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A GENERAL INEQUALITY OF NGÔ-THANG-DAT-TUAN TYPE

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ABSTRACT. In the present note a general integral inequality is proved in the direction that was initiated by Q. A. Ngô et al [Note on an integral inequality, *J. Inequal. Pure and Appl. Math.*, 7(4) (2006), Art.120].

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1. INTRODUCTION

In their paper [7] Ngô, Tang, Dat, and Tuan proved the following inequalities. If f is a nonnegative, continuous function on $[0,1]$ satisfying

$$\int_x^1 f(t)\,dt \geq \int_x^1 t\,dt, \quad \forall x \in [0,1],$$

then

$$\int_0^1 f(x)^{\alpha+1}\,dx \geq \int_0^1 x^\alpha f(x)\,dx, \quad \int_0^1 f(x)^{\alpha+1}\,dx \geq \int_0^1 x\,f(x)^\alpha\,dx$$

for every positive number α.

This result has initiated a series of papers containing various extensions and generalizations [1, 2, 3, 5, 6]. Among others, it turns out that the conditions above imply

$$\int_0^1 f(x)^{\alpha+\beta}\,dx \geq \int_0^1 x^\alpha f(x)^\beta\,dx$$

for every $\alpha > 0$, $\beta \geq 1$, which answered an open question of Ngô et al. in the positive [3].

The aim of this note is to formulate and prove a further generalization. It is presented in Section 2. Section 3 contains corollaries, which are immediate extensions of a couple of known results.

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2. MAIN RESULT

Theorem 2.1. *Let $u, v : [0, +\infty) \to \mathbb{R}$ be nonnegative, differentiable, increasing functions. Suppose that $u'(t)$ is positive and increasing, and $\frac{v'(t)u(t)}{u'(t)}$ is increasing for $t > 0$. Let f and g be nonnegative, integrable functions defined on the interval $[a, b]$. Suppose g is increasing, and*

$$\int_x^b g(t)\,dt \le \int_x^b f(t)\,dt \tag{2.1}$$

holds for every $x \in [a, b]$. Then

$$\int_a^b u(g(t))v(g(t))\,dt \le \int_a^b u(f(t))v(g(t))\,dt \le \int_a^b u(f(t))v(f(t))\,dt, \tag{2.2}$$

$$\int_a^b u(g(t))v(f(t))\,dt \le \int_0^1 u(f(t))v(f(t))\,dt, \tag{2.3}$$

provided the integrals are finite.

Remark 1.

(1) Here and throughout, by *increasing* we always mean *nondecreasing*.
(2) Note that continuity of f or g is not required.
(3) Unfortunately, the other inequality

$$\int_0^1 u(g(t))v(g(t))\,dt \le \int_0^1 u(g(t))v(f(t))\,dt, \tag{2.4}$$

which seems to be missing from (2.3), is not necessarily valid. Set $[a, b] = [0, 1]$, $u(t) = t^\beta$, $v(t) = t^\alpha$, with $\alpha > 0$, $\beta > 1$. Let $g(t) = t$, and $f(t) = 1$, if $1/2 \le t \le 1$, and zero otherwise. Then all the conditions of Theorem 2.1 are satisfied, and

$$\int_a^b u(g(t))v(g(t))\,dt = \int_0^1 t^{\alpha+\beta}\,dt = \frac{1}{\alpha+\beta+1},$$
$$\int_a^b u(g(t))v(f(t))\,dt = \int_{1/2}^1 t^{\beta}\,dt = \frac{1}{\beta+1}\left(1 - \frac{1}{2^{\beta+1}}\right).$$

It is easy to see that (2.4) does not hold if $\alpha < \frac{\beta+1}{2^{\beta+1}-1}$.

Although f is discontinuous in this counterexample, it is not continuity that can help, for f can be approximated in L_1 with continuous (piecewise linear) functions.

For the proof we shall need the following lemmas of independent interest.

