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AN INEQUALITY FOR THE CLASS NUMBER

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ABSTRACT. We prove in an elementary way a new inequality for the average order of the Piltz divisor function with application to class number of number fields.

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1. INTRODUCTION

It could be interesting to use tools from analytic number theory to solve problems of algebraic number theory. For example, let $\mathbb{K}$ be a number field of degree n, signature (r_1, r_2), class number $h_{\mathbb{K}}$, regulator $\mathcal{R}_{\mathbb{K}}$, and $w_{\mathbb{K}}$ is the number of roots of unity in $\mathbb{K}$, $\zeta_{\mathbb{K}}$ the Dedekind zeta function, $A_{\mathbb{K}} := 2^{-r_2}\pi^{-n/2}d_{\mathbb{K}}^{1/2}$ where $d_{\mathbb{K}}$ is the absolute value of the discriminant of $\mathbb{K}$. The following formula, valid for any real number $\sigma > 1$,

$$A_{\mathbb{K}}^{\sigma}\Gamma^{r_1}\left(\frac{\sigma}{2}\right)\Gamma^{r_2}(\sigma)\zeta_{\mathbb{K}}(\sigma) = \frac{2^{r_1}h_{\mathbb{K}}\mathcal{R}_{\mathbb{K}}}{\sigma(\sigma-1)w_{\mathbb{K}}} + \sum_{\mathfrak{a}\neq 0}\int_{\|y\|\geq 1}\left\{\|y\|^{\sigma/2} + \|y\|^{\frac{1-\sigma}{2}}\right\}e^{-g(\mathfrak{a},y)}\frac{dy}{y}, \tag{1.1}$$

where $g(\mathfrak{a}, y)$ is a certain function depending on a nonzero integral ideal $\mathfrak{a}$ and vector $y := (y_1, \dots, y_{r_1+r_2}) \in (\mathbb{R}_+)^{r_1+r_2}$ (here $\|y\| := \max|y_i|$), is the generalization of the well-known formula

$$\pi^{-\sigma/2}\Gamma\left(\frac{\sigma}{2}\right)\zeta(\sigma) = \frac{1}{\sigma(\sigma-1)} + \sum_{n=1}^{\infty}\int_1^{\infty}\left\{y^{\sigma/2} + y^{\frac{1-\sigma}{2}}\right\}e^{-\pi n^2 y}\frac{dy}{y}$$

for the classical Riemann zeta function. Since the integrand in (1.1) is positive, we get

$$h_{\mathbb{K}}\mathcal{R}_{\mathbb{K}} \leq \sigma(\sigma-1)w_{\mathbb{K}}2^{-r_1}A_{\mathbb{K}}^{\sigma}\Gamma^{r_1}\left(\frac{\sigma}{2}\right)\Gamma^{r_2}(\sigma)\zeta_{\mathbb{K}}(\sigma) \tag{1.2}$$

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for any real number $\sigma > 1$. The study of the function on the right-hand side of (1.2) provides upper bounds for $h_{\mathbb{K}}\mathcal{R}_{\mathbb{K}}$ (see [3] for example) .

In a more elementary way, one can connect the class number $h_{\mathbb{K}}$ with the Piltz divisor function τ_n by using the following result ([1]):

Lemma 1.1. *Let $b_{\mathbb{K}} > 0$ be a real number such that every class of ideals of $\mathbb{K}$ contains a nonzero integral ideal with norm $\leq b_{\mathbb{K}}$. If τ_n is the Piltz divisor function, then:*

$$h_{\mathbb{K}} \leq \sum_{m \leq b_{\mathbb{K}}} \tau_n(m).$$

Recall that τ_n is defined by the relations $\tau_1(m) = m$ and $\tau_n(m) = \sum_{d|m} \tau_{n-1}(d)$ $(n \geq 2)$. This function has been studied by many authors (see [6] for a good survey of its properties). A standard argument from analytic number theory gives if $n \geq 4$

$$\sum_{m \leq x} \tau_n(m) = x\mathcal{P}_{n-1}(\log x) + O_\varepsilon\left(x^{\frac{n-1}{n+2}+\varepsilon}\right),$$

