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MONOTONICITY AND CONVEXITY OF FOUR SEQUENCES ORIGINATING FROM NANSON'S INEQUALITY

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ABSTRACT. In the short note, four sequences originating from Nanson's inequality are introduced, their monotonicities and convexities are obtained, and Nanson's inequality is refined.

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1. INTRODUCTION

A real sequence $\{a_i\}_{i=1}^{k}$ for $k > 2$ is called convex if

$$a_i + a_{i+2} \geq 2a_{i+1} \tag{1.1}$$

for $i \in \mathbb{N}$ with $i + 2 \leq k$.

The Nanson's inequality (see [3, p. 465] and [1, 2, 4]) reads that if $\{a_i\}_{i=1}^{2n+1}$ is a convex sequence, then

$$\frac{1}{n}\sum_{k=1}^{n} a_{2k} \leq \frac{1}{n+1}\sum_{k=0}^{n} a_{2k+1}. \tag{1.2}$$

The equality in (1.2) holds only if $\{a_i\}_{i=1}^{2n+1}$ is an arithmetic sequence.

It is clear that inequality (1.2) can be rewritten as

$$H(n) \triangleq n\sum_{k=0}^{n} a_{2k+1} - (n+1)\sum_{k=1}^{n} a_{2k} \geq 0. \tag{1.3}$$

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Similar to $H(n)$, it can be introduced for given $n \in \mathbb{N}$ that

$$h(m) = (n-m+1)\sum_{k=m-1}^{n} a_{2k+1} - (n-m+2)\sum_{k=m}^{n} a_{2k} \quad \text{for } 1 \le m \le n+1, \tag{1.4}$$

$$C(m) = \frac{1}{n(n+1)}\left[m\sum_{i=0}^{m} a_{2i+1} + (n-m)\sum_{i=1}^{m} a_{2i} + (n+1)\sum_{i=m+1}^{n} a_{2i}\right] \tag{1.5}$$

for $0 \le m \le n$, and

$$c(m) = \frac{1}{n(n+1)}\left[(n-m+1)\sum_{i=m-1}^{n} a_{2i+1} + (n+1)\sum_{i=1}^{m-1} a_{2i} + (m-1)\sum_{i=m}^{n} a_{2i}\right] \tag{1.6}$$

for $1 \le m \le n+1$, where $\sum_{i=q+1}^{q} b_i = 0$ is assumed for any $b_i \in \mathbb{R}$ and $q \in \mathbb{N}$.

The aim of this paper is to study monotonicity and convexity of H, h, C and c. From this, some new inequalities and refinements of (1.2) are deduced.

Our main results are the following two theorems.

Theorem 1.1. *Let $\{a_i\}_{i=1}^{2n+1}$ for $n \ge 1$ be a convex sequence. Then*

(1) *the sequence $\{H(j)\}_{j=1}^{n}$ is increasing and convex,*
(2) *the sequence $\{C(j)\}_{j=0}^{n}$ satisfies*

$$\frac{1}{n}\sum_{i=1}^{n} a_{2i} = C(0) \le C(1) \le \cdots \le C(n-1) \le C(n) = \frac{1}{n+1}\sum_{i=0}^{n} a_{2i+1}. \tag{1.7}$$

Theorem 1.2. *Let $\{a_i\}_{i=1}^{2n+1}$ for $n \ge 1$ be a convex sequence. Then*

(1) *the sequence $\{h(j)\}_{j=1}^{n+1}$ is decreasing and convex,*
(2) *the sequence $\{c(j)\}_{j=1}^{n+1}$ satisfies*

$$\frac{1}{n}\sum_{i=1}^{n} a_{2i} = c(n+1) \le c(n) \le \cdots \le c(2) \le c(1) = \frac{1}{n+1}\sum_{i=0}^{n} a_{2i+1}, \tag{1.8}$$

(3) *and*

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} a_{2i} &= \frac{C(0)+c(n+1)}{2} \\ &\le \frac{C(1)+c(n)}{2} \le \cdots \\ &\le \frac{C(n-1)+c(2)}{2} \\ &\le \frac{C(n)+c(1)}{2} = \frac{1}{n+1}\sum_{i=0}^{n} a_{2i+1}.\end{aligned} \tag{1.9}$$

Remark 1.3. Inequalities (1.7), (1.8) and (1.9) are refinements of (1.2).

2. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. If $\{a_i\}_{i=1}^n$ is convex, then it is easy to see that

$$
\begin{aligned}
(2.1)\quad & a_i - a_{i+1} - a_{n-1} + a_n \\
&= (a_i - 2a_{i+1} + a_{i+2}) + (a_{i+1} - 2a_{i+2} + a_{i+3}) + \cdots \\
&\quad + (a_{n-4} - 2a_{n-3} + a_{n-2}) + (a_{n-3} - 2a_{n-2} + a_{n-1}) \\
&\quad + (a_{n-2} - 2a_{n-1} + a_n) \ge 0.
\end{aligned}
$$

