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A STABILITY VERSION OF HÖLDER'S INEQUALITY FOR $0 < p < 1$

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ABSTRACT. We use a refinement of Hölder's inequality for $1 < p < \infty$ to obtain the corresponding refinement when $r \in (0, 1)$. This in turn allows us to sharpen the reverse triangle inequality on the nonnegative functions in L^r, for $r \in (0, 1)$.

Key words and phrases: Hölder's inequality, Reverse triangle inequality.

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By $\|F\|_t := \left(\int |F|^t\right)^{1/t}$ we do not mean to imply that this quantity is finite, nor do we assume that $t > 0$; in fact, in this note negative exponents are unavoidable.

It is well known that Hölder's inequality can be extended to the range $0 < r < 1$, by an argument that essentially amounts to a clever rewriting of the case $1 < p < \infty$, cf. [2, pg. 191]. We denote the conjugate exponent of r by $s := r/(r-1)$, and the conjugate exponent of p by $q := p/(p-1)$ (of course, to go from the range $(0, 1)$ to $(1, \infty)$ and viceversa, one sets $r = 1/p$). Hölder's inequality for $0 < r < 1$ tells us that if h and k are nonnegative functions in L^r and L^s respectively, then $\int hk \geq \left(\int h^r\right)^{1/r} \left(\int k^s\right)^{1/s}$. This entails that given functions $h, w \geq 0$ in L^r, the reverse triangle inequality $\|h + w\|_r \geq \|h\|_r + \|w\|_r$ holds. Nonnegativity is of course crucial.

Here we extend to the range $(0, 1)$ the following stability version of Hölder's inequality, which appears in [1]:

Let $1 < p < \infty$ and let $q = p/(p-1)$ be its conjugate exponent. If $f \in L^p$, $g \in L^q$ are nonnegative functions with $\|f\|_p, \|g\|_q > 0$, and $1 < p \leq 2$, then

$$\text{(1)} \quad \|f\|_p\|g\|_q \left(1 - \frac{1}{p}\left\|\frac{f^{p/2}}{\|f^{p/2}\|_2} - \frac{g^{q/2}}{\|g^{q/2}\|_2}\right\|_2^2\right)_+ \leq \|fg\|_1 \leq \|f\|_p\|g\|_q \left(1 - \frac{1}{q}\left\|\frac{f^{p/2}}{\|f^{p/2}\|_2} - \frac{g^{q/2}}{\|g^{q/2}\|_2}\right\|_2^2\right),$$

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while if $2 \le p < \infty$, *the terms* $1/p$ *and* $1/q$ *exchange their positions in the preceding inequalities.*

Inequality (1) essentially states that $\|fg\|_1 \approx \|f\|_p \|g\|_q$ if and only if the angle between the L^2 vectors $f^{p/2}$ and $g^{q/2}$ is small (in this sense it is a stability result). To see that on the cone of nonnegative functions (1) extends the parallelogram identity, rearrange the latter, for nonzero x and y in a real Hilbert space, as follows (cf. [1, formula (2.0.2)]):

$$(x, y) = \|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right). \tag{2}$$

Writing (2) as a two sided inequality, adequately replacing some of the Hilbert space norms by p and q norms, and the terms $1/2$ by $1/p$ and $1/q$, we see that (1) indeed generalizes (2). Note also that $\|f^{p/2}\|_2 = \|f\|_p^{p/2}$. Save in the case where $p = q = 2$, the nonnegative functions $f \in L^p$ and $g \in L^q$ will in principle belong to different spaces, so to compare them L^2 is retained in (1) as the common measuring ground; to go from L^p and L^q into L^2 we use the Mazur map, which for nonnegative functions of norm 1 in L^p is simply $f \mapsto f^{p/2}$ (cf. [1] for more details).

