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SOME NEW INEQUALITIES FOR THE GAMMA, BETA AND ZETA FUNCTIONS

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Abstract

An inequality involving a positive linear operator acting on the composition of two continuous functions is presented. This inequality leads to new inequalities involving the Beta, Gamma and Zeta functions and a large family of functions which are Mellin transforms.

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Contents

1. Introduction

Let I be the interval $(0, 1)$ or $(0, +\infty)$ and let f and g be functions which are strictly increasing, strictly positive and continuous on I. To fix ideas, we shall suppose that $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0+$. Suppose also that $f\,/g$ is strictly increasing.

Let L be a positive linear functional defined on a subspace $C^*(I) \subset C(I)$; see Note below. Supposing that $f, g \in C^*(I)$, define the function ϕ by

$$\phi = g\frac{L(f)}{L(g)}. \tag{1.1}$$

Next, let F be defined on the ranges of f and g so that the compositions $F(f)$ and $F(g)$ each belong to $C^*(I)$.

Note. In our applications the functional L will involve an integral over the interval I, and so that L will be well-defined, it is necessary to require extra end conditions to be satisfied by the members of $C(I)$. The subspace arrived at in this way will be denoted by $C^*(I)$ and this will be the domain of L.

The subspace $C^*(I)$ may vary from case to case but, for technical reasons, it will always be supposed that the functions e_k,where $e_k(x) = x^k$ $(k = 0, 1, 2)$, are in $C^*(I)$.

Our object is to prove the results:

Theorem 1.1.

(a) If F is convex then

$$L[F(f)] \geq L[F(\phi)]. \tag{1.2a}$$

(b) If F is concave then

$$L[F(f)] \leq L[F(\phi)]. \tag{1.2b}$$

Clearly it is sufficient to consider only (1.2a) and, prior to Section 3 where we present our applications, we shall proceed with this understanding.

In the note [1] this result was proved for the case in which I was $[0, 1]$, $g(x)$ was x, and F was differentiable but it has since been realised that the more general results of the present theorem are a source of interesting inequalities involving the Gamma, Beta and Zeta functions.

The method of proof in [1] could possibly be adapted to the present case but, instead, we shall give a proof which is entirely different. As well as using the more general $g(x)$ it allows the less stringent hypothesis that F is merely convex and deals with intervals other than $[0, 1]$. We also believe that this proof is of some interest in its own right.

2. Proofs

First, we need the following lemma:

Lemma 2.1.

$$L(f^2) - L(\phi^2) \geq 0. \tag{2.1}$$

Proof. It is seen from (1.1) that

$$L(f) - L(\phi) = 0.$$

Since L is positive, this negates the possibility that

$$f(x) - \phi(x) > 0 \quad \text{or} \quad f(x) - \phi(x) < 0 \quad \text{for all } x \in I.$$

Hence $f - \phi$ changes sign in I and since

$$f - \phi = f - g\frac{L(f)}{L(g)}$$

and

$$\frac{f}{g} \text{ is strictly increasing in } I,$$

this change of sign is from $-$ to $+$.

We suppose that the change of sign occurs at $x = \gamma$ and that $f(\gamma) = \phi(\gamma) = K$ (say).

Since $f - \phi$ is non-negative on $x \geq \gamma$ and $f + \phi \geq 2K$ there, then

$$(f - \phi)(f + \phi) \geq 2K(f - \phi) \text{ on } x \geq \gamma.$$

Since $f - \phi$ is negative on $x < \gamma$ and $f + \phi < 2K$ there then

$$(f - \phi)(f + \phi) > 2K(f - \phi) \text{ on } x < \gamma.$$

Hence

$$f^2 - \phi^2 = (f - \phi)(f + \phi) \geq 2K(f - \phi) \quad \text{on } I.$$

Applying L we get the result of the lemma. □

Proof of the theorem (part (a)). Let us introduce the functional Λ defined on $C^*(I)$ by

$$\Lambda(G) = L[G(f)] - L[G(\phi)],$$

in which f and ϕ are fixed. It is easily seen that Λ is a continuous linear functional.

According to the theorem, we will be interested in those F for which $F \in S$ where S is the subset of $C^*(I)$ consisting of continuous convex functions.

Now the set S is itself convex and closed so that the maximum and/or minimum values of Λ, when acting on S, will be taken in its set of extreme points, say $Ext(S)$.

But

$$Ext(S) = \{Ae_0 + Be_1\},$$

where $e_k(x) = x^k$ $(k = 0, 1, 2)$.

