators $E_{A,B}(X)$ and $\widetilde{E}_{A,B}(X)$.

The above corollary generalizes Theorem 1 in [6]and Lemma 2 in [3].

Now by using Theorem 2.5 , Corollary 2.6, Remark 2.7 and the following Lemma 2.8 and Lemma 2.10 we obtain some interesting results see also ([3], [13]). Let $S = U\,|S|$ be the polar decomposition of S.

Lemma 2.8. *Let $A, B \in B(H)$ and $T \in C_p$ such that $\ker \delta_{A,B}(T) \subseteq \ker \delta^*_{A,B}(T)$.*

If $A\,|S|^{p-1}\,U^* = |S|^{p-1}\,U^*B$, where $p > 1$ and $S = U\,|S|$ is the polar decomposition of S, then $A\,|S|\,U^* = |S|\,U^*B$.

Proof. If $T = |S|^{p-1}$, then

$$ATU^* = TU^*B. \tag{2.6}$$

We prove that

$$AT^nU^* = T^nU^*B, \tag{2.7}$$

for all $n \geq 1$. If $S = U\,|S|$, then

$$\ker U = \ker |S| = \ker |S|^{p-1} = \ker T$$

and

$$(\ker U)^\perp = (\ker T)^\perp = \overline{R(T)}.$$

This shows that the projection U^*U onto $(\ker T)^\perp$ satifies $U^*UT = T$ and $TU^*UT = T^2$. By taking the adjoints of (2.6) and since $\ker \delta_{A,B}(T) \subseteq \ker \delta^*_{A,B}(T)$, we get $BUT = UTA$ and

$$AT^2 = ATU^*UT = TU^*BUT = TU^*UTA = T^2A.$$

Since A commutes with the positive operator T^2, A commutes with its square roots, that is,

$$AT = TA \tag{2.8}$$

By (2.6) and (2.8) we obtain (2.7). Let $f(t)$ be the map defined on $\sigma(T) \subset R^+$ by $f(t) = t^{\frac{1}{p-1}}$; $1 < p < \infty$. Since f is the uniform limit of a sequence (P_i) of polynomials without constant term (since $f(0) = 0$), it follows from (2.8) that $AP_i(T)U^* = P_i(T)U^*B$. Therefore $AT^{\frac{1}{p-1}}U^* = U^*T^{\frac{1}{p-1}}B$. □

Theorem 2.9. *Let A, B be operators in $B(H)$ such that $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$. Then $T \in \ker \delta_{A,B} \cap C_p$, if and only if*

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \,,$$

for all $X \in C_p$.

Proof. If $T \in \ker \Delta_{A,B}$ then by applying Theorem 3.4 in [9] it follows that

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \,,$$

for all $X \in C_p$. Conversely, if

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \,,$$

for all $X \in C_p$, then from Corollary 2.6

$$A\,|T|\,U^* = |S|\,U^*B.$$

Since $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$, $B^*\,|T|^{p-1}\,U^* = |T|^{p-1}\,U^*A^*$. By taking adjoints we get $AU\,|T|^{p-1} = U\,|T|^{p-1}\,B$. From Lemma 2.8 it follows that $AU\,|T| = U\,|T|\,B$. i.e., $T \in \ker \delta_{A,B}$. □

Note that the above theorem still holds if we consider $\Delta_{A,B}$ instead of $\delta_{A,B}$.

Let $A = (A_1, A_2, \ldots, A_n)$, $B = (B_1, B_2, \ldots, B_n)$ be $n-$tuples of operators in $B(H)$. In the following Theorem 2.12 we will characterize $T \in C_p$ for $1 < p < \infty$, which are orthogonal to $R(E_{A,B} \mid C_p)$ (the range of $E_{A,B} \mid C_p$) for a general pair of operators A, B. For this let $S = U\,|S|$ be the polar decomposition of S. We start by the following lemma or the case where $E_C = \sum C_iXC_i - X$ which will be used in the proof of Theorem 2.12.

Let $S = U\,|S|$ be the polar decomposition of S.