where $\mathcal{P}_{n-1}$ is a polynomial of degree $n-1$ and leading coefficient $\frac{1}{(n-1)!}$. For some improvements of the error term and related results, see [4]. Note that the Lindelöf Hypothesis is equivalent to $\alpha_n = (n-1)/(2n)$ for any $n = 2, 3, \dots$ where α_n is the least number such that

$$\sum_{m \leq x} \tau_n(m) - x\mathcal{P}_{n-1}(\log x) = O_\varepsilon\left(x^{\alpha_n+\varepsilon}\right).$$

If we are interested in finding upper bounds of the form

$$\sum_{m \leq x} \tau_n(m) \ll_n x(\log x)^{n-1},$$

one mostly uses arguments based upon induction and the following inequality:

Lemma 1.2. *We set $S_n(x) := \sum_{m \leq x} \tau_n(m)$. Then:*

$$S_{n+1}(x) \leq S_n(x) + x\int_1^x t^{-2} S_n(t)\,dt.$$

Proof. It suffices to use the definition above, interchange the summations and integrate by parts. □

Using this lemma, it is easy to show by induction the following bound:

$$\sum_{m \leq x} \tau_n(m) \leq \frac{x}{(n-1)!}(\log x + n - 1)^{n-1}$$

which enables us to obtain Lenstra's bound again (see [2]), namely:

$$h_{\mathbb{K}} \leq \frac{b_{\mathbb{K}}}{(n-1)!}(\log b_{\mathbb{K}} + n - 1)^{n-1}. \tag{1.3}$$

In what follows, n is a positive integer and we set

$$S_n(x) := \sum_{m \leq x} \tau_n(m)$$

for any real number $x \geq 1$. $b_{\mathbb{K}}$ is a positive real number always satisfying the hypothesis of Lemma 1.1. $\mathbb{K}$ is a number field of degree n and class number $h_{\mathbb{K}}$. $d_{\mathbb{K}}$ is the absolute value of the discriminant of $\mathbb{K}$. For some tables giving values of $b_{\mathbb{K}}$, see [7]. The functions ψ and ψ_2 are defined by

$$\psi(t) = t - [t] - \frac{1}{2},$$
$$\psi_2(t) = \int_0^t \psi(u)\,du + \frac{1}{8} = \frac{\psi^2(t)}{2},$$

where $[t]$ denotes the integral part of t. Recall that we have for all real numbers t :

$$|\psi(t)| \leq \frac{1}{2},$$
$$0 \leq \psi_2(t) \leq \frac{1}{8}.$$

We denote by γ and γ_1 the Euler-Mascheroni constant and the first Stieltjes constant, defined respectively by:

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right),$$
$$\gamma_1 = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\log k}{k} - \frac{(\log n)^2}{2}\right).$$

The following results are well-known (see [5] for example):

$$0.577215 < \gamma < 0.577216,$$
$$-0.072816 < \gamma_1 < -0.072815,$$

and

$$\gamma = \frac{1}{2} - 2\int_1^\infty \frac{\psi_2(t)}{t^3}dt \tag{1.4}$$

and

$$\gamma_1 = -\int_1^\infty \frac{2\log t - 3}{t^3}\psi_2(t)\,dt. \tag{1.5}$$

2. RESULTS

Theorem 2.1. *Let $n \geq 3$ be an integer. For any real number $x \geq 13$, we have:*

$$\sum_{m\leq x}\tau_n(m) \leq \frac{x}{(n-1)!}(\log x + n - 2)^{n-1}.$$

Applying this result with Lemma 1.1 allows us to improve upon (1.3) :

Theorem 2.2. *Let $\mathbb{K}$ be a number field of degree $n \geq 3$. If $b_{\mathbb{K}} \geq 13$ satisfies the hypothesis of Lemma 1.1, then:*

$$h_{\mathbb{K}} \leq \frac{b_{\mathbb{K}}}{(n-1)!}(\log b_{\mathbb{K}} + n - 2)^{n-1}.$$