Lemma 2.2. *Let f and g be nonnegative integrable functions on $[a, b]$ that satisfy (2.1). Let $h : [a, b] \to \mathbb{R}$ be nonnegative and increasing. Then*

$$\int_a^b h(t)g(t)\,dt \le \int_a^b h(t)f(t)\,dt. \tag{2.5}$$

Proof. We can suppose that u is right continuous, because it can only have countably many discontinuities, so replacing $u(t)$ with $u(t+)$ in these points does not affect the integrals. Clearly, $h(t) = h(a) + \int_{(a,t]} dh(s)$, hence

$$\int_a^b h(t)g(t)\,dt = \int_a^b \left(h(a) + \int_{a+}^{t+} dh(s)\right) g(t)\,dt$$
$$= h(a)\int_a^b g(t)\,dt + \int_a^b\int_a^b I(s \le t)g(t)\,dh(s)\,dt,$$

where $I(\cdot)$ stands for the characteristic function of the set in brackets. By Fubini's theorem we can interchange the order of the integration, obtaining

$$\begin{aligned}\int_a^b h(t)g(t)\,dt &= h(a)\int_a^b g(t)\,dt + \int_a^b\int_a^b I(s\le t)g(t)\,dt\,dh(s)\\ &= h(a)\int_a^b g(t)\,dt + \int_a^b\left(\int_t^b g(s)\,ds\right)dh(s).\end{aligned}$$

Remembering condition (2.1), we can write

$$\begin{aligned}\int_a^b h(t)g(t)\,dt &\le h(a)\int_a^b f(t)\,dt + \int_a^b\left(\int_t^b f(s)\,ds\right)dh(s)\\ &= \int_a^b h(t)f(t)\,dt,\end{aligned}$$

as required. □

Lemma 2.3. *Let f and g be as in Theorem 2.1, and let $v : [0,+\infty) \to \mathbb{R}$ be a nonnegative increasing function. Define $V(x) = \int_0^x v(t)\,dt$, $x \ge 0$. Then*

$$\int_a^b V(g(t))\,dt \le \int_a^b V(f(t))\,dt. \tag{2.6}$$

Equivalently, we can say that inequality (2.6) is valid for all increasing convex functions $V : [0,+\infty) \to \mathbb{R}$.

Proof. We can suppose that the right-hand side is finite, for the integrand on the left-hand side is bounded. Let V^* denote the Legendre transform of V, that is, $V^*(x) = \int_0^x v^{-1}(t)\,dt$, where $v^{-1}(t) = \inf\{s : v(s) \ge t\}$ is the (right continuous) generalized inverse of v. Then by the Young inequality [4] we have that $xy \le V(x) + V^*(y)$ holds for every $x, y \ge 0$, with equality if and only if $v(x-) \le y \le v(x+)$. Hence, by substituting $x = f(t)$ and $y = v(g(t))$ we obtain

$$f(t)v(g(t)) \le V(f(t)) + V^*(v(g(t))) = V(f(t)) + g(t)v(g(t)) - V(g(t)). \tag{2.7}$$

By integrating this we get that

$$\int_a^b f(t)v(g(t))\,dt \le \int_a^b V(f(t))\,dt + \int_a^b g(t)v(g(t))\,dt - \int_a^b V(g(t))\,dt. \tag{2.8}$$

With $h(t) = v(g(t))$ Lemma 2.2 yields

$$\int_a^b g(t)v(g(t))\,dt \le \int_0^1 f(t)v(g(t))\,dt. \tag{2.9}$$

Combining (2.8) with (2.9) we arrive at (2.6). □

Proof of Theorem 2.1. First we prove for the case where $u(t) = t$. Then $t\,v'(t)$ has to be increasing.