Next we extend inequality (1) to the range $0 < r < 1$, keeping the role of L^2. Unlike the case of Hölder's inequality for $1 < p < \infty$, here we assume that $hk \in L^1$. In exchange, we do not need to suppose a priori that $h \in L^r$; this will be part of the conclusion.

Theorem 1. *Let* $0 < r < 1$, *and let* $s = s/(s-1)$ *be its conjugate exponent. If* $k \in L^s$, $hk \in L^1$, $\|h\|_r, \|k\|_s > 0$, *and* $1/2 \le r < 1$, *then*

$$\|hk\|_1 \left(1 - r\left\|\frac{h^{1/2}k^{1/2}}{\|h^{1/2}k^{1/2}\|_2} - \frac{k^{s/2}}{\|k^{s/2}\|_2}\right\|_2^2\right)_+^{\frac{1}{r}} \tag{3a}$$

$$\le \|h\|_r\|k\|_s \le \|hk\|_1 \left(1 - (1-r)\left\|\frac{h^{1/2}k^{1/2}}{\|h^{1/2}k^{1/2}\|_2} - \frac{k^{s/2}}{\|k^{s/2}\|_2}\right\|_2^2\right)^{\frac{1}{r}}, \tag{3b}$$

while if $0 < r \le 1/2$, *the terms* r *and* $1 - r$ *exchange their positions in the preceding inequalities.*

Proof. Suppose $1/2 \le r < 1$. Set $p = 1/r$ and use q and s to denote the conjugate exponents of p and r respectively. Since $1 < p \le 2$, we can apply (1) to the functions $f := h^r k^r$ and $g = k^{-r}$, which belong to L^p and L^q respectively: $\int f^p = \int hk < \infty$ and $\int g^q = \int k^s < \infty$. Now the inequalities (3) immediately follow. If $0 < r \le 1/2$, then $2 \le p < \infty$, so just interchange the terms $1/p$ and $1/q$ in (1). □

Note that from (3b), together with the hypothesis $\|h\|_r\|k\|_s > 0$, we get

$$0 < 1 - (1-r)\left\|\frac{h^{1/2}k^{1/2}}{\|h^{1/2}k^{1/2}\|_2} - \frac{k^{s/2}}{\|k^{s/2}\|_2}\right\|_2^2 \tag{4}$$

for all $r \in [1/2, 1)$ (for $r \in (1/2, 1)$ this already follows from $\left\|\frac{x}{\|x\|_2} - \frac{y}{\|y\|_2}\right\|_2^2 \le 2$, which is immediate from (2) when $x, y \ge 0$). The analogous result, with r instead of $1 - r$, holds when $0 < r \le 1/2$. Thus, (3b) can be rewritten as

$$\|h\|_r\|k\|_s \left(1 - (1-r)\left\|\frac{h^{1/2}k^{1/2}}{\|h^{1/2}k^{1/2}\|_2} - \frac{k^{s/2}}{\|k^{s/2}\|_2}\right\|_2^2\right)^{-\frac{1}{r}} \le \|hk\|_1 \tag{5}$$

when $1/2 \leq r < 1$, while if $0 < r \leq 1/2$, the same formula holds but with r replacing $1 - r$.

Now we are ready to obtain a sharpening of the reverse triangle inequality for nonnegative functions.

Theorem 2. *Let* $0 < r < 1$. *Given nonnegative functions* $h, w \in L^r$ *with* $\|h\|_r, \|w\|_r > 0$, *set* $k := (h+w)^{r-1}/\|(h+w)^{r-1}\|_s$. *Then, if* $1/2 \leq r < 1$, *we have*

$$\|h+w\|_r \geq \|h\|_r \left(1 - (1-r)\left\|\frac{h^{1/2}k^{1/2}}{\|h^{1/2}k^{1/2}\|_2} - k^{s/2}\right\|_2^2\right)^{-\frac{1}{r}} + \|w\|_r \left(1 - (1-r)\left\|\frac{w^{1/2}k^{1/2}}{\|w^{1/2}k^{1/2}\|_2} - k^{s/2}\right\|_2^2\right)^{-\frac{1}{r}}, \tag{6}$$

while if $0 < r \leq 1/2$, *the same inequality holds but with* $1 - r$ *replaced by* r.