Now

$$\Lambda(e_0) = L[e_0(f)] - L[e_0(\phi)] = L(1) - L(1) = 0$$
$$\Lambda(e_1) = L[e_1(f)] - L[e_1(\phi)] = L(f) - L(\phi) = 0 \quad \text{by (1.1)}$$

so that zero is the (unique) extreme value of Λ.

Next

$$\Lambda(e_2) = L[e_2(f)] - L[e_2(\phi)] \ = L(f^2) - L(\phi^2) \geq 0 \ \text{ by } \ (2.1)$$

so this extreme value is a minimum. That is to say that

$$\Lambda(F) = L[F(f)] - L[F(\phi)] \ \geq 0 \text{ for all } F \in S$$

and this concludes the proof of the theorem. □

3. Preparation for the Applications

In (1.2a) and (1.2b) take

$$F(u) = u^{\alpha},$$

which is convex if $(\alpha < 0$ or $\alpha > 1)$ and concave if $0 < \alpha < 1$. So now we have

$$L(f^{\alpha}) \gtrless L(\phi^{\alpha})$$

with $\gtrless$ (upper and lower) respectively, in the cases 'convex', 'concave'. There is equality in case $\alpha = 0$ or $\alpha = 1$.

Substituting for ϕ this reads:

$$\frac{[L(g)]^{\alpha}}{L(g^{\alpha})} \gtrless \frac{[L(f)]^{\alpha}}{L(f^{\alpha})}. \tag{3.1}$$

Finally, take

$$f(x) = x^{\beta} \quad \text{and} \quad g(x) = x^{\delta} \quad \text{with} \quad \beta > \delta > 0.$$

Then (3.1) becomes (using incorrect, but simpler, notation):

$$\frac{[L(x^{\delta})]^{\alpha}}{L(x^{\alpha\delta})} \gtrless \frac{[L(x^{\beta})]^{\alpha}}{L(x^{\alpha\beta})}. \tag{3.2}$$

The inequality (3.2) is the source of our various examples.

4. Applications

Note. To avoid repetition in the examples below (except at (4.8)) it is to be understood that $\gtrless$ correspond to the cases ($\alpha < 0$ or $\alpha > 1$) and $(0 < \alpha < 1)$ respectively. There will be equality if $\alpha = 0$ or 1. Furthermore, it will always be the case that $\beta > \delta > 0$.

4.1. The Gamma function

Referring back to the Note in the Introduction, the subspace $C^*(I)$ for this application is obtained from $C(I)$ by requiring its members to satisfy:

(i) $w(x) = O(x^\theta)$ (for any $\theta > -1$) as $x \to 0$

(ii) $w(x) = O(x^\varphi)$ (for any finite φ) as $x \to +\infty$.

Then we define

$$L(w) = \int_0^\infty w(x)e^{-x}dx.$$

In this case (3.2) gives:

$$\frac{[\Gamma(1+\delta)]^\alpha}{\Gamma(1+\alpha\delta)} \gtrless \frac{[\Gamma(1+\beta)]^\alpha}{\Gamma(1+\alpha\beta)} \tag{4.1}$$

in which, $\alpha\beta > -1$ and $\alpha\delta > -1$.

In [2] this result was obtained partially in the form

$$\frac{[\Gamma(1+y)]^n}{\Gamma(1+ny)} > \frac{[\Gamma(1+x)]^n}{\Gamma(1+nx)},$$

where $1 \geq x > y > 0$ and $n = 2, 3, \ldots$.

Then, in [3] this was improved to

$$\frac{[\Gamma(1+y)]^{\alpha}}{\Gamma(1+\alpha y)} > \frac{[\Gamma(1+x)]^{\alpha}}{\Gamma(1+\alpha x)},$$

where $1 \geq x > y > 0$ and $\alpha > 1$.

The methods used in [2] and [3] to obtain these results are quite different from that used here.

4.2. The Beta function

The subspace $C^*(I)$ for this application is obtained from $C(I)$ by requiring its members to satisfy:

$$w(x) = O(x^{\theta}) \ \text{(for any } \theta > -1) \ \text{ as } x \to 0,$$
$$w(x) = O(1) \ \text{ as } \ x \to 1.$$

Then we define

$$L(w) = \int_0^1 w(x)(1-x)^{\zeta-1} dx : (\zeta > 0).$$

From (3.2) we have

$$\frac{[B(1+\delta,\zeta)]^{\alpha}}{B(1+\alpha\delta,\zeta)} \gtrless \frac{[B(1+\beta,\zeta)]^{\alpha}}{B(1+\alpha\beta,\zeta)}, \tag{4.2}$$

in which $\alpha\delta > -1$, $\alpha\beta > -1$ and $\zeta > 0$.