The first inequality of (2.2) has already been proved in (2.9). On the other hand, from the Young inequality, similarly to (2.7) we can derive that

$$\begin{aligned}f(t)v(g(t)) &\le V(f(t)) + V^*(v(g(t)))\\ &= V^*(v(g(t))) + f(t)v(f(t)) - V^*(v(f(t))).\end{aligned}$$

Therefore,

$$\int_a^b f(t)v(g(t))\,dt \le \int_a^b V^*(v(g(t)))\,dt + \int_a^b f(t)v(f(t))\,dt - \int_a^b V^*(v(f(t)))\,dt. \tag{2.10}$$

Here

$$V^*(v(x)) = xv(x) - V(x) = \int_0^x \big[(tv(t))' - v(t)\big]dt = \int_0^x tv'(t)\,dt,$$

thus Lemma 2.3 can be applied with $V^*(v(x))$ in place of $V(x)$.

$$\int_a^b V^*(v(g(t)))\,dt \le \int_a^b V^*(v(f(t)))\,dt. \tag{2.11}$$

Now we can complete the proof of the second inequality of (2.2) by plugging (2.11) back into (2.10).

Next, since $[f(t) - g(t)][v(f(t)) - v(g(t))] \ge 0$, we obtain that

$$\int_0^1 f(t)v(f(t))\,dt - \int_0^1 g(t)v(f(t)\,dt \ge \int_0^1 f(t)v(g(t))\,dt - \int_0^1 g(t)v(g(t) \ge 0,$$

by (2.2). This proves (2.3).

For the general case, we first apply Lemma 2.3 on the interval $[x, b]$, with $u(t)$ in place of $V(t)$. We can see that $u(f(t))$ and $u(g(t))$ satisfy condition (2.1). Now, u is invertable. Let $w(t) = v(u^{-1}(t))$, then

$$w'(t) = \frac{v'(u^{-1}(t))}{u'(u^{-1}(t))},$$

hence, by the conditions of Theorem 2.1, $tw'(t)$ is increasing. The proof can be completed by applying the particular case just proved to the functions $u(f(t))$ and $u(g(t))$, with w in place of v. □

3. COROLLARIES, PARTICULAR CASES

In this section we specialize Theorem 2.1 to obtain some well known results that were mentioned in the Introduction. First, let $u(x) = x^\beta$ and $v(x) = x^\alpha$ with $\alpha > 0$ and $\beta \ge 1$. They clearly satisfy the conditions of Theorem 2.1.

Corollary 3.1. *Let f and g be nonnegative, integrable functions defined on the interval $[a, b]$. Suppose g is increasing, and*

$$\int_x^b g(t)\,dt \le \int_x^b f(t)\,dt \tag{3.1}$$

holds for every $x \in [a, b]$. Then, for arbitrary $\alpha > 0$ and $\beta \ge 1$ we have

$$\int_a^b g(t)^{\alpha+\beta}\,dt \le \int_a^b g(t)^\alpha f(t)^\beta\,dt \le \int_a^b f(t)^{\alpha+\beta}\,dt, \tag{3.2}$$

$$\int_a^b f(t)^\alpha g(t)^\beta\,dt \le \int_a^b f(t)^{\alpha+\beta}\,dt. \tag{3.3}$$

Next, change α, β, $f(t)$, and $g(t)$ in Corollary 3.1 to α/β, 1, $f(t)^\beta$ and $g(t)^\beta$, respectively.

Corollary 3.2. *Let α and β be arbitrary positive numbers. Let f and g satisfy the conditions of Corollary 3.1, but, instead of* (3.1) *suppose that*

$$\int_x^b g(t)^\beta\,dt \le \int_x^b f(t)^\beta\,dt \tag{3.4}$$

holds for every $x \in [a, b]$. Then inequalities (3.2) *and* (3.3) *remain valid.*

In particular, for the case of $[a, b] = [0, 1]$, $g(t) = t$ Corollary 3.1 yields Theorem 2.3 of [3], and Corollary 3.2 implies Theorem 2.1 of [5]. If, in addition, we set $\beta = 1$, Corollary 3.1 gives Theorems 3.2 and 3.3 of [7].

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