Lemma 2.10. *Let $C = (C_1, C_2, \ldots, C_n)$ be an $n-$tuple of operators in $B(H)$ such that $\sum_{i=1}^n C_iC_i^* \leq 1$, $\sum_{i=1}^n C_i^*C_i \leq 1$ and $\ker E_C \subseteq \ker E_{C^*}$. If*

$$\sum_{i=1}^{n} C_iU\,|S|^{p-1}\,C_i = U\,|S|^{p-1} \,,$$

where $p > 1$, then

$$\sum_{i=1}^{n} C_iU\,|S|\,C_i = U\,|S| \,.$$

Proof. If $T = |S|^{p-1}$, then

$$\sum_{i=1}^{n} C_iUTC_i = UT. \tag{2.9}$$

We prove that

$$\sum_{i=1}^{n} C_iUT^nC_i = UT^n, \tag{2.10}$$

It is known that if $\sum_{i=1}^n C_iC_i^* \leq 1$, $\sum_{i=1}^n C_i^*C_i \leq 1$ and $\ker E_c \subseteq \ker E_c^*$ that the eigenspaces corresponding to distinct non-zero eigenvalues of the compact positive operator $|S|^2$ reduces each C_i (see [4], Theorem 8), ([18], Lemma 2.3)). In particular, $|S|$ commutes with C_i for all $1 \leq i \leq n$. This implies also that $|S|^{p-1} = T$ commutes with each C_i for all $1 \leq i \leq n$. Hence $C_i |T| = |T| C_i$ and $C_iT^2 = T^2C_i$. □

Since C_i commutes with the positive operator T^2, then C_i commutes with its square roots, that is,

$$C_iT = TC_i. \tag{2.11}$$

By the same arguments used in the proof of Lemma 2.8 the proof of this lemma can be completed.

Theorem 2.11. *Let $C = (C_1, C_2, \ldots, C_n)$ be an $n-$tuple of operators in $B(H)$ such that $\sum_{i=1}^n C_iC_i^* \leq 1$, $\sum_{i=1}^n C_i^*C_i \leq 1$ and $\ker E_C \subseteq \ker E_{C^*}$ then $S \in \ker E_C \cap C_p$ $(1 < p < \infty)$, if and only if,*

$$\|S + E_C(X)\|_p \geq \|S\|_p \,,$$

for all $X \in C_p$.

Proof. If $S \in \ker E_C$ then from ([18], Theorem 2.4) it follows that

$$\|S + E_C(X)\|_p \geq \|S\|_p \,,$$

for all $X \in C_p$. Conversely, if

$$\|S + E_C(X)\|_p \geq \|S\|_p \,,$$

for all $X \in C_p$. then from Corollary 2.6 applied for the elementary operator $E(X)$, we get

$$\sum_{i=1}^n C_i |S|^{p-1} U^*C_i = |S|^{p-1} U^*.$$

Since $\ker E_C \subseteq \ker E_{C^*}$,

$$\sum_{i=1}^n C_i^* |S|^{p-1} U^*C_i^* = |S|^{p-1} U^*.$$

Taking the adjoint we get $\sum_{i=1}^n C_iU |S|^{p-1} C_i = U |S|^{p-1}$ and from Lemma 2.10 it follows that

$$\sum_{i=1}^n C_iU |S| C_i = U |S| \,,$$

i.e., $S \in \ker E_C$. □

Theorem 2.12. *Let $A = (A_1, A_2, \ldots, A_n)$, $B = (B_1, B_2, \ldots, B_n)$ be $n-$tuples of operators in $B(H)$ such that $\sum_{i=1}^n A_iA_i^* \leq 1$, $\sum_{i=1}^n A_i^*A_i \leq 1$, $\sum_{i=1}^n B_iB_i^* \leq 1$, $\sum_{i=1}^n B_i^*B_i \leq 1$ and $\ker E_{A,B} \subseteq \ker E_{A^*,B^*}$.*

Then $S \in \ker E_{A,B} \cap C_p$, if and only if,

$$\|S + E_{A,B}(X)\|_p \geq \|S\|_p$$

for all $X \in C_p$.

Proof. It suffices to take the Hilbert space $H \oplus H$, and operators

$$C_i = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \quad S = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$$

and apply Theorem 2.12. □

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