4.3. The Zeta function (i)

For this example the subspace $C^*(I)$ is the same as for the Gamma function case above. L is defined by

$$L(w) = \int_0^\infty w(x)\frac{xe^{-x}}{1-e^{-x}}dx.$$

We recall here (see [4]) that when s is real and $s > 1$ then

$$\Gamma(s)\zeta(s) = \int_0^\infty x^{s-1}\frac{e^{-x}}{1-e^{-x}}dx.$$

Using (3.2) this leads to

$$\frac{[\Gamma(2+\delta)\zeta(2+\delta)]^\alpha}{\Gamma(2+\alpha\delta)\zeta(2+\alpha\delta)} \gtrless \frac{[\Gamma(2+\beta)\zeta(2+\beta)]^\alpha}{\Gamma(2+\alpha\beta)\zeta(2+\alpha\beta)}, \tag{4.3}$$

in which $\alpha\beta > -1$ and $\alpha\delta > -1$.

The number of examples of this nature could be enlarged considerably. For example, the formula

$$\Gamma(s)\eta(s) = \int_0^\infty x^{s-1}\frac{e^{-x}}{1+e^{-x}}dx, \quad s > 0,$$

where

$$\eta(s) = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^s}$$

leads, via (3.2), to similar inequalities.

Indeed, recalling that the Mellin transform [5] of a function q is defined by

$$Q(s) = \int_0^\infty q(x) x^{s-1} dx,$$

we see that the Mellin transform of any non-negative function satisfies an inequality of the type (3.2). In fact, (4.1) and (4.3) are examples of this.

4.4. The Zeta function (ii)

We conclude by presenting a family of inequalities in which the Zeta function appears alone, in contrast with (4.3).

With $a > 1$ define the non-decreasing function $w_N \in [0,1]$ as follows:

$$\begin{aligned} w_N(x) &= 0 \quad \left(0 \le x < \frac{1}{N}\right) \\ &= \sum_{k=m}^{\infty} \frac{1}{k^a} \quad \left(\frac{1}{m} \le x < \frac{1}{m-1}\right), \quad m = N, N-1, \ldots, 2 \\ &= \sum_{k=1}^{\infty} \frac{1}{k^a} \quad (x = 1) \end{aligned}$$

Then we have

$$\text{(4.4)} \qquad \int_0^1 x^s dw_N(x) = \sum_{k=1}^{N-1} \frac{1}{k^{s+a}} + \frac{1}{N^s} \sum_{k=N}^{\infty} \frac{1}{k^a}$$

and we note that

$$\sum_{k=N}^{\infty} \frac{1}{k^a} < \frac{1}{a-1} \cdot \frac{1}{N^{a-1}}. \tag{4.5}$$

Writing

$$V_N(s) = \int_0^1 x^s dw_N(x) \quad \left(\equiv \int_{\frac{1}{N}}^1 x^s dw_N(x) \right)$$

and defining L on $C[0,1]$[1] by

$$L(v) = \int_0^1 v(x) dw_N(x)$$

then (3.2) gives the inequalities

$$\frac{[V_N(\delta)]^\alpha}{V_N(\alpha\delta)} \gtrless \frac{[V_N(\beta)]^\alpha}{V_N(\alpha\beta)}. \tag{4.6}$$

But, from (4.4) and (4.5), letting $N \to \infty$ shows that $V_N(s) \to \zeta(s+a)$ provided that $a > 1$ and $s > 0$ and so (4.6) gives the Zeta function inequality:

$$\frac{[\zeta(a+\delta)]^\alpha}{\zeta(a+\alpha\delta)} \gtrless \frac{[\zeta(a+\beta)]^\alpha}{\zeta(a+\alpha\beta)}, \tag{4.7}$$

provided $a > 1$, $\alpha\beta > 0$ and $\alpha\delta > 0$.

[1] Not a subspace of $C(0,1)$ but the theorem is true in this context also.

Finally, since the $\zeta(s)$ is known to be continuous for $s > 1$ we can now let $a \to 1$ in (4.7) provided that we keep $\alpha > 0$ when we get

$$\frac{[\zeta(1+\delta)]^{\alpha}}{\zeta(1+\alpha\delta)} \gtrless \frac{[\zeta(1+\beta)]^{\alpha}}{\zeta(1+\alpha\beta)}, \tag{4.8}$$

in which $\beta > \delta > 0$ and $\alpha > 0$. Regarding the directions of the inequalities here, we note that the option $\alpha \leq 0$ does not arise.

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