From (1.1) and (2.1), it follows that

$$
\begin{aligned}
H(j) - H(j-1) &= j\sum_{i=0}^{j} a_{2i+1} - (j+1)\sum_{i=1}^{j} a_{2i} - (j-1)\sum_{i=0}^{j-1} a_{2i+1} + j\sum_{i=1}^{j-1} a_{2i} \\
&= \left(j\sum_{i=0}^{j} a_{2i+1} - (j-1)\sum_{i=0}^{j-1} a_{2i+1}\right) + \left(j\sum_{i=1}^{j-1} a_{2i} - (j+1)\sum_{i=1}^{j} a_{2i}\right) \\
&= \left(ja_{2j+1} + \sum_{i=0}^{j-1} a_{2i+1}\right) - \left(ja_{2j} + \sum_{i=1}^{j} a_{2i}\right) \\
&= \sum_{i=1}^{j} (a_{2i-1} - a_{2i} - a_{2j} + a_{2j+1}) \\
&\ge 0,
\end{aligned}
$$

which implies the increasing monotonicity of $H(j)$ for $1 \le j \le n$.

It is obvious that

$$
C(k) = \frac{1}{n(n+1)}\left[H(k) + (n+1)\sum_{i=1}^{n} a_{2i}\right] = \frac{H(k)}{n(n+1)} + \frac{1}{n}\sum_{i=1}^{n} a_{2i}. \tag{2.2}
$$

From the increasingly monotonic property of $H(j)$ for $1 \le j \le n$, inequalities in (1.7) are concluded.

For $j = 1, 2, \ldots, n-2$, direct calculation gives

$$
\begin{aligned}
H(j)&-2H(j+1) + H(j+2) \\
&= \left(j\sum_{i=0}^{j} a_{2i+1} - (j+1)\sum_{i=1}^{j} a_{2i}\right) - 2\left((j+1)\sum_{i=0}^{j+1} a_{2i+1} - (j+2)\sum_{i=1}^{j+1} a_{2i}\right) \\
&\quad + \left((j+2)\sum_{i=0}^{j+2} a_{2i+1} - (j+3)\sum_{i=1}^{j+2} a_{2i}\right) \\
&= \left(j\sum_{i=0}^{j} a_{2i+1} - (j+1)\sum_{i=0}^{j+1} a_{2i+1}\right) + \left((j+2)\sum_{i=0}^{j+2} a_{2i+1} - (j+1)\sum_{i=0}^{j+1} a_{2i+1}\right) \\
&\quad + \left((j+2)\sum_{i=1}^{j+1} a_{2i} - (j+1)\sum_{i=1}^{j} a_{2i}\right) + \left((j+2)\sum_{i=1}^{j+1} a_{2i} - (j+3)\sum_{i=1}^{j+2} a_{2i}\right)
\end{aligned}
$$

$$= \left(-ja_{2j+3} - \sum_{i=0}^{j+1} a_{2i+1} \right) + \left((j+1)a_{2j+5} + \sum_{i=0}^{j+2} a_{2i+1} \right)$$
$$+ \left((j+1)a_{2j+2} + \sum_{i=1}^{j+1} a_{2i} \right) + \left(-(j+2)a_{2j+4} - \sum_{i=1}^{j+2} a_{2i} \right)$$
$$= (j+1)a_{2j+2} - ja_{2j+3} - (j+2)a_{2j+4} + (j+1)a_{2j+5}$$
$$+ \left(\sum_{i=1}^{j+1} a_{2i} - \sum_{i=1}^{j+2} a_{2i} \right) + \left(\sum_{i=0}^{j+2} a_{2i+1} - \sum_{i=0}^{j+1} a_{2i+1} \right)$$
$$= (j+1)a_{2j+2} - ja_{2j+3} - (j+3)a_{2j+4} + (j+2)a_{2j+5}$$
$$= (j+1)(a_{2j+2} - 2a_{2j+3} + a_{2j+4}) + (j+2)(a_{2j+3} - 2a_{2j+4} + a_{2j+5}) \geq 0$$

which implies that the sequence $\{H(j)\}_{j=1}^{n}$ is convex. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. By the same arguments as in Theorem 1.1, the decreasing and convex properties of the sequences $\{h(j)\}_{j=1}^{n+1}$ and $\{c(j)\}_{j=1}^{n+1}$ are immediately obtained.

Adding (1.7) and (1.8) yields (1.9). The proof of Theorem 1.2 is complete. □

REFERENCES

[1] D.D. ADAMOVIĆ AND J.E. PEČARIĆ, On Nanson's inequality and on some inequalities related to it, *Math. Balkanica (N. S.)*, **3**(1) (1989), 3–11.

[2] J. CHEN AND ZH.-H. YE, *Chūděng Shùxué Qiányán* (*Frontier of Elelmentary Mathematics*), Vol. 1, Jiāngsū Jiàoyù Chūbǎn Shè (Jiangsu Education Press), Nanjing City, China, 1996. (Chinese)

[3] J.-CH. KUANG, *Chángyòng Bùděngshì* (*Applied Inequalities*), 3rd ed., Shāndōng Kēxué Jìshù Chūbǎn Shè (Shandong Science and Technology Press), Jinan City, Shandong Province, China, 2004. (Chinese)

[4] I.Ž. MILOVANOVIĆ, J.E. PEČARIĆ AND GH. TOADER, On an inequality of Nanson, *Anal. Numér. Théor. Approx.*, **15**(2) (1986), 149–151.