3. THE CASE $n = 3$

The aim of this section is to show that the result of Theorem 2.1 is true for $n = 3$. Hence we will prove the following inequality for S_3 :

Lemma 3.1. *For any real number $x \geq 13$, we have:*

$$S_3(x) \leq \frac{x}{2}(\log x + 1)^2.$$

We first check this result for $13 \leq x \leq 670$ with the PARI/GP system [8], and then suppose $x > 670$. The lemma will be a direct consequence of the following estimation:

Lemma 3.2. *For any real number $x > 670$, we have:*

$$S_3(x) = x\left\{\frac{(\log x)^2}{2} + (3\gamma - 1)\log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1\right\} + R(x)$$

where:

$$|R(x)| \leq 2.36\, x^{2/3}\log x.$$

The proof of this lemma needs some technical results:

Lemma 3.3. *Let $x, y \geq 1$ be real numbers.*

(i) *If $e^{3/2} \leq y \leq x$, then we have:*

$$\sum_{k \leq y} \frac{1}{k}\log\left(\frac{x}{k}\right) = \log x \log y - \frac{(\log y)^2}{2} + \gamma \log x - \gamma_1 + R_1(x, y)$$

with:

$$|R_1(x, y)| \leq \frac{\log(x/y)}{2y} + \frac{\log x}{4y^2}.$$

(ii)

$$S_2(y) = y\log y + (2\gamma - 1)\,y + R_2(y)$$

with:

$$|R_2(y)| \leq y^{1/2} + \frac{1}{2}.$$

(iii)

$$\sum_{n \leq y} \frac{\tau(n)}{n} = \frac{(\log y)^2}{2} + 2\gamma\log y + \gamma^2 - 2\gamma_1 + R_3(y)$$

with:

$$|R_3(y)| \leq \frac{1}{y^{1/2}} + \frac{1}{y}.$$

Proof. (i) By the Euler-MacLaurin summation formula, we get:

$$\begin{aligned}
&\sum_{k\le y}\frac{1}{k}\log\left(\frac{x}{k}\right)\\
&=\frac{\log x}{2}+\int_1^y\frac{1}{t}\log\left(\frac{x}{t}\right)dt-\frac{\psi(y)}{y}\log\left(\frac{x}{y}\right)\\
&\qquad-\frac{\psi_2(y)}{y^2}\left(\log\left(\frac{x}{y}\right)+1\right)-\int_1^y\frac{2\log(x/t)+3}{t^3}\psi_2(t)\,dt\\
&=\log x\log y-\frac{(\log y)^2}{2}+\left(\frac{1}{2}-2\int_1^\infty\frac{\psi_2(t)}{t^3}dt\right)\log x\\
&\qquad+\int_1^\infty\frac{2\log t-3}{t^3}\psi_2(t)\,dt-\frac{\psi(y)}{y}\log\left(\frac{x}{y}\right)-\frac{\psi_2(y)}{y^2}\left(\log\left(\frac{x}{y}\right)+1\right)\\
&\qquad+2\log x\int_y^\infty\frac{\psi_2(t)}{t^3}dt-\int_y^\infty\frac{2\log t-3}{t^3}\psi_2(t)\,dt
\end{aligned}$$

and using (1.4) and (1.5) we get:

$$\sum_{k\le y}\frac{1}{k}\log\left(\frac{x}{k}\right)=\log x\log y-\frac{(\log y)^2}{2}+\gamma\log x-\gamma_1+R_1(x,y)$$

and since $e^{3/2}\le y\le x$, we have:

$$\begin{aligned}
|R_1(x,y)|&\le\frac{\log(x/y)}{2y}+\frac{\log(x/y)+1}{8y^2}+\frac{\log x}{8y^2}+\frac{\log y-1}{8y^2}\\
&=\frac{\log(x/y)}{2y}+\frac{\log x}{4y^2}.
\end{aligned}$$