Proof. Suppose $1/2 \leq r < 1$, and note that k is a unit vector in L^s. Hence, so is $k^{s/2}$ in L^2. By the nonnegativity of h and w we have

$$\|h+w\|_r = \int \frac{(h+w)^{r-1}}{\|(h+w)^{r-1}\|_s}(h+w) = \int hk + \int wk. \tag{7}$$

Since the left hand side of the preceding equality is finite, so are both integrals on the right hand side, and now the result follows by applying (4). If $0 < r \leq 1/2$, we argue in the same way, but with r replacing $1 - r$ in (4). □

Let us write $\theta(x,y) := \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$. To conclude, we make some comments on the size of $\theta(h^{1/2}k^{1/2}, k^{s/2})$, which also apply to $\theta(w^{1/2}k^{1/2}, k^{s/2})$. On a real Hilbert space, $\theta(x,y)$ is comparable to the angle between the vectors x and y. In particular, $\theta(h^{1/2}k^{1/2}, k^{s/2})$ is zero if and only if there exists a $t > 0$ such that $h = tw$, in which case $\|h+w\|_r = \|h\|_r + \|w\|_r$. Under any other circumstance, the inequality given by (6) is strictly better that the standard reverse triangle inequality.

On the other hand, if we ask how small

$$\left(1 - (1-r)\left\|\frac{h^{1/2}k^{1/2}}{\|h^{1/2}k^{1/2}\|_2} - k^{s/2}\right\|_2^2\right)^{\frac{1}{r}}$$

can be for $r \in [1/2, 1)$, the obvious bound $\theta(h^{1/2}k^{1/2}, k^{s/2}) \leq \sqrt{2}$ is informative when r is close to 1, but useless if $r = 1/2$. The analogous remark holds for

$$\left(1 - r\left\|\frac{h^{1/2}k^{1/2}}{\|h^{1/2}k^{1/2}\|_2} - k^{s/2}\right\|_2^2\right)^{\frac{1}{r}}$$

when $0 < r \leq 1/2$. However, nontrivial bounds also hold near $1/2$, since for every $r \in (0,1)$, $\|h+w\|_r \leq 2^{1/r-1}\left(\|h\|_r + \|w\|_r\right)$ (see for instance Exercise 13.25 a), [2, pg. 199]). Thus, $\theta(h^{1/2}k^{1/2}, k^{s/2})$ and $\theta(w^{1/2}k^{1/2}, k^{s/2})$ cannot be simultaneously large. More precisely, if $1/2 \leq r < 1$, then either

$$\theta^2(h^{1/2}k^{1/2}, k^{s/2}) \leq \frac{1-2^{r-1}}{1-r}$$

or

$$\theta^2(w^{1/2}k^{1/2}, k^{s/2}) \leq \frac{1-2^{r-1}}{1-r},$$

while if $0 < r \leq 1/2$, then either

$$\theta^2(h^{1/2}k^{1/2}, k^{s/2}) \leq \frac{1 - 2^{r-1}}{r}$$

or

$$\theta^2(w^{1/2}k^{1/2}, k^{s/2}) \leq \frac{1 - 2^{r-1}}{r}.$$

REFERENCES

[1] J.M. ALDAZ, A stability version of Hölder's inequality, *J. Math. Anal. Applics.*, to appear. doi:10.1016/j.jmaa.2008.01.104. [ONLINE: http://arxiv.org/abs/0710.2307].

[2] E. HEWITT AND K. STROMBERG, *Real and Abstract Analysis. A Modern Treatment of the Theory of Functions of a Real Variable*. Second printing corrected, Springer-Verlag, New York-Berlin, 1969.