(ii) This result is well-known (see [1] for example).
(iii) Using a result from [5], we have for any real number $y\ge 1$:

$$-y^{-1/2}-\left(\frac{3}{4}+\frac{1}{8e^3}\right)y^{-1}-\frac{y^{-3/2}}{8}-\frac{y^{-2}}{64}\le R_3(y)\le y^{-1/2}+\left(\frac{1}{2}+\frac{1}{8e^3}\right)y^{-1}$$

which concludes the proof of Lemma 3.3. □

Proof of Lemmas 3.1 and 3.2. The Dirichlet hyperbola principle and the estimations of Lemma 3.3 give, for any real number $e^{3/2}\le T<x$:

$$\begin{aligned}
S_3(x)&=\sum_{n\le T}S_2\left(\frac{x}{n}\right)+\sum_{n\le x/T}\tau(n)\left[\frac{x}{n}\right]-[T]\,S_2\left(\frac{x}{T}\right)\\
&=\sum_{n\le T}\left(\frac{x}{n}\log\left(\frac{x}{n}\right)+(2\gamma-1)\frac{x}{n}+R_4(x,n)\right)+x\sum_{n\le x/T}\frac{\tau(n)}{n}-\frac{1}{2}S_2\left(\frac{x}{T}\right)\\
&\qquad-\sum_{n\le x/T}\tau(n)\,\psi\left(\frac{x}{n}\right)-TS_2\left(\frac{x}{T}\right)+\frac{1}{2}S_2\left(\frac{x}{T}\right)+\psi(T)\,S_2\left(\frac{x}{T}\right)\\
&=\sum_{n\le T}\left(\frac{x}{n}\log\left(\frac{x}{n}\right)+(2\gamma-1)\frac{x}{n}+R_4(x,n)\right)\\
&\qquad+x\sum_{n\le x/T}\frac{\tau(n)}{n}-TS_2\left(\frac{x}{T}\right)+R_5(x,T)
\end{aligned}$$

with

$$|R_4(x,n)| \le \sqrt{\frac{x}{n}} + \frac{1}{2}$$
$$|R_5(x,T)| \le S_2\left(\frac{x}{T}\right) \le \frac{x}{T}\log\left(\frac{x}{T}\right) + (2\gamma-1)\frac{x}{T} + \sqrt{\frac{x}{T}} + \frac{1}{2}$$

and hence:

$$\begin{aligned} S_3(x) = x\Bigg\{ & \log x \log T - \frac{(\log T)^2}{2} + \gamma \log x \\ & - \gamma_1 + R_6(x,T) + (2\gamma-1)(\log T + \gamma + R_7(T)) \Bigg\} \\ & + \sum_{n \le T} R_4(x,n) + x\left\{ \frac{(\log(x/T))^2}{2} + 2\gamma\log\left(\frac{x}{T}\right) + \gamma^2 - 2\gamma_1 + R_8(x,T) \right\} \\ & + R_5(x,T) - x\log\left(\frac{x}{T}\right) - (2\gamma-1)x - TR_9(x,T) \end{aligned}$$

with, if $e^{3/2} \le T < x$:

$$|R_6(x,T)| \le \frac{\log(x/T)}{2T} + \frac{\log x}{4T^2}$$
$$|R_7(T)| \le \frac{1}{T}$$
$$|R_8(x,T)| \le \sqrt{\frac{T}{x}} + \frac{T}{x}$$
$$|R_9(x,T)| \le \sqrt{\frac{x}{T}} + \frac{1}{2}$$

and thus:

$$\begin{aligned} S_3(x) = x & \left\{ \frac{(\log x)^2}{2} + (3\gamma-1)\log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1 \right\} \\ & + xR_6(x,T) + (2\gamma-1)xR_7(T) + R_{10}(x,T) + xR_8(x,T) \\ & + R_5(x,T) - TR_9(x,T) \end{aligned}$$

with

$$\begin{aligned} |R_{10}(x,T)| & \le \sum_{n \le T} |R_4(x,n)| \\ & \le \sqrt{x}\sum_{n \le T}\frac{1}{\sqrt{n}} + \frac{T}{2} \\ & \le 2\sqrt{xT} - \sqrt{x} + \frac{T}{2} \end{aligned}$$

and therefore:

$$S_3(x) = x\left\{ \frac{(\log x)^2}{2} + (3\gamma-1)\log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1 \right\} + R_{11}(x,T)$$

with:

$$\begin{aligned}|R_{11}(x,T)| \le \frac{x\log(x/T)}{2T} + \frac{x\log x}{4T^2} + 4\sqrt{xT} - \sqrt{x} \\ + \frac{2x}{T}\log\left(\frac{x}{T}\right) + 2(2\gamma-1)\frac{x}{T} + \sqrt{\frac{x}{T}} + 2T + \frac{1}{2}.\end{aligned}$$

We choose:

$$T = x^{1/3},$$

which gives:

$$S_3(x) = x\left\{\frac{(\log x)^2}{2} + (3\gamma-1)\log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1\right\} + R_{12}(x),$$

where:

$$\begin{aligned}|R_{12}(x)| &\le \frac{5}{3}x^{2/3}\log x + 2(2\gamma+1)x^{2/3} - x^{1/2} + \frac{1}{4}x^{1/3}\log x + 3x^{1/3} + \frac{1}{2} \\ &\le 2.36\, x^{2/3}\log x\end{aligned}$$

since $x > 670$. This concludes the proof of Lemma 3.2, and then of Lemma 3.1. □

4. PROOF OF THEOREM 2.1

We first need the following simple bounds:

Lemma 4.1. *For any integer $n \ge 3$, we have:*

$$\int_1^{13} t^{-2} S_n(t)\,dt < \frac{n^3}{4} \le \frac{1}{n!}\left(n + \frac{1}{2}\right)^n.$$

Proof. This follows from straightforward computations which give:

$$\begin{aligned}\int_1^{13} t^{-2} S_n(t)\,dt &= \frac{7}{624}n^3 + \frac{2281}{9360}n^2 + \frac{90283}{90090}n + 1 - \frac{1}{13} \\ &< \frac{n^3}{4}\end{aligned}$$

since $n \ge 3$. The second inequality follows from studying the sequence (u_n) defined by

$$u_n = \frac{n^3 \times n!}{4(n+1/2)^n}.$$

We get:

$$\begin{aligned}\frac{u_{n+1}}{u_n} &= \frac{2(n+1)^4}{n^3(2n+3)}\left(\frac{2n+1}{2n+3}\right)^n \\ &\le \frac{512}{243}\left(1 - \frac{2}{2n+3}\right)^n \le \frac{512e^{-1}}{243} < 1\end{aligned}$$

and hence (u_n) is decreasing, and thus:

$$u_n \le u_3 = \frac{324}{343} \le 1,$$

which concludes the proof of Lemma 4.1. □

Proof of Theorem 2.1. We use induction, the result being true for $n = 3$ by Lemma 3.1. Now suppose the inequality is true for some integer $n \geq 3$. By Lemmas 1.2, 4.1 and the induction hypothesis, we get:

$$\begin{aligned}
&S_{n+1}(x)\\
&\leq S_n(x) + x\int_1^{13} t^{-2} S_n(t)\,dt + x\int_{13}^{x} t^{-2} S_n(t)\,dt\\
&\leq x\left\{\frac{(\log x + n - 2)^{n-1}}{(n-1)!} + \frac{1}{n!}\left(n+\frac{1}{2}\right)^n + \frac{1}{(n-1)!}\int_{13}^{x}\frac{(\log t + n - 2)^{n-1}}{t}dt\right\}\\
&= x\left\{\frac{(\log x + n - 2)^{n}}{n!} + \frac{(\log x + n - 2)^{n-1}}{(n-1)!} + \frac{1}{n!}\left(\left(n+\frac{1}{2}\right)^n - \left(n + \log\left(13e^{-2}\right)\right)^n\right)\right\}\\
&\leq \frac{x}{n!}\left\{(\log x + n - 2)^n + (n-1)(\log x + n - 2)^{n-1}\right\}\\
&\leq \frac{x}{n!}(\log x + n - 1)^n.
\end{aligned}$$

The proof of Theorem 2.1 is now complete. □

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